Time Translation and the Taylor Structure of Kinematics

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Recent discussions have emphasized the use of Taylor series to explain the familiar kinematic relations in the introductory calculus sequence [1]. In this note we show that the connection can also be made in the opposite direction. By examining how the familiar kinematic relations behave under translation in time, the series structure appears naturally within elementary kinematics. In this way, extensions of motion such as jerk become accessible at the pre-calculus level, while also providing students with a natural motivation for the Taylor series encountered later in calculus.

In elementary kinematics the motion of a particle with uniform acceleration is described by simple polynomial relations in time: the acceleration is constant, the velocity varies linearly, and the position varies quadratically. A natural extension introduces a uniform jerk j_0, so that the acceleration varies linearly with time. In this case the velocity acquires an additional quadratic term, while the position acquires a cubic term. Writing the cubic coefficient in the general form $1/N$, we obtain

$$a(t) = a_0 + j_0 t, \qquad v(t) = v_0 + a_0 t + \frac{1}{2} j_0 t^2, \qquad x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{N} j_0 t^3. \tag{1}$$

Here $x_0 = x(0)$, $v_0 = v(0)$, and $a_0 = a(0)$ denote the position, velocity, and acceleration at the reference time $t = 0$.

Determining the coefficient of the cubic term is often associated with calculus, either through repeated integration of the kinematic relations or through Taylor series expansions. Consequently, introductory kinematics treatments rarely proceed beyond the constant-acceleration formulas, and extensions involving jerk and higher rates of change of motion are seldom developed explicitly at the elementary level [2].

Time translation in kinematics. The cubic coefficient in (1) can be determined directly by examining how the kinematic relations transform under a translation in time. We translate the time variable according to $t \to t + h$, treating h as a new dynamic variable. Conceptually, we pause the clock at time t and begin measuring time again from $h = 0$. Substituting $t + h$ into the expression for $x(t)$ and recombining in powers of h gives

$$x(t+h) = \left(x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{N} j_0 t^3\right) + \left(v_0 + a_0 t + \frac{3}{N} j_0 t^2\right) h + \frac{1}{2}\left(a_0 + \frac{6}{N} j_0 t\right) h^2 + \frac{1}{N} j_0 h^3. \tag{2}$$

Because the kinematic relations must retain the same form under a shift of the time origin, the translated expression for $x(t + h)$ must reproduce the velocity and acceleration evaluated at time t. For consistency with the corresponding velocity and acceleration relations, the parenthesized quantities must equal those variables evaluated at time t. This requirement is satisfied only when $N = 6$, in which case Eq. (2) reproduces the relations in Eq. (1) and thereby determines the unknown coefficient in the position relation. Hence the above simplifies to

$$x(t+h) = x_t + v_t h + \frac{1}{2} a_t h^2 + \frac{1}{6} j_0 h^3. \tag{3}$$

For clarity we write $x_t = x(t)$, $v_t = v(t)$, $a_t = a(t)$, and similarly for higher rates of change of motion. Equation (3) expresses the motion relative to an arbitrary reference instant t, with h measuring the elapsed interval from that instant. The familiar kinematic relations are recovered when $t = 0$, while setting $h = 0$ simply reproduces the motion at the reference instant.

Higher powers and the Taylor structure. Time translation extends naturally to higher rates of change of motion. Continuing beyond jerk—sometimes whimsically named snap, crackle, pop, and similar terms—produces successively higher powers with factorial coefficients. The pattern arises from the binomial expansion of $(t + h)^n$: when a new motion quantity is introduced, the highest power of h appears with the largest binomial coefficient, and consistency with the translated relations produces the factorial normalization. Thus the higher terms appear with coefficients $1/4!$, $1/5!$, $1/6!$, and so forth.

In general, writing $x_t^{(n)}$ for the nth rate of change of position evaluated at time t, the translated motion takes the form

$$x(t+h) = \sum_{n=0}^{\infty} \frac{1}{n!} x_t^{(n)} h^n. \tag{4}$$

This is precisely the Taylor expansion of the function $x(t)$ about the time t. The Taylor series structure thus appears directly from the translation properties of the kinematic equations, reversing the usual presentation in which these relations are derived from Taylor expansions.

The same argument applies to every quantity in the kinematic chain. For example, applying it to velocity and acceleration gives

$$v(t+h) = v_t + a_t h + \frac{1}{2} j_t h^2 + \frac{1}{6} s_t h^3 + \cdots, \tag{5}$$

$$a(t+h) = a_t + j_t h + \frac{1}{2} s_t h^2 + \frac{1}{6} c_t h^3 + \cdots. \tag{6}$$

Recovering initial conditions. The translated motion relations also allow earlier states of motion to be recovered from measurements taken at a later time. To illustrate, suppose a particle moves with constant jerk j_0, but the initial quantities x_0, v_0, and a_0 are unknown. At a later time p the position, velocity, and acceleration are measured, giving the values x_p, v_p, and a_p. The task is to determine the initial conditions in terms of these known quantities.

One could proceed by writing the kinematic formulas for motion with constant jerk, substituting $t = p$, and solving the resulting system. While this works, it involves several substitutions and a fair amount of algebra.

Instead, the shifted relations allow the answer to be obtained directly. Taking the measured instant $t = p$ as the base point and shifting backward by setting $h = -p$ in Eqs. (3), (5), and (6) gives

$$x_0 = x_p - v_p p + \frac{1}{2} a_p p^2 - \frac{1}{6} j_0 p^3, \qquad v_0 = v_p - a_p p + \frac{1}{2} j_0 p^2, \qquad a_0 = a_p - j_0 p. \tag{7}$$

Connection with calculus. The central ideas of differential calculus are closely tied to quantities that arise in the study of motion. Historically, questions about velocity—how the position of a moving body changes from one instant to the next—played an important role in motivating the development of derivatives. Indeed, the notion of instantaneous velocity was investigated well before the formal theory of limits was established [3].

The translated motion relations developed above are particularly well suited to illustrate this transition to calculus. They describe the motion relative to an arbitrary reference instant t, with h representing the elapsed time from that instant. It is therefore natural to ask what happens as the reference instant is updated more frequently, or equivalently, as the interval h becomes arbitrarily small. This question leads directly to a limiting form of the translated relations.

Dividing Eq. (4) by h gives

$$\frac{x(t+h) - x_t}{h} = v_t + \frac{1}{2} a_t h + \frac{1}{6} j_t h^2 + \cdots. \tag{8}$$

The left-hand side represents the average rate of change of position over the interval from t to $t + h$. As the interval shrinks, the higher powers of h vanish, leaving only the velocity term. In the limiting case,

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = v(t), \tag{9}$$

which is precisely the definition of instantaneous velocity in differential calculus. In this way, the derivative emerges naturally from the translated kinematic relations.

Disclosure statement

The authors report there are no competing interests to declare.

Data availability

No datasets were generated or analyzed for this study.

Data deposition

No data were deposited in external repositories for this study.

Supplemental material

No supplemental material accompanies this writing.

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