Trigonometry
MANUSCRIPT

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[1]

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Contents

Chapter 1

Trigonometry

1 Angles and Triangles

Trigonometry is the mathematical study of triangles. A triangle is defined as the intersection of three non-parallel straight lines as shown in Figure 1.1. The place where two lines intersect is called a *vertex*, and a triangle has three vertices. At each vertex, the triangle has an *interior* angle A, B, C. The vertex-to-vertex distance is a *side* of the triangle, AB, BC or CA, respectively.

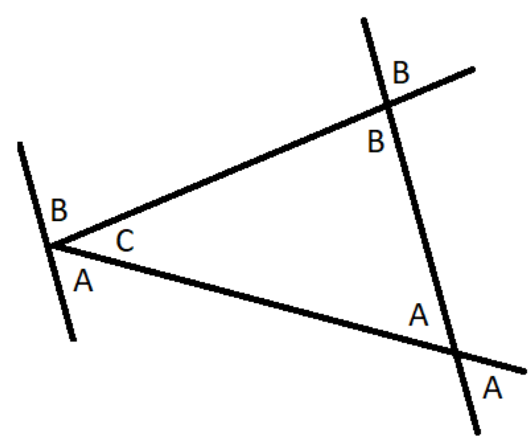

Figure 1.1: Triangle made from three lines.

Using the rules of Euclidean geometry, the angles A, B are portrayed with their *opposite* angles outside the triangle. Also from Euclidean geometry, we can imagine translating the segment AB to the left (as shown) until touching vertex C. From here, we see that the sum of angles $B+C+A$ is equivalent to the the angle represented by a straight line.

1.1 Angles

Any angle A, B, C, etc. is generally represented by the symbol θ (Greek *theta*), a parameter that must be a *dimensionless quantity*. That is, θ must be a pure number such as 3 or -17.5 , but never some measure of meters, seconds, or pounds.

Degrees and Radians

There are two standard units for representing angle, namely *degrees* and *radians*. By convention, a triangle encloses 180 degrees, also written $180°$, which is equivalent to π radians:

$$A + B + C = 180°$$
$$A + B + C = \pi$$

From these, we have a pair of unit conversion factors

$$1° = \frac{\pi}{180}\,\text{rad} \qquad 1\,\text{rad} = \frac{180°}{\pi}\,,$$

which extrapolates to the following:

$$\begin{aligned} 0° &= 0\,\text{rad} \\ 45° &= \pi/4\,\text{rad} \\ 90° &= \pi/2\,\text{rad} \\ 135° &= 3\pi/4\,\text{rad} \\ 360° &= 2\pi\,\text{rad} \end{aligned}$$

The primary domain for angles is represented by

$$0 \le \theta < 360°$$
$$0 \le \theta < 2\pi\,,$$

which is to say that any quantity depending on angle regards $0°$ and $360°$ to be synonymous. Or, an angle of $375°$ is effectively the same as $15°$.

1.2 Taxonomy of Triangles

Area of a Triangle

From geometry, we know that the area of any triangle is given by

$$\text{Area} = \frac{\text{Base} \times \text{Height}}{2}\,.$$

Equilateral Triangle

An *equilateral* triangle has all three sides of the same length. It follows too that all three angles must be the same, particularly $60°$ or $\pi/3$ rad, regardless of the size of the triangle. An equilateral triangle exhibits three-fold symmetry about its center, which is to say, there are three orientations of an equilateral triangle that appear identical.

Isosceles Triangle

An *isosceles* triangle has two equal sides and two equal angles. The third side and third angle are allowed to be larger or smaller than the other sides and angles. An isosceles triangle exhibits mirror symmetry about a line through the vertex of the two equal sides and the center of the triangle.

Scalene Triangle

A *scalene* triangle has no equal sides, no equal angles, and no symmetry. It's a typical 'unplanned' triangle.

Acute Triangle

An *acute* triangle has all internal angles less than $90°$.

Obtuse Triangle

An *obtuse* triangle has any one internal angle greater than $90°$.

1.3 Right Triangles

A *right triangle* is any triangle that has two sides meeting at $90° = \pi/2$ rad. Labeling either of the 'unused' angles as θ, the sides of the right triangle take on unique names as shown in Figures 1.2, 1.3.

- The side across from the ninety-degree angle is the *Hypotenuse*.
- The side across from θ is the *Opposite*.
- The side touching θ is the *Adjacent*.

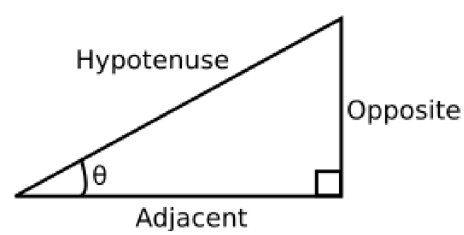

Figure 1.2: Right triangle.

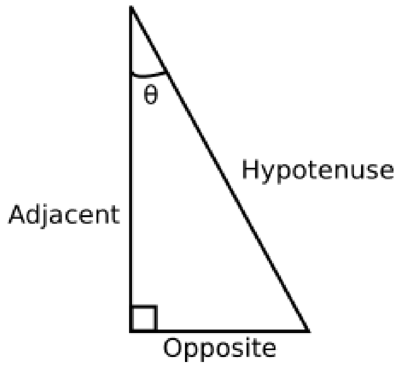

Figure 1.3: Right triangle.

Area of a Right Triangle

For the case of right triangles, it's convenient to associate Base, Height with Opposite, Adjacent (or vice versa):

$$A_{\text{Right}} = \frac{\text{Opposite} \times \text{Adjacent}}{2}$$

1.4 Pythagorean Theorem

The *Pythagorean theorem* is an equation that relates the sides of a right triangle to one another, and happens to also be the backbone equation of trigonometry. Figure 1.4 shows a typical right triangle with hypotenuse c, opposite a, and adjacent b.

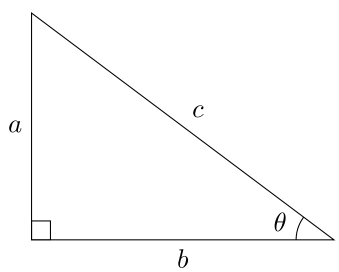

Figure 1.4: Right triangle.

To derive the Pythagorean theorem, imagine a line that extends from the ninety-degree vertex and intersects the hypotenuse at a right angle as shown in Figure 1.5.

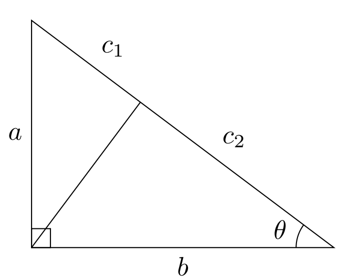

Figure 1.5: Similar right triangles.

With this, the hypotenuse is broken into two segments obeying

$$c_1 + c_2 = c\,.$$

Then, using similar triangles, we can write two observations:

$$\frac{c_1}{a} = \frac{a}{c}$$
$$\frac{c_2}{b} = \frac{b}{c}$$

Rearrange these and rewrite to get

$$c_1 c = a^2$$
$$c_2 c = b^2\,,$$

and then sum the two equations

$$c\,(c_1 + c_2) = a^2 + b^2\,,$$

and replace $c_1 + c_2$ with c to finish the job:

$$a^2 + b^2 = c^2 \tag{1.1}$$

Using the triangle side names in place of a, b, c yields the equivalent statement:

$$\text{Opposite}^2 + \text{Adjacent}^2 = \text{Hypotenuse}^2$$

Inverted Pythagorean Theorem

In the above construction, one can find straightforwardly that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c_1 c_2}\,.$$

As it turns out, the product $c_1 c_2$ has a significance.

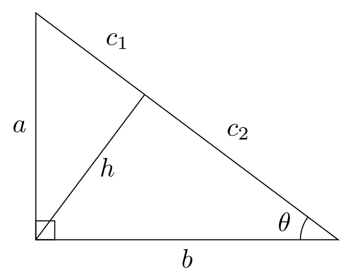

Figure 1.6: Similar right triangles.

Let the line connecting the ninety-degree corner to the hypotenuse be denoted h as shown in Figure 1.6. Using the Pythagorean theorem twice, we write

$$c_1^2 + h^2 = a^2$$
$$c_2^2 + h^2 = b^2\,.$$

Add the pair of equations an simplify to find

$$h^2 = c_1 c_2\,,$$

leading to an 'inverted' Pythagorean theorem:

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2} \tag{1.2}$$

1.5 Sine, Cosine, Tangent

On a right triangle, the opposite, adjacent, and hypotenuse can be stacked into ratios. These ratios have designated names:

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \tag{1.3}$$

$$\text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \tag{1.4}$$

$$\text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\text{Sine}}{\text{Cosine}} \tag{1.5}$$

The ratio of sides of a triangle, i.e. the sine, cosine, or tangent, is equivalent governed by the angle θ formed between the hypotenuse and the adjacent. It's customary to include the θ-dependence into the notation and create the abbreviations:

$$\begin{aligned} \text{Sine} &= \sin(\theta) \\ \text{Cosine} &= \cos(\theta) \\ \text{Tangent} &= \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \end{aligned}$$

SohCahToa

A useful mnemonic for recovering Equations (1.3)-(1.5) on the fly is the fictitious name:

SohCahToa

In this, the letter a stands for adjacent, o for opposite, and h for hypotenuse. Meanwhile S is for the sine, C for cosine, and T for tangent. Then, the SocCahToa shorthand expands to:

$$\begin{aligned} S &= o/h \\ C &= a/h \\ T &= o/a \end{aligned}$$

Fundamental Trigonometric Identity

Immediately from the Pythagorean theorem, we can write the most important equation in trigonometry, known as the *fundamental identity*. Starting with Equations (1.3), (1.4), square each and take the sum:

$$(\sin(\theta))^2 + (\cos(\theta))^2 = \frac{\text{Opposite}^2 + \text{Adjacent}^2}{\text{Hypotenuse}^2}$$

The right side is identically one due to the Pythagorean theorem. On the left is the sum of sine squared and cosine squared, which is conventionally written with the exponent before the parentheses:

$$(\sin(\theta))^2 + (\cos(\theta))^2 = \sin^2(\theta) + \cos^2(\theta)$$

In concise form, the fundamental trigonometric identity reads:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{1.6}$$

1.6 Special Right Triangles

Right Isosceles Triangle

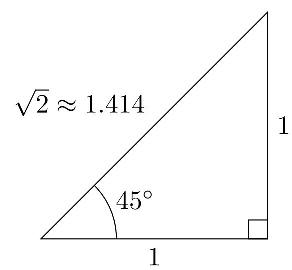

Figure 1.7: Right triangle: isosceles (unit sides)

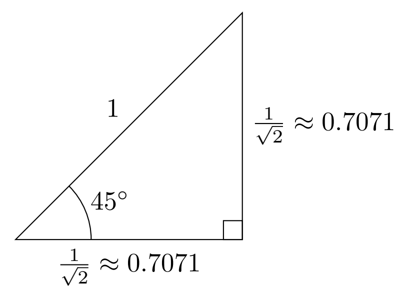

Figure 1.8: Right triangle: isosceles (unit hypotenuse)

Right Pi/8 Triangle

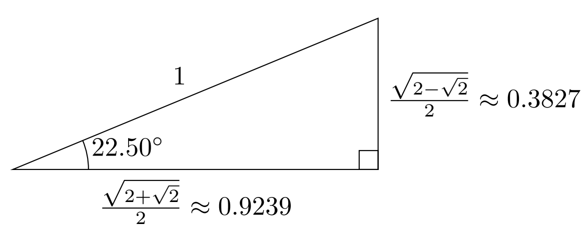

Figure 1.9: Right triangle: 22.5° (opposing angle 67.5°, unit hypotenuse)

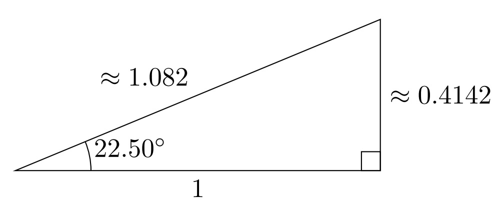

Figure 1.10: Right triangle: 22.5° (opposing angle 67.5°, unit adjacent)

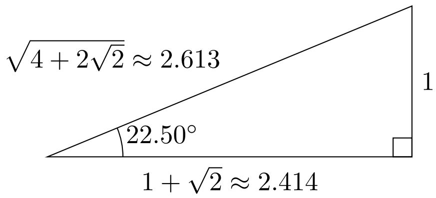

Figure 1.11: Right triangle: 22.5° (opposing angle 67.5°, unit opposite)

Right Pi/16 Triangle

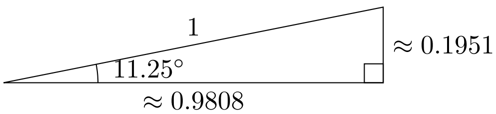

Figure 1.12: Right triangle: 11.25° (opposing angle 78.75°, unit hypotenuse)

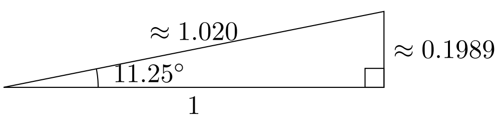

Figure 1.13: Right triangle: 11.25° (opposing angle 78.75°, unit adjacent)

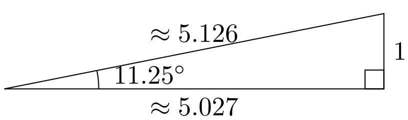

Figure 1.14: Right triangle: 11.25° (opposing angle 78.75°, unit opposite)

Right 30-60-90 Triangle

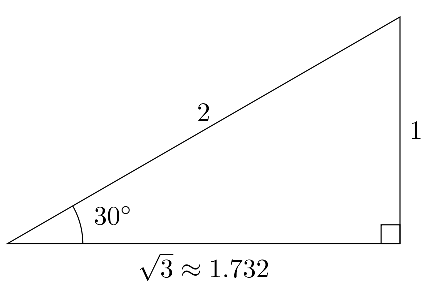

Figure 1.15: Right triangle: 30-60-90

Right 3-4-5 Triangle

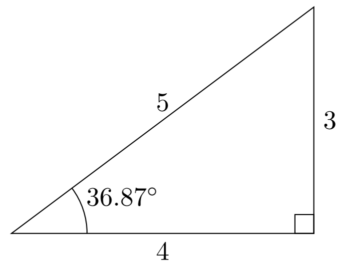

Figure 1.16: Right triangle: 3-4-5

Right 5-12-13 Triangle

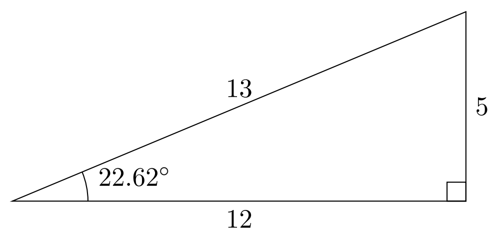

Figure 1.17: Right triangle: 5-12-13

Pythagorean Triplets

A *Pythagorean triplet* is any set of three integers that obey the Pythagorean theorem. In addition to the triplet $(3, 4, 5)$, $(5, 12, 13)$ detailed above, one can also have (but not limited to):

$$7, 24, 25$$
$$8, 5, 17$$
$$9, 40, 41$$

2 Circles

In the Cartesian plane, a circle is most generally described by

$$(x-h)^2 + (y-k)^2 = R^2 \,, \tag{1.7}$$

where the center of the circle is located at (h, k) and the radius is R as depicted in Figure 1.18.

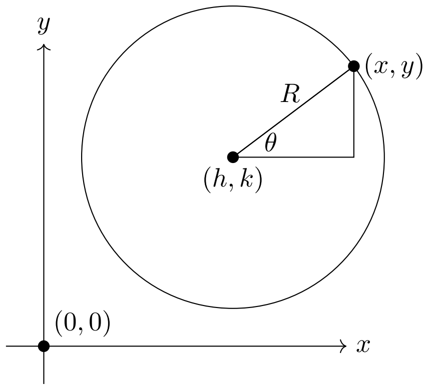

Figure 1.18: Offset circle.

Taking a second look at this construction, notice that the line joining the center of the circle to a point (x, y) on its perimeter is the hypotenuse of a right triangle where θ is defined to 'rise off' a line parallel to the x-axis.

Theta Convention

By tradition, the angle θ always 'opens up' in the counter-clockwise direction, starting from $\theta = 0$, measured from a ray parallel to the positive x-axis.

2.1 Taxonomy of Circles

As we've seen a circle is entirely characterized by its center (h, k) and its radius R.

Diameter

The *diameter* of any shape is distance between its maximally-separated points on its perimeter. On a circle, any point one chooses has a 'twin' across the circle precisely distance $2R$ away. It follows that the diameter of the circle is 2r:

$$\text{Diameter} = 2R$$

Circumference

The *circumference* of a circle is the total length of its perimeter. This is in fact where the definition of π originates:

$$\text{Circumference} = 2\pi R$$

Area

It's straightforward to show, although not using trigonometry alone, that the area of a circle is

$$A = \pi R^2 \; .$$

Arc Length

In terms of θ, the distance along a circular perimeter is given by

$$S = R\theta \; ,$$

where S is called *arc length*. Let $\theta = 2\pi$ for the arc length to recover the circumference.

Inscription Problem

Let ABC be a triangle with right angle A and hypotenuse $|BC|$ as shown in Figure 1.19. If the inscribed circle of radius R touches the hypotenuse at D, show that:

$$|CD| = \frac{|AC| + |BC| - |AB|}{2}$$

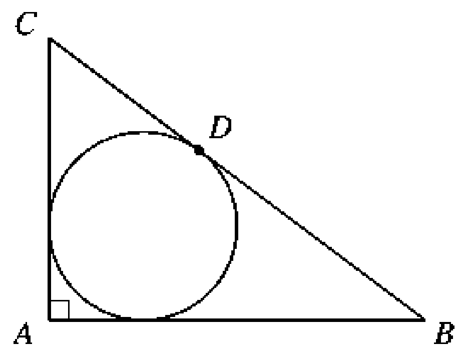

Figure 1.19: Inscribed Circle in Right Triangle

By inspecting the Figure, there are two ways to write the radius of the circle:

$$R = |AB| - |BD|$$
$$R = |AC| - |CD|$$

Eliminating R tells us

$$|AB| - |BD| = |AC| - |CD| \; .$$

Replace $|BD|$ using

$$|BD| = |BC| - |CD| \; ,$$

and solve for $|CD|$ to get the answer.

2.2 Parameterized Circle

A triangle having a fixed hypotenuse with continuously adjustable opposite and adjacent sides is all one needs to trace out a circle. By letting θ sweep from 0 to 2π, the endpoint of the hypotenuse, having location (x, y) in the plane, is described by:

$$x(\theta) = h + R\cos(\theta) \tag{1.8}$$
$$y(\theta) = k + R\sin(\theta) \tag{1.9}$$

The above represents the *parameterized* equation of a circle. To quickly recover Equation (1.7), solve for $\cos(\theta)$, $\sin(\theta)$, respectively, and exploit Equation (1.6).

2.3 Unit Circle

A circle centered at the origin with unit radius, i.e. $(h, k) = (0, 0)$ and $R = 1$, is called the *unit circle*. The unit circle is a special case where the adjacent side is equal to $\cos(\theta)$ and the opposite side is equal to $\sin(\theta)$. In the Cartesian plane, the unit circle is:

$$x^2 + y^2 = 1$$

The unit circle is most useful as a data structure to help remember the sine and cosine values of key angles, as shown in Figure 1.20.

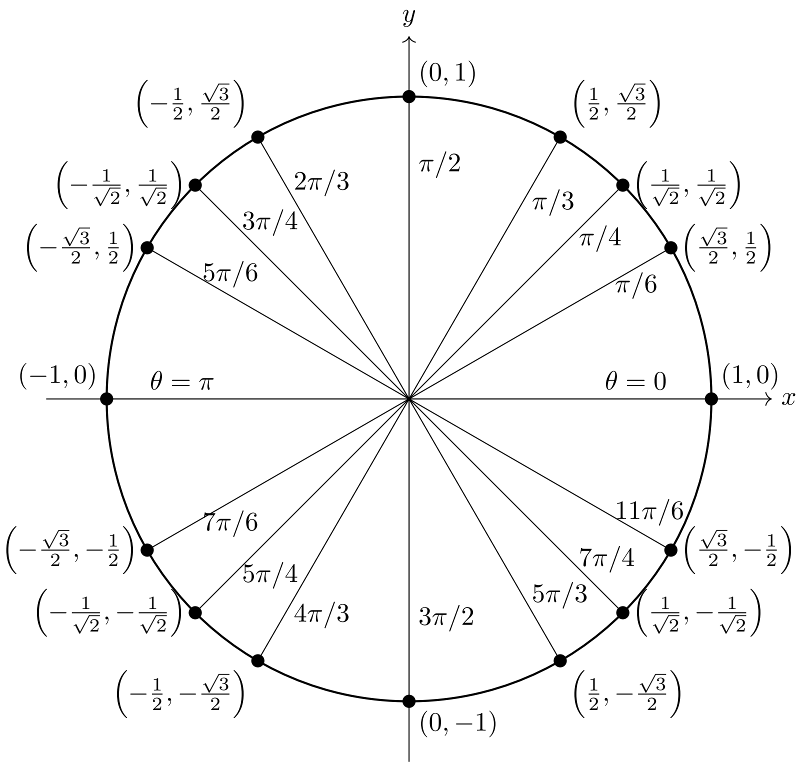

Figure 1.20: Unit circle.

2.4 Tangent Line

In the general sense, a *tangent* line is a straight line that touches a curve (locally) in one place, and the slope of the curve at the point of contact equals the slope of the line. When it comes to circles, the line is relatively straightforward to analyze.

Consider the unit circle (radius one, centered at the origin) with a point (x_0, y_0) selected somewhere on the perimeter. The slope of the 'position line' from the origin to (x_0, y_0) is naturally y_0, x_0, which is identically $\tan(\theta)$:

$$\tan(\theta) = \frac{y_0}{x_0}$$

The tangent line to the unit circle at (x_0, y_0) has slope $-x_0/y_0$, and is sketched in Figure 1.21. By standard straight line analysis, the equation of the tangent line obeys

$$\frac{y - y_0}{x - x_0} = -\frac{x_0}{y_0}\ .$$

More concisely, the same equation can be written

$$x x_0 + y y_0 = 1\ . \tag{1.10}$$

It just happens that the length of the tangent line segment from (x_0, y_0) to its intersection with the x-axis is equal to $\tan(\theta)$. To prove this, note that the line's intersection with the x-axis occurs at $(1/x_0, 0)$, and then construct the distance

$$\sqrt{(x_0 - 1/x_0)^2 + y_0^2}\ ,$$

which simplifies to x_0/y_0, the definition of $\tan(\theta)$ on the unit circle.

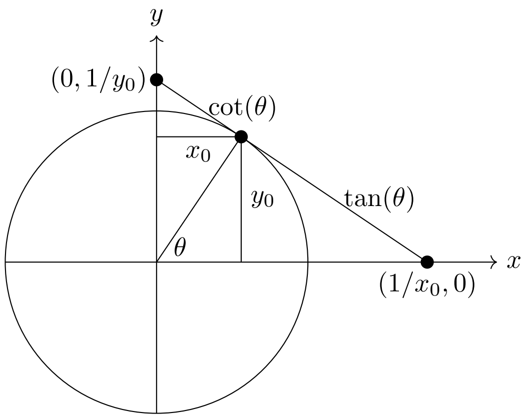

Figure 1.21: Unit circle with tangent line.

2.5 Cotangent

The *cotangent* (of angle θ) is defined as

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \,. \tag{1.11}$$

On the unit circle, $\cot(\theta)$ is the segment of the tangent line extending from $(0, 1/y_0)$ to (x_0, y_0). To establish this, much like the tangent case, simplify the quantity

$$\sqrt{(1/y_0 - y_0)^2 + x^2} \,,$$

which comes out to the ratio x_0/y_0, the definition of $\cot(\theta)$ on the unit circle.

2.6 Secant

Continuing with Figure 1.21, the distance from the origin to the point $(1/x_0, 0)$ is called the *secant*. This is not to be confused with a 'secant line', which is the extension of a chord through the circle.

To relate the secant to the existing items of trigonometry, observe from the Figure that

$$(\cot(\theta) + \tan(\theta))\sin(\theta) = \frac{1}{x_0} \,,$$

which, using Equation (1.5) and Equation (1.6), simplifies to

$$\frac{1}{\cos(\theta)} = \frac{1}{x_0} \,.$$

Since x_0 is already claimed as the cosine of theta, we have:

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{1.12}$$

2.7 Cosecant

The distance from the origin to the point $(0, 1/y_0)$ is called the *cosecant*.

Much like the secant case, observe from the Figure that

$$(\cot(\theta) + \tan(\theta))\cos(\theta) = \frac{1}{y_0} \,,$$

which simplifies to

$$\frac{1}{\sin(\theta)} = \frac{1}{y_0} \,.$$

Since x_0 is already claimed as the sine of theta, we have:

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{1.13}$$

Inverted Pythagorean Theorem

For a proverbial a checksum on Figure 1.21, recall the inverted Pythagorean theorem

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2} \,.$$

In this case, identify

$$\begin{aligned} a &= 1/x_0 \\ b &= 1/y_0 \\ h &= \sqrt{x_0^2 + y_0^2} = 1 \end{aligned}$$

2.8 Periodicity

Due to the confined domain $[0 : 2\pi)$ of the θ-variable, it follows that quantities like $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$ are only unique in this interval. It's just fine, however, to feed θ-values outside the standard domain. Before $\theta = 0$ or after $\theta = 2\pi$, everything repeats via

$$\sin(\theta \pm 2n\pi) = \sin(\theta) \tag{1.14}$$
$$\cos(\theta \pm 2n\pi) = \cos(\theta) \tag{1.15}$$
$$\tan(\theta \pm 2n\pi) = \tan(\theta) \,, \tag{1.16}$$

where n is any integer. This property is called *periodicity*.

2.9 Phase

A *phase shift* occurs when any quantity is added to θ.

Negative Angles

If we replace θ by $-\theta$, the symmetry of the unit circle demands:

$$\sin(-\theta) = -\sin(\theta) \tag{1.17}$$
$$\cos(-\theta) = \cos(\theta) \tag{1.18}$$
$$\tan(-\theta) = -\tan(\theta) \tag{1.19}$$

Phase Shift Pi

A phase shift of π radians jumps exactly across the unit circle. Accordingly, we have:

$$\sin(\theta \pm \pi) = -\sin(\theta) \tag{1.20}$$
$$\cos(\theta \pm \pi) = -\cos(\theta) \tag{1.21}$$

Phase Shift Pi/2

As they're defined, it turns out that $\sin(\theta)$ and $\cos(\theta)$ are related by the phase $\pi/2$:

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta) \tag{1.22}$$
$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta) \tag{1.23}$$
$$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta) \tag{1.24}$$
$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta) \tag{1.25}$$

Similar equations apply when θ is replaced with $-\theta$:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \tag{1.26}$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta) \tag{1.27}$$

3 Trigonometric Identities

It turns out that $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$, along with their reciprocated counterparts, fit into a slew of equations called *trigonometric identities*. In practice, the so-called 'trig identities' are bits of algebra that can be used to elaborate on or simplify a given situation.

3.1 Fundamental Trig Identities

The fundamental trigonometric identity first documented as Equation (1.6), namely

$$\sin^2(\theta) + \cos^2(\theta) = 1 ,$$

can be exploited to yield several more. Divide by $\sin^2(\theta)$ or by $\cos^2(\theta)$ to yield the following:

$$1 + \cot^2(\theta) = \csc^2(\theta) \tag{1.28}$$
$$\tan^2(\theta) + 1 = \sec^2(\theta) \tag{1.29}$$

For an interesting sanity check, take the sum of the two above equations to come up with

$$(\tan(\theta) + \cot(\theta))^2 = \csc^2(\theta) + \sec^2(\theta) , \tag{1.30}$$

which is the summary of Figure 1.21.

3.2 Angle-Sum Formulas

Consider the sum of two angles α, β, as they embed in the unit circle as shown in Figure 1.22. The triangle swept out by β has adjacent side $\cos(\beta)$ and opposite side $\cos(\beta)$. Each of these sides is the hypotenuse of a pair of right triangles whose sides are the products denoted in the Figure.

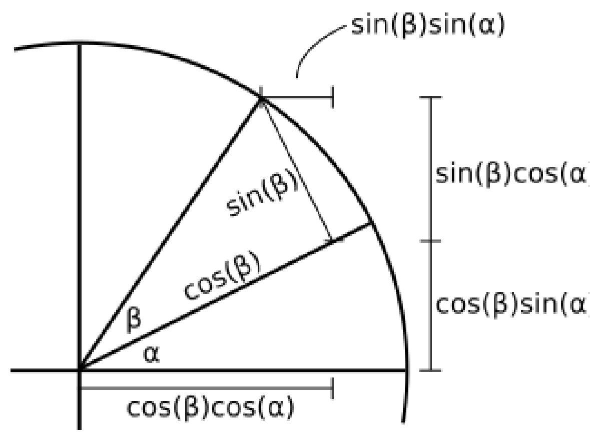

Figure 1.22: Angle-sum analysis.

Expressing $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$ in terms of the products of individual terms, we find by inspection the *angle-sum formulas*:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{1.31}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{1.32}$$

Using these two results, we can easily calculate the tangent of $\alpha + \beta$:

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)} \tag{1.33}$$

3.3 Product Formulas

Starting with the angle-sum formulas (1.31), (1.32), it's straightforward to derive the *product formulas*:

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{1.34}$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{1.35}$$

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta) \tag{1.36}$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{1.37}$$

3.4 Double-Angle Formulas

Starting with the product formulas, let $\alpha = \beta = \theta$ to derive the *double-angle formulas*:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \tag{1.38}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \tag{1.39}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} \tag{1.40}$$

3.5 Half-Angle Formulas

Starting with equation (1.39), replace the $\sin^2(\theta)$ term and also replace $\theta \to \theta/2$ to write

$$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \; .$$

From here, it's a small matter of algebra to generate the *half-angle formulas*:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2} - \frac{\cos(\theta)}{2}} \tag{1.41}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2} + \frac{\cos(\theta)}{2}} \tag{1.42}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)} \tag{1.43}$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1-\cos(\theta)} \tag{1.44}$$

$$\sec\left(\frac{\theta}{2}\right) = \frac{2\cos(\theta/2)}{1+\cos(\theta)} \tag{1.45}$$

3.6 Superposition Relationships

Superposition of Sines

Consider the sum $\alpha+\beta$ and difference $\alpha-\beta$ of two angles. Take the sine and cosine, respectively, of each quantity and take their product

$$\begin{aligned}\sin(\alpha+\beta)\cos(\alpha-\beta) = \\ &(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \times \\ &(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \; ,\end{aligned}$$

simplifying to, after a bit of work,

$$\sin(\alpha+\beta)\cos(\alpha-\beta) = \frac{\sin(2\alpha)}{2} + \frac{\sin(2\beta)}{2} \; .$$

Refactor the α, β variables to get the first result:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.46}$$

Replace β with $-\beta$ to get the second superposition relationship for free:

$$\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{1.47}$$

These are both called *superposition relationships*.

Superposition of Cosines

Starting with the superposition relationships above, introduce the phase shifts:

$$\begin{aligned}\alpha &\to \alpha + \pi/2 \\ \beta &\to \beta - \pi/2\end{aligned}$$

Inserting these and simplifying gives two more superposition relationships for the cosine:

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.48}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{1.49}$$

4 Inverse Trigonometry

For each quantity $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\csc(\theta)$, $\sec(\theta)$, $\cot(\theta)$, there exists an *inverse* trigonometric quantity that does the job of 'solving for' θ. These are called the arc-sine, arc-cosine, arc-tangent, and so on, defined as follows:

$$\arcsin(\sin(\theta)) = \theta \tag{1.50}$$
$$\arccos(\cos(\theta)) = \theta \tag{1.51}$$
$$\arctan(\tan(\theta)) = \theta \tag{1.52}$$
$$\operatorname{arccsc}(\csc(\theta)) = \theta \tag{1.53}$$
$$\operatorname{arcsec}(\sec(\theta)) = \theta \tag{1.54}$$
$$\operatorname{arccot}(\cot(\theta)) = \theta \tag{1.55}$$

Inverse Trig Notation

Confusingly enough, there is another way to write $\arcsin(\theta)$, $\arccos(\theta)$, etc., using the nomenclature

$$\begin{aligned} \sin^{-1}(\theta) &= \arcsin(\theta) \\ \cos^{-1}(\theta) &= \arccos(\theta) \ , \end{aligned}$$

and so on. This overloading of notation does not mean at all, for instance, that the $\arcsin(\theta)$ is equal to the reciprocal of $\sin(\theta)$.

4.1 Inverse Reciprocal Identities

Some handy identities we can establish early are:

$$\arcsin(1/x) = \operatorname{arccsc}(x) \tag{1.56}$$
$$\operatorname{arccsc}(1/x) = \arcsin(x) \tag{1.57}$$
$$\arccos(1/x) = \operatorname{arcsec}(x) \tag{1.58}$$
$$\operatorname{arcsec}(1/x) = \arccos(x) \tag{1.59}$$
$$\arctan(1/x) = \operatorname{arccot}(x) \tag{1.60}$$
$$\operatorname{arccot}(1/x) = \arctan(x) \tag{1.61}$$

To prove any of the above will demonstrate how to handle the rest. Choosing the arctan case,

$$\begin{aligned} A &= \arctan(1/x) \\ B &= \operatorname{arccot}(x) \ , \end{aligned}$$

and then

$$\begin{aligned} \tan(A) &= 1/x \\ \cot(B) &= x \ . \end{aligned}$$

From this we see $\tan(A) = \tan(B)$, meaning $A = B$, and the proof is done.

4.2 Inverse Triangle Identities

Arccosine

Consider a right triangle with hypotenuse 1, adjacent side x, and opposite side $\sqrt{1-x^2}$. (This is just the unit circle centered at the origin.) From this, we can gather

$$\begin{aligned} x &= \cos(\theta) \\ \arccos(x) &= \theta \\ \sin(\arccos(x)) &= \sin(\theta) \ , \end{aligned}$$

resulting in:

$$\sin(\arccos(x)) = \sqrt{1-x^2} \tag{1.62}$$

Divide through by x to get a second result:

$$\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x} \tag{1.63}$$

Arcsine

Now we modify the triangle slightly. Suppose the hypotenuse of another right triangle is 1, and the opposite side is x, making the adjacent equal to $\sqrt{1-x^2}$. From this, we can gather

$$\begin{aligned} x &= \sin(\theta) \\ \arcsin(x) &= \theta \\ \cos(\arcsin(x)) &= \cos(\theta) \ , \end{aligned}$$

resulting in:

$$\cos(\arcsin(x)) = \sqrt{1-x^2} \tag{1.64}$$

Similarly we can establish:

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}} \tag{1.65}$$

Arctangent

Consider a new right triangle with hypotenuse $\sqrt{x^2+1}$, adjacent side 1, and opposite side x. For this case, we have

$$\begin{aligned} x &= \tan(\theta) \\ \arctan(x) &= \theta \ , \end{aligned}$$

implying

$$\begin{aligned} \cos(\arctan(x)) &= \cos(\theta) \\ \sin(\arctan(x)) &= \sin(\theta) \ . \end{aligned}$$

From these, conclude:

$$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}} \tag{1.66}$$
$$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}} \tag{1.67}$$

Arccosecant

The reciprocal trig quantities are a little harder to analyze. For the arc-cosecant, consider a right triangle with hypotenuse x, opposite side 1, and adjacent side $\sqrt{x^2-1}$. Following this, we find:

$$\begin{aligned} x &= \csc(\theta) \\ \operatorname{arccsc}(x) &= \theta \,, \end{aligned}$$

implying

$$\begin{aligned} \sin(\operatorname{arccsc}(x)) &= \sin(\theta) \\ \cos(\operatorname{arccsc}(x)) &= \cos(\theta) \ . \end{aligned}$$

From these, conclude:

$$\sin(\operatorname{arccsc}(x)) = \frac{1}{x} \tag{1.68}$$

$$\cos(\operatorname{arccsc}(x)) = \frac{\sqrt{x^2-1}}{x} \tag{1.69}$$

$$\tan(\operatorname{arccsc}(x)) = \frac{1}{\sqrt{x^2-1}} \tag{1.70}$$

Arcsecant

To handle the arc-secant case, swap the role of the opposite and adjacent sides in the right triangle used for the arc-cosecant case:

$$\sin(\operatorname{arcsec}(x)) = \frac{\sqrt{x^2-1}}{x} \tag{1.71}$$

$$\cos(\operatorname{arcsec}(x)) = \frac{1}{x} \tag{1.72}$$

$$\tan(\operatorname{arcsec}(x)) = \sqrt{x^2-1} \tag{1.73}$$

Arccotangent

To complete the ensemble, consider a right triangle with hypotenuse $\sqrt{x^2+1}$, opposite side 1, and adjacent side x. Running through the standard exercise gives three new results:

$$\tan(\operatorname{arccot}(x)) = \frac{1}{x} \tag{1.74}$$

$$\sin(\operatorname{arccot}(x)) = \frac{1}{\sqrt{x^2+1}} \tag{1.75}$$

$$\cos(\operatorname{arccot}(x)) = \frac{x}{\sqrt{x^2+1}} \tag{1.76}$$

5 Trigonometry Tables

Trigonometry tables are lists of data containing key values of $\sin(\theta)$, $\cos(\theta)$. Contained in the tables that follow are the data generated by a trip around the unit circle.

5.1 Standard Trigonometry Tables

First Quadrant

Angle (rad)	Angle (°)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
0	0	0	1	0	$\mp\infty$	1	$\mp\infty$
$\pi/16$	11.25	0.195	0.981	0.198	5.142	1.020	5.081
$\pi/8$	22.5	0.383	0.924	0.414	2.610	1.086	2.414
$3\pi/16$	33.75	0.556	0.831	0.671	1.795	1.202	1.486
$\pi/4$	45	0.707	0.707	1	1.414	1.414	1
$5\pi/16$	56.25	0.831	0.556	1.496	1.202	1.795	0.671
$3\pi/8$	67.5	0.924	0.383	2.414	1.086	2.610	0.414
$7\pi/16$	78.75	0.981	0.195	5.081	1.020	5.142	0.198
$\pi/2$	90	1	0	$\pm\infty$	1	$\pm\infty$	0

Second Quadrant

Angle (rad)	Angle (°)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$9\pi/16$	101.25	0.981	-0.195	-5.081	1.020	-5.142	-0.198
$5\pi/8$	112.5	0.924	-0.383	-2.414	1.086	-2.610	-0.414
$11\pi/16$	123.75	0.831	-0.556	-1.496	1.202	-1.795	-0.671
$3\pi/4$	135	0.707	-0.707	-1	1.414	-1.414	-1
$13\pi/16$	146.25	0.556	-0.831	-0.671	1.795	-1.202	-1.486
$7\pi/8$	157.5	0.383	-0.924	-0.414	2.610	-1.086	-2.414
$15\pi/16$	168.75	0.195	-0.981	-0.198	5.142	-1.020	-5.081
π	180	0	-1	0	$\pm\infty$	-1	$\mp\infty$

Third Quadrant

Angle (rad)	Angle (°)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$17\pi/16$	191.25	-0.195	-0.981	0.198	-5.142	-1.020	5.081
$9\pi/8$	202.5	-0.383	-0.924	0.414	-2.610	-1.086	2.414
$19\pi/16$	213.75	-0.556	-0.831	0.671	-1.795	-1.202	1.486
$5\pi/4$	225	-0.707	-0.707	1	-1.414	-1.414	1
$21\pi/16$	236.25	-0.831	-0.556	1.496	-1.202	-1.795	0.671
$11\pi/8$	247.5	-0.924	-0.383	2.414	-1.086	-2.610	0.414
$23\pi/16$	258.75	-0.981	-0.195	5.081	-1.020	-5.142	0.198
$3\pi/2$	270	-1	0	$\pm\infty$	-1	$\mp\infty$	0

Fourth Quadrant

Angle (rad)	Angle (°)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$25\pi/16$	281.25	-0.981	0.195	-5.081	-1.020	5.142	-0.198
$13\pi/8$	292.5	-0.924	0.383	-2.414	-1.086	2.610	-0.414
$27\pi/16$	303.75	-0.831	0.556	-1.496	-1.202	1.795	-0.671
$7\pi/4$	315	-0.707	0.707	-1	-1.414	1.414	-1
$29\pi/16$	326.25	-0.556	0.831	-0.671	-1.795	1.202	-1.486
$15\pi/8$	337.5	-0.383	0.924	-0.414	-2.610	1.086	-2.414
$31\pi/16$	348.75	-0.195	0.981	-0.198	-5.142	1.020	-5.081
2π	360	0	1	0	$\mp\infty$	1	$\mp\infty$

5.2 Generating Trigonometry Tables

A scientific calculator should be able to generate values for $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ in degrees or radians. Sticking just with tables for a moment, there arises the question of, what if we need information for a value not explicitly listed, i.e., what's the cosine of 83 degrees? Although slightly ahead of the standard trigonometry regimen, we can explore two answers to this question.

Trigonometry from Polynomials

As it turns out, it's possible to show that the sine and cosine can be calculated *exactly* for any θ using the expansions:

$$\begin{aligned}\sin(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\ \cos(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\end{aligned}$$

The expansions on the right obey all of the rules of the sine and cosine, respectively. From these, all other quantities can be generated. Note that θ must occur in radians, not degrees.

With the key terms $\sin(\theta)$, $\cos(\theta)$ in hand, all of the others can be derived from these as a matter of definition.

Small-Angle Approximation

To prepare for a second means of generating trigonometry tables, particularly intermediate values in an existing table, we'll need to borrow ahead from calculus and write the so-called 'small-angle approximation'. In essence, note that for very small angles θ, the sine and cosine become:

$$\begin{aligned}\sin(\theta) &\approx \theta - \cancel{\frac{\theta^3}{3!} + \cdots} \\ \cos(\theta) &\approx 1 - \cancel{\frac{\theta^2}{2!} + \cdots}\,,\end{aligned}$$

or move concisely,

$$\begin{aligned}&\theta \text{ small:} \qquad \sin(\theta) \approx \theta \\ &\theta \text{ small:} \qquad \cos(\theta) \approx 1\,.\end{aligned}$$

The small angle approximation can be discovered a number of ways, not necessarily from calculus. However, the notation surely couches best in a calculus framework.

Trig Tables by Interpolation

Now, recall the so-called *angle-sum formulas* via Equations (1.31), (1.32), namely

$$\begin{aligned} \sin(\alpha+\beta) &= \\ &\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ \cos(\alpha+\beta) &= \\ &\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\ , \end{aligned}$$

and put the following restrictions on α, β:

$$\begin{aligned} 0 \le \alpha < 2\pi \\ |\beta| \ll \alpha \end{aligned}$$

In other words, α is treated like a regular angle, and β is a very small angle. Using the small-angle approximation on β, the above can be approximately restated:

$$\sin(\alpha+\beta) \approx \sin(\alpha)+\cos(\alpha)\,\beta \tag{1.77}$$

$$\cos(\alpha+\beta) \approx \cos(\alpha)-\sin(\alpha)\,\beta \tag{1.78}$$

To run through an example, suppose we want the cosine of 83 degrees. Step one is to look at the closest entry in the existing trigonometry table, where we find

$$\cos\left(\frac{7\pi}{16}\right) = \cos(78.75^\circ) = 0.195\ .$$

The difference between 83 degrees and 78.75 degrees is assigned to β:

$$\beta = 83^\circ - 78.75^\circ = 4.25^\circ = 0.0742\,\text{rad}$$

With β in radians, we can plug into Equation (1.78) straightforwardly:

$$\begin{aligned} &\cos(83^\circ) \approx \cos(78.75^\circ) - \sin(78.75^\circ)\,(0.0742) \\ &\cos(83^\circ) \approx 0.1223 \end{aligned}$$

For a sanity check the 'exact' value of $\cos(83^\circ)$ is about 0.1219. Of course, had we chosen a smaller β to begin with, the approximation would be more accurate. In this same spirit, by choosing β in small increments, trigonometry tables of any size can be calculated with enough patience or resources.

Example 1

Given $\sin(20^\circ) = 0.342$ and $\cos(20^\circ) = 0.940$, estimate $\sin(22^\circ)$ and $\cos(22^\circ)$.

Let

$$\begin{aligned} \alpha &= 20^\circ = 20^\circ\left(\frac{\pi}{180^\circ}\right) = 0.349 \\ \beta &= 2^\circ = 2^\circ\left(\frac{\pi}{180^\circ}\right) = 0.0349\ , \end{aligned}$$

so then:

$$\begin{aligned} \sin(22^\circ) &\approx 0.342 + 0.0349\,(0.940) \approx 0.375 \\ \cos(22^\circ) &\approx 0.940 + 0.0349\,(0.342) \approx 0.928 \end{aligned}$$

Insanity Check

As a matter of brutal curiosity, it's worth checking (once, if ever) that the polynomial expansions for $\sin(\theta)$, $\cos(\theta)$ obey the fundamental identity

$$\sin^2(\theta) + \cos^2(\theta) = 1\ .$$

This chore involves squaring two infinite polynomials, a technically impossible task, but the idea is to spot a pattern in the algebra so the work doesn't go forever.

For a shorthand notation let $S = \sin(\theta)$, $C = \cos(\theta)$, and square each of these separately:

$$\begin{aligned} S^2 &= S\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots\right) \\ &= (1)\,x^2 - \left(\frac{2}{3!}\right)x^4 + \left(\frac{2}{5!} + \frac{1}{3!3!}\right)x^6 \\ &\quad - \left(\frac{2}{7!} + \frac{2}{3!5!}\right)x^8 + \cdots \end{aligned}$$

$$\begin{aligned} C^2 &= C\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots\right) \\ &= 1 - \left(\frac{2}{2!}\right)x^2 + \left(\frac{2}{4!} + \frac{1}{2!2!}\right)x^4 \\ &\quad - \left(\frac{2}{6!} + \frac{2}{2!4!}\right)x^6 \\ &\quad + \left(\frac{2}{8!} + \frac{2}{2!6!} + \frac{1}{4!4!}\right)x^8 - \cdots \end{aligned}$$

This is a mess, but the results for S^2, C^2 share a few similarities. First, there is a 1 on the right side of the C^2 quantity, which means all other terms in the sum must cancel the entire right side of the S^2 sum. Looking at the powers in x, we see both sums have only even powers (not surprisingly), and moreover their signs are equal and opposite.

For the fundamental trig identity to hold, it must be that all of the coefficients attached to similar power of x are equal. Checking this, we indeed find:

$$\begin{aligned} (1) &= \left(\frac{2}{2!}\right) = 1 \\ \left(\frac{2}{3!}\right) &= \left(\frac{2}{4!} + \frac{1}{2!2!}\right) = \frac{1}{3} \\ \left(\frac{2}{5!} + \frac{1}{3!3!}\right) &= \left(\frac{2}{6!} + \frac{2}{2!4!}\right) = \frac{2}{45} \\ \left(\frac{2}{7!} + \frac{2}{3!5!}\right) &= \left(\frac{2}{8!} + \frac{2}{2!6!} + \frac{1}{4!4!}\right) = \frac{1}{315} \end{aligned}$$

Finally, piece it all together to write

$$\begin{aligned}\sin^2(\theta) &= \theta^2 - \frac{1}{3}\theta^4 + \frac{2}{45}\theta^6 - \frac{1}{315}\theta^8 + \cdots \\ \cos^2(\theta) &= 1 - \theta^2 + \frac{1}{3}\theta^4 - \frac{2}{45}\theta^6 + \frac{1}{315}\theta^8 - \cdots ,\end{aligned}$$

summing together to one.

5.3 Trigonometry Plots

Using trigonometry tables as a database allows for graphing the values of $\sin(\theta)$, $\cos(\theta)$, etc, in near-continuous fashion. The following plots in the Cartesian plane represent the continuous limit of trigonometry tables:

Sine, Cosine, Tangent

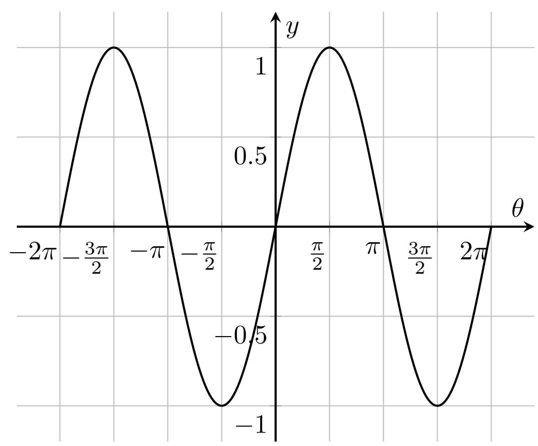

Figure 1.23: $y = \sin(\theta)$.

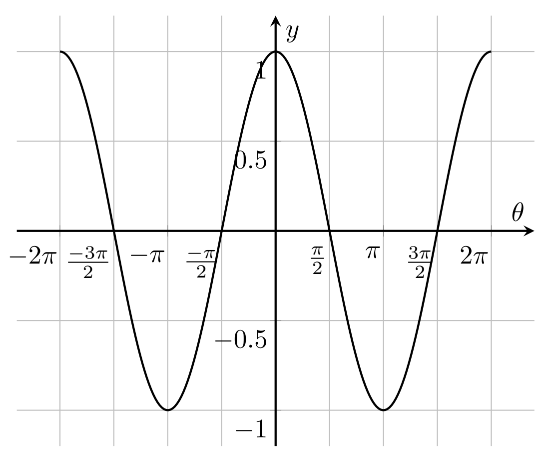

Figure 1.24: $y = \cos(\theta)$.

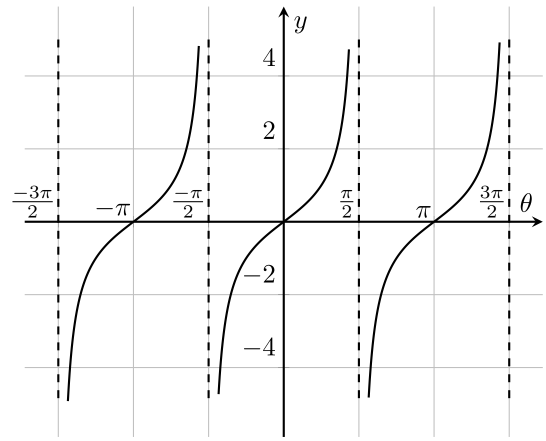

Figure 1.25: $y = \tan(\theta)$.

Cosecant, Secant, Cotangent

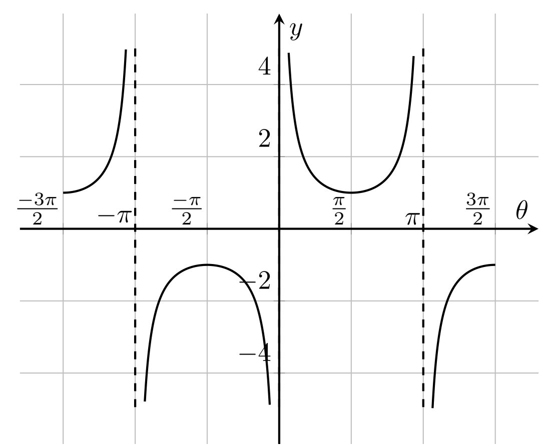

Figure 1.26: $y = \csc(\theta) = 1/\sin(\theta)$.

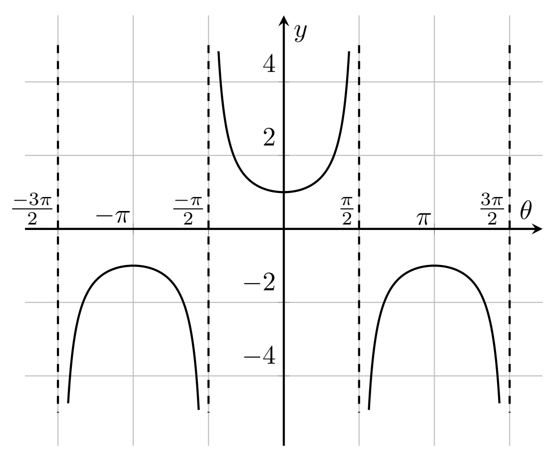

Figure 1.27: $y = \sec(\theta) = 1/\cos(\theta)$.

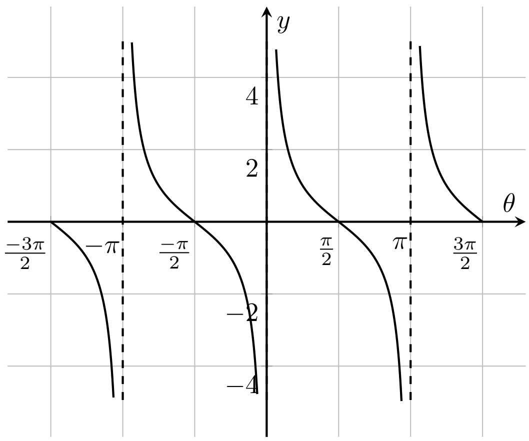

Figure 1.28: $y = \cot(\theta) = 1/\tan(\theta)$.

5.4 Inverse Trigonometry Analysis

The inverse trigonometric quantities are a little more awkward to deal with, i.e. to generate corresponding inverse trigonometry tables. For reasons that ultimately come from computational efficiency arguments, none of which we'll repeat here, it makes sense to get everything in terms of the arctangent.

Arcsine, Arccosine

Recall Equations (1.66), (1.67) and invert these to solve for the arcsine and arccosine, respectively:

$$\arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right),\ x^2 \leq 1$$
$$\arccos(x) = \arctan\left(\frac{\sqrt{1-x^2}}{x}\right),\ 0 < x \leq 1$$

The arcsine-equation is valid $|x| \leq 1$, which happens to be the cover all cases one may throw at the arcsine.

The arccosine equation is valid as-is for $|x| > 0$, and needs a correction to cover the whole domain. It turns out that

$$\arccos(x) = \pi + \arctan\left(\frac{\sqrt{1-x^2}}{x}\right),\ -1 \leq x < 0$$

does the job for $|x| < 0$, which is easy to verify. Neither equation for the arccosine handles the exact case $x = 0$, which by definition corresponds to $\pi/2$.

The summary of our findings is listed in the table below and also in Figures 1.29, 1.30.

x	arcsin(x)	arccos(x)
-1.0	$-\pi/2$	π
-0.8	-0.9273	2.4981
-0.6	-0.6435	2.1859
-0.4	-0.4115	2.9845
-0.2	-0.2014	2.3698
0.0	0	$\pi/2$
0.2	0.2014	1.3694
0.4	0.4115	1.1593
0.6	0.6435	0.9273
0.8	0.9273	0.6435
1.0	$\pi/2$	0

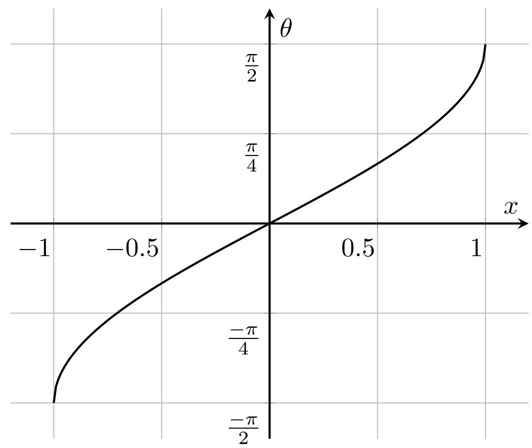

Figure 1.29: $\theta = \arcsin(x)$, $|x| \leq 1$.

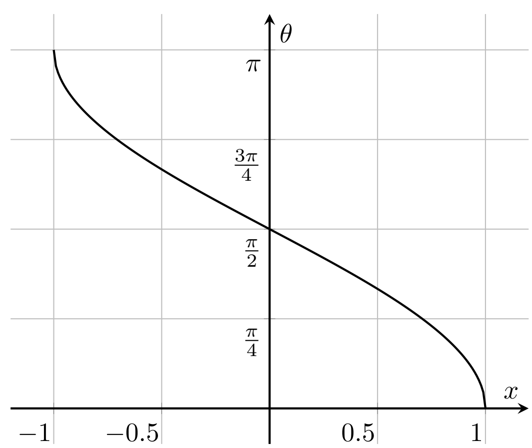

Figure 1.30: $\theta = \arccos(x)$, $|x| \leq 1$.

Arccosecant, Arcsecant

For the arccosecant and arcsecant we repeat a similar analysis to the above starting with Equations (1.70), (1.73). This exercise results in:

$$\operatorname{arccsc}(x) = \arctan\left(\frac{1}{\sqrt{x^2-1}}\right), \ x > 0$$
$$\operatorname{arccsc}(x) = -\arctan\left(\frac{1}{\sqrt{x^2-1}}\right), \ x < 0$$

$$\operatorname{arcsec}(x) = \arctan\left(\sqrt{x^2-1}\right), \ x > 1$$
$$\operatorname{arcsec}(x) = \pi - \arctan\left(\sqrt{x^2-1}\right), \ x < -1$$

The summary of our findings is listed in the table below and also in Figures 1.29, 1.30.

x	arccsc(x)	arcsec(x)
$-\infty$	0	$\pi/2$
-100	-0.0100	1.5808
-3.2	-0.3178	1.8886
-1.6	-0.6751	2.2459
-1.2	-0.9851	2.5559
-1.0	$-\pi/2$	π
0.0		
1.0	$\pi/2$	0
1.2	0.9851	0.5857
1.6	0.6751	0.8957
3.2	0.3178	1.2530
100	0.0100	1.5808
∞	0	$\pi/2$

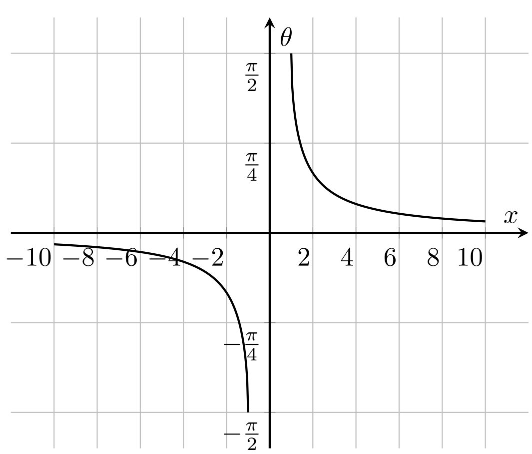

Figure 1.31: $\theta = \operatorname{arccsc}(x)$, $|x| \geq 1$.

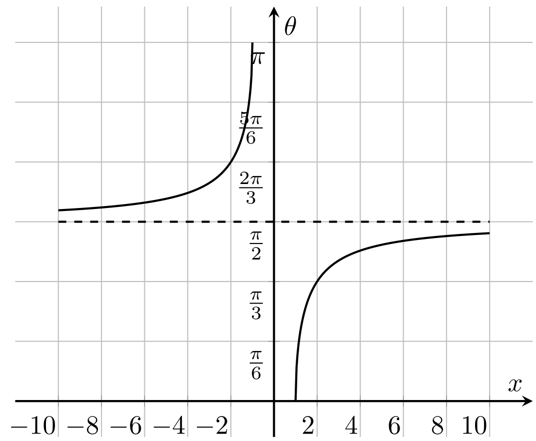

Figure 1.32: $\theta = \operatorname{arcsec}(x)$, $|x| \geq 1$.

Arctangent, Arccotangent

Finally we get to the case of arctangent and its reciprocal. There are numerous methods for grinding out the arctangent of any angle, one being called the Taylor expansion, a trick from calculus, which works in the domain $x^2 < 1$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The Taylor expansion still leaves the question of what to do about the case $x^2 > 1$. For this, it's straightforward to show using trigonometric identities that

$$\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$$

holds for any x, which turns the problem of say, calculating the arctangent of 4 into a problem of calculating the arctangent of 1/4. Taken together, the pair of above equations can be used to calculate any arctangent value.

For the arccotangent, invert Equation (1.74) to write

$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right),$$

which can make direct use of the work previously done with the arctangent.

The summary of our findings is listed in the table below and also in Figures 1.33, 1.34.

x	$\arctan(x)$	$\text{arccot}(x)$
$-\infty$	$-\pi/2$	0
-100	-1.5608	-0.0099
-4.0	-1.3258	-0.2450
-1.6	-1.0122	-0.5586
-0.4	-0.3805	-1.1903
0.0	0	$\mp\pi/2$
0.4	0.3805	1.1903
1.6	1.0122	0.5586
4.0	1.3258	0.2450
100	1.5608	0.0099
∞	$\pi/2$	0

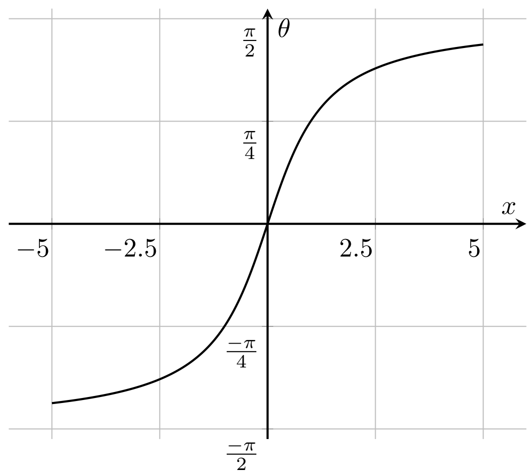

Figure 1.33: $\theta = \arctan(x)$

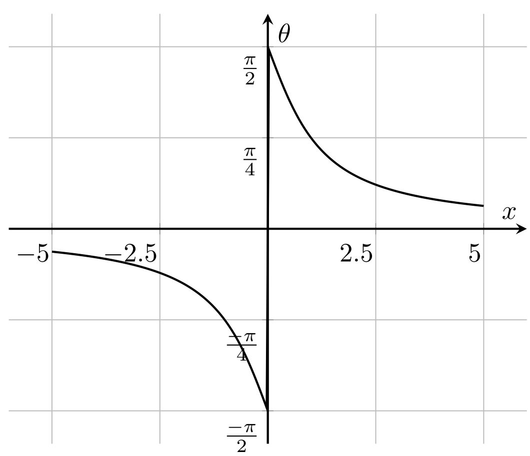

Figure 1.34: $\theta = \text{arccot}(x)$

6 Trigonometry and Geometry

6.1 Law of Cosines

Trigonometry allows one to answer an age-old question from geometry which seeks to find a Pythagorean-like theorem for any arbitrarily-shaped triangle. This is answered by a special formula called the *law of cosines*.

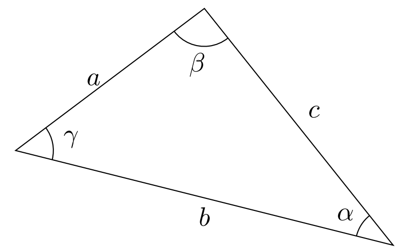

Figure 1.35: Arbitrary triangle.

Consider a triangle with three sides and three angles as labeled in Figure 1.35. Proceed by choosing any vertex, such as where sides a, c come together, and draw a line that intersects the third side, i.e. side b, at a ninety-degree angle. Note that β is not changed despite being omitted from Figure 1.36. Note too that side b is now broken into the sum $b_1 + b_2 = b$.

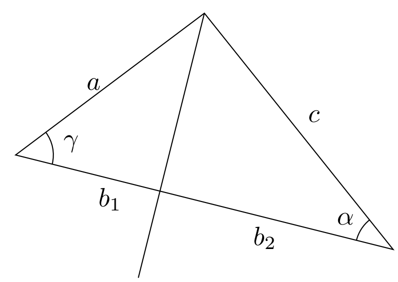

Figure 1.36: Arbitrary triangle with line intersecting side b at a right angle.

From standard trigonometry analysis, we can write a few true statements from the latter Figure:

$$
\begin{aligned}
a\cos(\gamma) &= b_1 \\
c\cos(\alpha) &= b_2 \\
a\sin(\gamma) &= c\sin(\alpha)
\end{aligned}
$$

Proceed by reconstructing the sum $b_1 + b_2$:

$$a\cos(\gamma) + c\cos(\alpha) = b\,,$$

and square both sides:

$$a^2\cos^2(\gamma) + c^2\cos^2(\alpha) + 2ac\cos(\gamma)\cos(\alpha) = b^2$$

Next make the replacements

$$
\begin{aligned}
\cos^2(\gamma) &= 1 - \sin^2(\gamma) \\
\cos^2(\alpha) &= 1 - \sin^2(\alpha)
\end{aligned}
$$

and simplify like mad.

At the end, arrive at the all-powerful law of cosines:

$$a^2 + c^2 - 2ac\cos(\beta) = b^2 \tag{1.79}$$

By symmetry, since we could have sliced the triangle two more ways, there two more expressions of the same law:

$$b^2 + c^2 - 2bc\cos(\alpha) = a^2 \tag{1.80}$$

$$a^2 + b^2 - 2ab\cos(\gamma) = c^2 \tag{1.81}$$

6.2 Inscribed Triangle

A tricky analysis starts with a circle of any radius R with a diameter $AB = 2R$ as shown in Figure 1.37. Choose a point C on the perimeter and draw lines from C to A, B to form an inscribed triangle. (Ignore all interior labels in the Figure until they're invoked.)

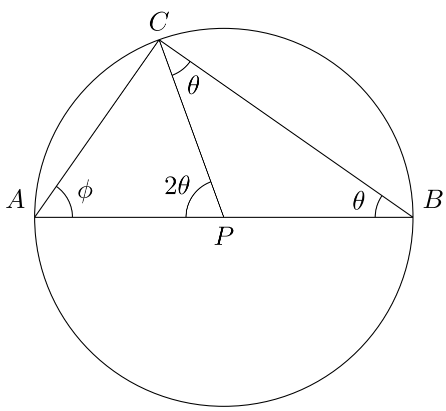

Figure 1.37: Inscribed triangle.

ACB is a Right Angle

First, it's possible to prove that $\angle ACB$ is always a right angle. Place the origin at A so that point C is located at (x, y), and denote the line AC as a variable r such that:

$$\begin{aligned} r\cos(\phi) &= x \\ r\sin(\phi) &= y \\ r^2 &= x^2 + y^2 \end{aligned}$$

Using these variables, the circle itself obeys

$$(x - R)^2 + y^2 = R^2 ,$$

readily simplifying as:

$$\begin{aligned} r^2 &= 2xR \\ r &= 2R\cos(\phi) \\ r &= 2R\sin\left(\frac{\pi}{2} - \phi\right) \end{aligned}$$

To see what we're shooting for, let's momentarily assume the thing we want to prove, i.e. that $\angle ACB = \pi/2$, and make sure no contradiction arises. Going with this, observe from the Figure that

$$r = 2R\sin(\theta) ,$$

and eliminating r gives

$$2R\sin\left(\frac{\pi}{2} - \phi\right) = 2R\sin(\theta) ,$$

which can only mean

$$\theta + \phi = \frac{\pi}{2} ,$$

and no contradiction arises. This result is also known as *Thales' theorem.*

For a proper proof, use the law of cosines to discern from the Figure that

$$\begin{aligned} \left(\overline{BC}\right)^2 &= \left(\overline{PC}\right)^2 + \left(\overline{PB}\right)^2 \\ &\quad - 2\left(\overline{PC}\right)\left(\overline{PB}\right)\cos(\angle BPC) \\ \left(\overline{AC}\right)^2 &= \left(\overline{PA}\right)^2 + \left(\overline{PC}\right)^2 \\ &\quad - 2\left(\overline{PA}\right)\left(\overline{PC}\right)\cos(2\theta) , \end{aligned}$$

simplifying to

$$\begin{aligned} \left(\overline{BC}\right)^2 &= 2R^2 - 2R^2\cos(\angle BPC) \\ \left(\overline{AC}\right)^2 &= 2R^2 - 2R^2\cos(2\theta) . \end{aligned}$$

Sum the two equations to get:

$$\begin{aligned} \left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2 &= (2R)^2 \\ &\quad - 2R^2\left(\cos(2\theta) + \cos(\pi - 2\theta)\right) \end{aligned}$$

The remaining cosine terms cancel exactly, and we find that the sides of the triangle obey the Pythagorean theorem:

$$\left(\overline{AC}\right)^2 + \left(\overline{BC}\right)^2 = (2R)^2 = \left(\overline{AB}\right)^2$$

Thus, ACB is a right triangle.

PCB Equals Theta

If the center of the circle is located at P, then the length PC is equal to length PB, both of which equal the radius R. This qualifies PCB as an isosceles triangle, having two equal sides, which also means two equal angles:

$$\angle PCB = \theta = \angle PBC$$

APC Equals Twice Theta

Using the properties of the isosceles triangle, the (unlabeled) angle BPC obeys

$$\angle BPC + 2\theta = \pi \; .$$

Being a straight line, the total angle across APB needs to be π, which means

$$\angle APC + \angle BPC = \pi \; .$$

Eliminating $\pi - \angle BPC$ from each equation yields the result we want:

$$\angle APC = 2\theta$$

Area of ACB

The area of triangle ACB is

$$A = \frac{1}{2}\left(\overline{AB}\right)\left(\overline{PC}\right)\sin\left(2\theta\right) \; .$$

In terms of the radis, the area is

$$A = R^2 \sin\left(2\theta\right) \; .$$

With the base fixed, we see that the height is maximized at $\theta = \pi/4$, and the maximal area is

$$A_{\max} = \frac{1}{2}\left(\overline{AB}\right)\left(\overline{PC}\right) = R^2 \; .$$

6.3 Law of Sines

There is another relationship called the *law of sines* that is obeyed by all triangles.

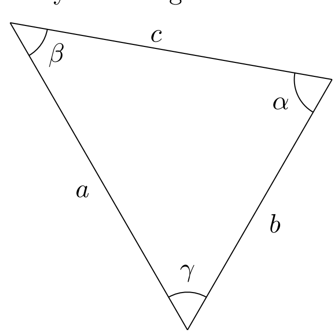

Figure 1.38: Arbitrary triangle.

Quick Derivation

Consider a triangle with three sides and three angles as labeled in Figure 1.38. Using the same process that led to Figure Figure 1.36, choose any vertex and draw a line perpendicular to the side opposite the vertex. We already did this once to yield

$$a\sin\left(\gamma\right) = c\sin\left(\alpha\right)$$

in deriving the law of cosines. This can be repeated for each vertex to yield two more similar relations:

$$c\sin\left(\beta\right) = b\sin\left(\gamma\right)$$
$$b\sin\left(\alpha\right) = a\sin\left(\beta\right)$$

Taking all three of the above equations together yields the (weakest statement of) the law of sines:

$$\frac{\sin\left(\alpha\right)}{a} = \frac{\sin\left(\beta\right)}{b} = \frac{\sin\left(\gamma\right)}{c} \tag{1.82}$$

Area-Based Derivation

A second derivation of the law of sines writes the total area T of triangle ABC three different ways. Analyzing similarly as above, we can write:

$$T = \frac{1}{2}ac\sin\left(\beta\right) = \frac{1}{2}ab\sin\left(\gamma\right)$$
$$T = \frac{1}{2}ba\sin\left(\gamma\right) = \frac{1}{2}bc\sin\left(\alpha\right)$$
$$T = \frac{1}{2}ca\sin\left(\beta\right) = \frac{1}{2}cb\sin\left(\alpha\right)$$

With these, we can not only re-derive Equation (1.82) but we can also interpret the law of sines as it relates to the area of the triangle:

$$\frac{2T}{abc} = \frac{\sin\left(\alpha\right)}{a} = \frac{\sin\left(\beta\right)}{b} = \frac{\sin\left(\gamma\right)}{c} \tag{1.83}$$

Circumcircle

Imagine the arbitrary triangle being enclosed by a carefully-placed circle so that all three vertices lie on the circle's perimeter. This is called a *circumcircle*, silly enough, but is nonetheless shown in Figure 1.83. Generally, such a circle has a radius R with a center point that may or may not lie within the confines of the triangle.

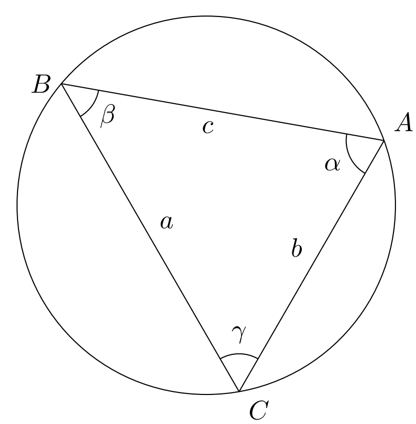

Figure 1.39: Arbitrary triangle with circumcircle.

Circumcircle Analysis

It turns out that the radius R of the circumcircle inscribing an arbitrary triangle relates to the law of sines. To prove this, start with any vertex, such as B, and draw a line through the circle's center until it intercepts the other side at point Q, i.e. draw a diameter $2R$ of the circle. Also, draw lines of length R from points A and C to the center of the circle as shown in Figure 1.40.

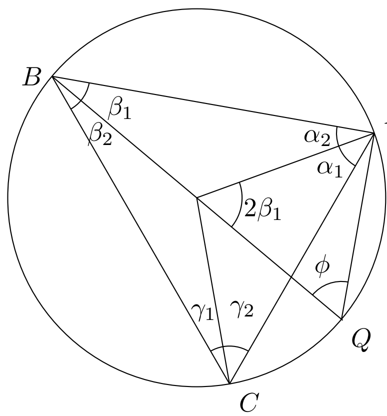

Figure 1.40: Arbitrary triangle with detailed circumcircle.

The angles α, β γ are now partitioned (not bisected) according to

$$\begin{aligned}\alpha &= \alpha_1 + \alpha_2\\ \beta &= \beta_1 + \beta_2\\ \gamma &= \gamma_1 + \gamma_2\,.\end{aligned}$$

By doing this work, the Figure highlights three isosceles triangles, each having two sides of length R and two identical angles:

$$\begin{aligned}\alpha_1 &= \gamma_2\\ \beta_1 &= \alpha_2\\ \gamma_1 &= \beta_2\end{aligned}$$

Phi Equals Gamma

To proceed, recall from inscribed angle analysis to notice that $\angle BAQ$ is exactly a right angle, and thus the angle $\phi = \angle BQA$ relates to β_1 by

$$\phi + \beta_1 = \frac{\pi}{2}\,.$$

This is enough to establish an important relationship between γ and ϕ via

$$\begin{aligned}\gamma &= \gamma_1 + \gamma_2 = \beta_2 + \alpha_1 = (\beta - \beta_1) + (\alpha - \alpha_2)\\ &= \alpha + \beta - 2\beta_1 = \pi - \gamma - 2\beta_1\\ \gamma &= -\gamma + 2\phi\,,\end{aligned}$$

finally revealing

$$\phi = \gamma\,.$$

Third Derivation

With $\phi = \gamma$ known, refer back to Figures 1.39, 1.40 to notice we can now write

$$2R\sin(\gamma) = c\,,$$

or

$$\frac{1}{2R} = \frac{\sin(\gamma)}{c}\,.$$

By symmetry, we could do the entire analysis twice more to land at yet another expression for the law of sines:

$$\frac{1}{2R} = \frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \tag{1.84}$$

Taking Equations (1.83), (1.84) together, we get a nifty formula for the area for a circle in terms of its sides and circumcircle radius:

$$\text{Area} = T = \frac{abc}{4R}$$

6.4 Triangle Inequality

There is an important property of triangles called the *triangle inequality*, stating that the sum of two side lengths is always greater than or equal to the remaining length. That is, for a triangle of sides x, y, z, it follows that

$$\begin{aligned}z &\le x + y\\ x &\le y + z\\ y &\le z + x\end{aligned}$$

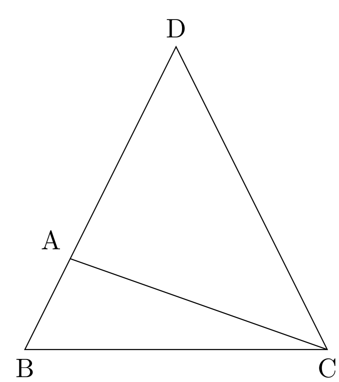

Figure 1.41: Triangle inequality.

To prove this, consider the triangle ABC depicted in Figure 1.41. Choose point D On the line through AB such that an isosceles triangle is formed with two equal lengths AC and AD. Then, angle BCD has greater measure than ACD, which means BD is greater than BC:

$$BD > BC$$

Next, note that

$$BD = AB + AD\ ,$$

which means

$$BD = AB + AC\ .$$

Reading right to left, we have that the sum $AB+AC$ equals BD, which we found is greater than BC. Thus we have

$$AB + AC > BC\ ,$$

completing the proof.

7 Polar Coordinate System

Recall momentarily the Cartesian coordinate system is the lattice on which all points in the plane are hung. There is no point in the plane that does not have a unique coordinate, and every coordinate corresponds to some point in the plane. As it turns out, there is another system called *polar coordinates* that can do the same job as Cartesian coordinates - to cover the plane completely.

7.1 Motivation

The polar coordinate system is built from the apparatus of trigonometry. Consider the unit circle centered at the origin represented by

$$\begin{aligned} x &= \cos(\theta) \\ y &= \sin(\theta)\ , \end{aligned}$$

or

$$x^2 + y^2 = 1\ .$$

By choosing any θ, (even those outside the standard domain), the point (x, y) lands somewhere in the plane a distance 1 from the origin.

Suppose next that the unit circle is replaced by any other circle of radius r, also centered at the origin. In the same way that θ allows freedom in the *angular* dimension, the varying radius allows for freedom in the *radial* dimension. For this reason, one can see that every point (x, y) in the Cartesian plane corresponds to some ordered pair (r, θ).

The mapping from (x, y) to (r, θ) defines the *polar coordinate system*:

$$x(r, \theta) = r\cos(\theta) \tag{1.85}$$

$$y(r, \theta) = r\sin(\theta) \tag{1.86}$$

The $x(r, \theta)$, $y(r, \theta)$ notation is there to remind us that x and y each depend on two variables as suggested in Figure 1.42.

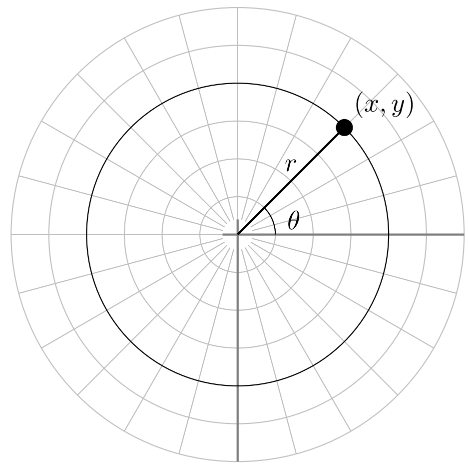

Figure 1.42: Polar coordinate system.

Radial and Angular Coordinates

In terms of x and y, the r- and θ-variables are straightforwardly isolated. Squaring Equations (1.85), polarcoordsy and taking their sum and square root gives the formula for r

$$r = \sqrt{x^2 + y^2}\ . \tag{1.87}$$

Note that the $\pm$ symbol is omitted from the front of the square root symbol. This is to mean that there is no such thing as negative distance from the origin when using polar coordinates.

Solve for θ by taking the ratio of the x- and y-equations, and then make use of the arctangent:

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{1.88}$$

As a consistency check, we should be able to apply $\sin()$ or $\cos()$ to both sides of Equation (1.88) to recover the x, y equations. Using the trig identities (1.66), (1.67), we have

$$\cos(\theta) = \cos\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{1}{\sqrt{y^2/x^2+1}} = \frac{x}{r}$$

and

$$\sin(\theta) = \sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{y/x}{\sqrt{y^2/x^2+1}} = \frac{y}{r}$$

as expected.

7.2 Lines

Navigating the plane in polar coordinates works out differently than when using Cartesian coordinates. For example, the Cartesian system makes trivial work out of straight lines, but things get ugly when it comes to tracing curves, such as circles, i.e.

$$y_{\text{circ}} = \pm\sqrt{R^2 - x^2} \, .$$

On the other hand, straight lines are a bit of a headache in polar coordinates, whereas y_{circ} simply $r = R$.

For the equation of a straight line in Cartesian coordinates

$$y = mx + b \, ,$$

use Equations (1.85), (1.86) for polar coordinates and the same line becomes

$$r = \frac{b}{\sin(\theta) - m\cos(\theta)} \, . \tag{1.89}$$

We can keep going, though. Express the slope m as the tangent of some new angle, say ϕ:

$$m = \tan(\phi)$$

With this, r becomes

$$r = \frac{b\cos(\phi)}{\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)} = \frac{b\cos(\phi)}{\sin(\theta - \phi)} \, .$$

Replace $\cos(\phi)$ using

$$\cos(\phi) = \frac{1}{1 + \tan^2(\phi)} = \frac{1}{\sqrt{1 + m^2}} \, ,$$

and arrive at another equation for the straight line:

$$r = \frac{b}{\sqrt{1 + m^2}} \csc(\theta - \arctan(m)) \, . \tag{1.90}$$

7.3 Scale and Rotation

In a similar way that the x- and y-directions in Cartesian coordinates are independent, i.e. a change in x is not a change in y, is also true in polar coordinates r, θ. Instead of vertical and horizontal changes, there are instead radial changes we'll call *scaling*, and angular changes addressed as *rotation*.

Scaling

The easy case is the radial one, where starting at some position (x_0, y_0) in the plane such that

$$\begin{aligned} x_0 &= r_0\cos(\theta_0) \\ y_0 &= r_0\sin(\theta_0) \, , \end{aligned}$$

we may multiply through by a positive constant λ to scale each coordinate and move to a new location (x, y):

$$\begin{aligned} x &= \lambda x_0 = (\lambda r_0)\cos(\theta_0) \\ y &= \lambda y_0 = (\lambda r_0)\sin(\theta_0) \end{aligned}$$

Inspecting this result, we see that the effect of scaling each coordinate by λ simply modifies the radius via

$$r = \lambda r_0$$

while leaving the angle the same.

Rotations

A way to move from (x_0, y_0) to a new location (x, y) independent of r is to rotate about the origin, which is to change the θ-variable only. Suppose some angle ϕ is added to θ such that

$$\begin{aligned} (x_0, y_0) &= (r\cos(\theta), r\sin(\theta)) \\ (x, y) &= (r\cos(\theta + \phi), r\sin(\theta + \phi)) \, , \end{aligned}$$

where the top pair is the $\phi = 0$ case of the bottom pair.

The trigonometry terms can be expanded using the angle-sum formulas (1.31), (1.32), resulting in

$$\begin{aligned} x &= r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi) \\ y &= r\sin(\theta)\cos(\phi) + r\cos(\theta)\sin(\phi) \, , \end{aligned}$$

simplifying to a set of equations (some may recognize as a rotation matrix):

$$\begin{aligned} x &= x_0\cos(\phi) - y_0\sin(\phi) \\ y &= x_0\sin(\phi) + y_0\cos(\phi) \end{aligned}$$

As a sanity check, one can check the sum

$$x^2 + y^2 = x_0^2 + y_0^2 \, ,$$

which assures the radius doesn't change under rotations.

7.4 Offset Circle

A circle offset from the origin is a bit messy in both Cartesian and polar coordinates. Consider a circle of radius a centered at the point (x_0, y_0), i.e.

$$(x - x_0)^2 + (y - y_0)^2 = a^2 \, ,$$

where (x, y) locates any point on the perimeter as shown in Figure 1.43.

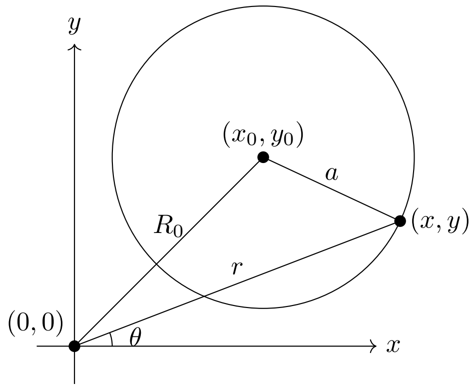

Figure 1.43: Offset circle.

Using polar coordinates, we have that the center of the circle is located by

$$\begin{aligned} x_0 &= R_0 \cos(\theta_0) \\ y_0 &= R_0 \sin(\theta_0) \ , \end{aligned}$$

and, of course, a the point (x, y) on the perimeter is at

$$\begin{aligned} x &= r \cos(\theta) \\ y &= r \sin(\theta) \ . \end{aligned}$$

Substituting the polar representations for x, y, x_0, y_0 into the equation of the offset circle and letting the algebra cook down results in something reminiscent of the law of cosines:

$$r^2 + R_0^2 - 2rR_0 \cos(\theta - \theta_0) = a^2$$

Note that $\theta - \theta_0$ is the angle formed between r and R_0. We can keep going, though. Isolate r using the quadratic formula and simplify again:

$$r = R_0 \cos(\theta - \theta_0) \pm \sqrt{a^2 - R_0^2 \sin^2(\theta - \theta_0)} \quad (1.91)$$

7.5 The Involute

A string is wrapped around a circle of radius a. Keeping the string tight, unwind the string and keep track of its endpoint. The shape traced out is called the *involute* as shown in Figure 1.44.

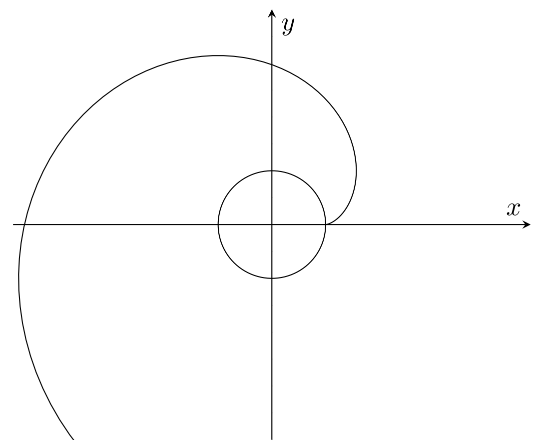

Figure 1.44: The involute.

Parameterize the unwinding using an angle ϕ, where $\phi = 0$ corresponds to the fully-wrapped string. In terms of ϕ, the endpoint of the string is given by

$$\begin{aligned} x(\phi) &= a \cos(\phi) + a\phi \sin(\phi) \\ y(\phi) &= a \sin(\phi) - a\phi \cos(\phi) \ . \end{aligned}$$

With x, y on hand, we can determine the absolute distance from the origin, i.e. the r-parameter for polar coordinates:

$$r = \sqrt{x^2 + y^2} = a\sqrt{1 + \phi^2}$$

The θ-parameter is a little more tricky. Using polar coordinates, start with

$$\begin{aligned} r \cos(\theta) &= a \cos(\phi) + a\phi \sin(\phi) \\ r \sin(\theta) &= a \sin(\phi) - a\phi \cos(\phi) \ . \end{aligned}$$

Proceed by letting

$$\phi = \tan(u)$$

for some new parameter u, which leads to

$$\begin{aligned} \cos(u) &= \pm a/r \\ \sin(u) &= \pm a\phi/r \ . \end{aligned}$$

Simplifying the above gives a tight relationship between the variables on hand:

$$\theta = \phi - u = \tan(u) - u$$

Note that u is confined to the domain $(\pi/2 : \pi/2)$.

8 Circular Motion

Consider a circular ring of radius R on which an object can slide freely. Using polar coordinates, we can write the Cartesian position (x, y) in terms of R and the angle θ characterizing the placement of the object:

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

Kinematics on a Circle

Motion on such a circle is completely characterized by $\theta(t)$. The rate at which θ changes in time is called the *angular frequency*, denoted ω (Greek *omega*) as:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

In the case that ω is a constant ω_0, we can write an equation for θ that is reminiscient of kinematics:

$$\theta = \theta_0 + \omega_0 t$$

Moreover, one can have the case where ω changes with time such that

$$\alpha = \frac{\Delta\omega}{\Delta t} \ ,$$

where α (Greek *alpha*) is the *angular acceleration*. For constant angular acceleration, we can write

$$\omega = \omega_0 + \alpha_0 t \ ,$$

and moreover,

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2 \ .$$

As it turns out, the entire splay of one-dimensional kinematics applies to circular motion by a swap of variables. You are encouraged to work out the following identities:

Identity:	Missing:
$\omega = \omega_0 + \alpha_0 t$	$\theta - \theta_0$
$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$	ω
$\theta = \theta_0 + (\omega_0 + \omega)\, t/2$	α_0
$\theta = \theta_0 + \omega t - \alpha_0 t^2/2$	ω_0
$\omega^2 = \omega_0^2 + 2\alpha_0(\theta - \theta_0)$	t

Velocity on a Circle

Given the position (x, y) for circular motion, we can work out the velocity v_x, v_y by first writing

$$x + \Delta x = R\cos(\theta + \Delta\theta)$$
$$y + \Delta y = R\sin(\theta + \Delta\theta) \ ,$$

where Δx, Δy, $\Delta\theta$ are all 'small' parameters that update the position.

The trigonometric terms can be simplified using the small-angle approximations (1.77), (1.78) to write

$$x + \Delta x = R\cos(\theta) - R\sin(\theta)\,\Delta\theta$$
$$y + \Delta y = R\sin(\theta) + R\cos(\theta)\,\Delta\theta \ ,$$

simplifying to

$$\Delta x = -R\sin(\theta)\,\Delta\theta$$
$$\Delta y = R\cos(\theta)\,\Delta\theta \ .$$

Per convenience of this setup, it is not required that ω is constant. Proceed by dividing through by Δt and simplify to arrive at

$$v_x(t) = \frac{\Delta x}{\Delta t} = -y(t)\,\omega(t)$$
$$v_y(t) = \frac{\Delta y}{\Delta t} = x(t)\,\omega(t) \ ,$$

and furthermore

$$v(t) = \sqrt{v_x^2 + v_y^2} = R\omega(t) \ .$$

To remind, the arc length traced on the circle for a given angular interval is written

$$\Delta S = R\Delta\theta \ ,$$

and dividing through by Δt delivers the same result $v = R\omega$.

8.1 Uniform Circular Motion

A special case called *uniform circular motion* has zero angular acceleration, characterized by $\omega(t) = \omega_0$. Importantly though, the linear acceleration $a(x, y)$ does not vanish. To see this, use the same 'small parameter' trick on the pair ov velocity equations to write

$$v_x + \Delta v_x = -(y + \Delta y)\,\omega_0$$
$$v_y + \Delta v_y = (x + \Delta x)\,\omega_0 \ ,$$

expanding to

$$v_x + \Delta v_x = v_x - R\cos(\theta)\,\Delta\theta\omega_0$$
$$v_y + \Delta v_y = v_y - R\sin(\theta)\,\Delta\theta\omega_0 \ ,$$

or

$$\Delta v_x = -R\cos(\theta)\,\Delta\theta\omega_0$$
$$\Delta v_y = -R\sin(\theta)\,\Delta\theta\omega_0 \ .$$

Divide through by Δt and simplify once more to attain the result:

$$a_x(t) = \frac{\Delta v_x}{\Delta t} = -x(t)\,\omega_0^2$$
$$a_y(t) = \frac{\Delta v_y}{\Delta t} = -y(t)\,\omega_0^2$$

Remarkably, the acceleration always points to the center of the circle. This should remind that the definition of acceleration is not only a change in speed, but a change in *direction* as well. The total linear acceleration is given by

$$a = -\frac{v^2}{R}\ .$$

Problem 1

A wheel of radius R is rotating with no angular acceleration. If a point on the rim of the wheel has tangential velocity v_0, what angular acceleration is required to bring the wheel to rest in 10 seconds? Answer:

$$\alpha_0 = -\frac{v_0/R}{10\ \mathrm{s}}$$

Problem 2

A bicycle with 700 mm diameter wheels is coasting by at 15 meters per second. The rider applies the handbrake that causes a constant angular deceleration $\alpha_0 = -0.15\ \mathrm{rad/s^2}$ over an interval of 4 seconds. What is the final speed of the bicycle? Answer:

$$v = 15\ \mathrm{m/s} + (700\ \mathrm{mm})\left(-0.15\ \mathrm{rad/s^2}\right)(4\ \mathrm{s})$$

Problem 3

Starting from rest, a wheel is given a constant angular acceleration $\alpha_0 = 0.250\ \mathrm{rad/s^2}$ for the first 10 seconds of its motion. At $t = 10\ \mathrm{s}$, the angular acceleration is switched to $\alpha_0 = 0$. How many revolutions has the wheel made in 20 seconds? Answer:

$$\omega_{10\ \mathrm{s}} = \alpha_0\,(10\ \mathrm{s})$$
$$\theta_{10\ \mathrm{s}} = \frac{1}{2}\alpha_0\,(10\ \mathrm{s})^2$$
$$\theta_{20\ \mathrm{s}} = \theta_{10\ \mathrm{s}} + \omega_{10\ \mathrm{s}}\,(10\ \mathrm{s})$$
$$N_{\mathrm{rev}} = \frac{\theta_{20\ \mathrm{s}}}{2\pi}$$

Problem 4

A traffic 'roundabout' or 'rotary' is approximately a circle. Suppose a car has an initial speed of 20 meters per second, and then the driver holds the brake for 3 seconds to decelerate at $-0.15\,g$ before entering a roundabout with a 20 meter radius. Calculate the linear acceleration of the car while in the roundabout. Answer:

$$v_R = 20\ \mathrm{m/s} + \left(-0.15 \times 9.8\mathrm{m/s^2}\right)(4\ \mathrm{s})$$
$$a = -\frac{v_R^2}{20\ \mathrm{m}}$$

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