Taylor Polynomial
MANUSCRIPT

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[1]

November 24, 2024

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Chapter 1

Taylor Polynomial

1 Uniform Acceleration

A study of kinematics often limits the acceleration to be constant, or *uniform*, which is convenient for describing many systems, including freefall motion near Earth's surface, or the motion of charges in a uniform electric field. In such a case, the student is provided with a hierarchy of kinematic formulas:

$$
\begin{aligned}
x(t) &= x_0 + v_0 t + \frac{1}{2} a t^2 \qquad (1.1)\\
v(t) &= v_0 + at \\
a(t) &= \text{constant}
\end{aligned}
$$

The initial values x_0, v_0 correspond to the position and velocity at time $t = 0$.

Kinematic Identities

The standard kinematic formulas are reinforced by a flurry of kinematic identities

$$
\begin{aligned}
x(t) &= x_0 + \overline{v} t \\
x(t) &= x_0 + v(t)\, t - \frac{1}{2} a t^2 \\
(v(t))^2 &= v_0^2 + 2a\Delta x \, ,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{v}(t) &= \frac{v_0 + v(t)}{2} \\
\Delta x &= x(t) - x_0 \, .
\end{aligned}
$$

Position Plot

The kinematic equation for position $x(t)$ is quadratic in the variable t, thus it's of interest to complete the square in t and write the position x as

$$
x(t) = \left(x_0 - \frac{v_0^2}{2a} \right) + \frac{a}{2} \left(t + \frac{v_0}{a} \right)^2 \, . \qquad (1.2)
$$

The vertex of the motion, occurring at $(t_{\text{vert}}, x_{\text{vert}})$, is calculated by setting $t = -v_0/a$, giving

$$
(t_{\text{vert}}, x_{\text{vert}}) = \left(\frac{-v_0}{a}, x_0 - \frac{v_0^2}{2a} \right) \, .
$$

There exists a condition for which the position returns to $x = x_0$, given by

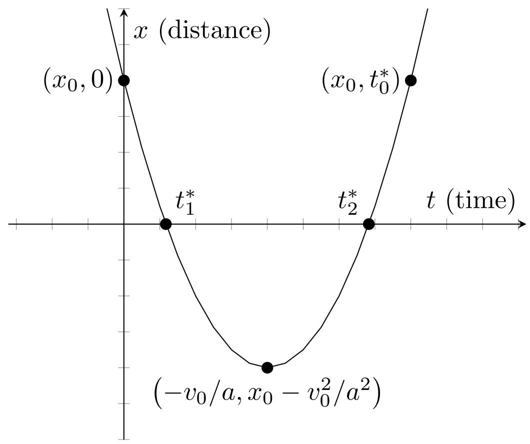
$$
t_0^* = \frac{-2v_0}{a} \, .
$$

Note that if v_0/a resolves to a positive number, the above condition is not met for positive time values.

We may also determine the t-intercepts, corresponding to the point(s) satisfying $x = 0$:

$$t^*_{1,2} = \frac{v_0}{a}\left(1 \pm \sqrt{1 - \frac{2x_0 a}{v_0^2}}\right)$$

The summary of our findings is contained in the following graph, choosing $x_0 > 0$, $v_0 < 0$, $a > 0$ for the sake of demonstration:



2 Uniform Jerk and Beyond

Inevitably during a study of kinematics, one wonders how things change if acceleration is itself allowed to vary with time, a phenomenon called *jerk*.

Identities

In the case of uniform jerk, represented by j, we may write

$$a(t) = a_0 + jt\ ,$$

perfectly analogous to $v = v_0 + at$ in the constant-acceleration regime. In the same analogy, certain 'acceleration-jerk' identities can be written, for instance:

$$v(t) = v_0 + a_0 t + \frac{1}{2} j t^2$$
$$a(t)^2 = a_0^2 + 2j\Delta v$$

2.1 Time-Shift Analysis

All is well until we try to come up with an equation for position $x(t)$. Going by pattern alone, it seems that the new 'jerk' term will depend on t^3, but we can't be sure which coefficient to write. Putting this uncertainty into the unknown coefficient A, we have:

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{A} j t^3$$

To proceed, introduce a shift of time such that

$$t \to t + h\ ,$$

where h can be of any value. Inserting this into the above gives

$$\begin{aligned} x(t+h) = x_0 &+ v_0(t+h) \\ &+ \frac{1}{2} a_0 (t+h)^2 + \frac{1}{A} j (t+h)^3\ , \end{aligned}$$

and now the job is to expand all factors involving $(t + h)$. Doing so, and then combining like terms in powers of h, results in something interesting:

$$\begin{aligned} x(t+h) = &\left(x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{A} j t^3\right) \\ &+ h\left(v_0 + a_0 t + \frac{3}{A} j t^2\right) \\ &+ \frac{1}{2} h^2 \left(a_0 + \frac{6}{A} j t\right) + \frac{1}{6} h^3 (j) \end{aligned}$$

From this, we see the *only* way to correctly recover the identities already written is to have

$$A = 6\ ,$$

and no other choice suffices.

2.2 Time-Shifted Kinematics

To tighten up the analysis above, define shorthand coefficients of motion x_t, v_t, and a_t such that:

$$\begin{aligned} x_t &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} j t^3 \\ v_t &= v_0 + a_0 t + \frac{1}{2} j t^2 \\ a_t &= a_0 + jt \end{aligned}$$

Then, the time-shifted position $x(t + h)$ can be written in condensed form that buries the explicit t-dependence in favor of h:

$$x(t+h) = x_t + v_t h + \frac{1}{2} a_t h^2 + \frac{1}{6} j h^3 \qquad (1.3)$$

This result kills two birds with one stone. Firstly, we arrive at a fully-adjustable equation of kinematics with any t as the 'initial' time value, shifting the burden of evolution to h.

Secondly, we see the additional 'uniform jerk term' in the role of kinematics arrives unambiguously as $(1/6)\, jt^3$:

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} j t^3$$

Inverse Relations

It's worthwhile to take the time-shifted kinematic identities for x_t, v_t, etc., and solve instead for x_0, v_0, etc., thereby *inverting* the set of equations. Starting with the a-terms and working back, we find:

$$a_0 = a_t - jt$$
$$v_0 = v_t - a_t t + \frac{1}{2} j t^2$$
$$x_0 = x_t - v_t t + \frac{1}{2} a_t t^2 - \frac{1}{6} j t^3$$

That is, the inverted version differs by the original up to a minus sign on the effective time variable.

Example 1

Suppose an experimental rocket moves with constant jerk J on a linear track (ignore gravity). A snapshot of the rocket's motion taken at time T tells us the position X, velocity V, and acceleration A at that instant. What must be the initial position, velocity, and acceleration at $t = 0$? What is the position of the rocket at any given time t?

The rocket's motion is represented by

$$x_t = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} J t^3$$
$$v_t = v_0 + a_0 t + \frac{1}{2} J t^2$$
$$a_t = a_0 + Jt \; ,$$

where x_0, v_0, a_0 are unknown. Let $t = T$ to bring X, V, and A into the picture:

$$X = x_0 + v_0 T + \frac{1}{2} a_0 T^2 + \frac{1}{6} J T^3$$
$$V = v_0 + a_0 T + \frac{1}{2} J T^2$$
$$A = a_0 + JT$$

The above is a linear system of three equations and three unknowns, easily solved by hand:

$$x_0 = X - VT + \frac{1}{2} A T^2 - \frac{1}{6} J T^3$$
$$v_0 = V - AT + \frac{1}{2} J T^2$$
$$a_0 = A - JT$$

For the path of motion, Equation (1.3) tells us

$$x(T+h) = X + Vh + \frac{1}{2} A h^2 + \frac{1}{6} J h^3 \; ,$$

where h is the time elapsed since T, or

$$T + h = t \; ,$$

so finally

$$x(t) = X + V(t-T) + \frac{A(t-T)^2}{2} + \frac{J(t-T)^3}{6} \; .$$

For two sanity checks, let $t = T$ to recover $x(T) = X$. Also let $t = 0$ to recover $x(0) = x_0$.

Uniform Snap

Having cracked the problem of uniform jerk, one wonders next what happens if jerk is allowed to vary in time, a situation describing *snap*. Indeed, if we introduce a uniform snap constant k, we have

$$j(t) = j_0 + kt \; ,$$

and the whole argument repeats. Then, there must be some fourth-order correction to the position such that

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} j_0 t^3 + \frac{1}{B} k t^4 \; .$$

Problem 1

Use time-shift analysis to figure out $B = 24$, and then appropriately append x_t, x_0, v_t, v_0, etc.

2.3 Numeric Coefficients

Looking at the equation $x(t)$ and the coefficient accompanying each term, we can't help but try to see a pattern to these. Each coefficient originates from expanding powers of $(t+h)^n$, discovered using either brute force or referring to Pascal's triangle.

Delving into polynomial expansion, one inevitably discovers the handiness of the *factorial* operator, (!) which is a way to express the product of descending integers starting with N:

$$N! = N(N-1)(N-2)\cdots(2)(1) \; ,$$

with the limit case $0! = 1$. With this, the coefficients in the kinematic equation for $x(t)$ can be written in tighter fashion:

$$x(t) = x_0 + v_0 t + \frac{1}{2!} a_0 t^2 + \frac{1}{3!} j_0 t^3 + \frac{1}{4!} k t^4$$

2.4 Change of Base Point

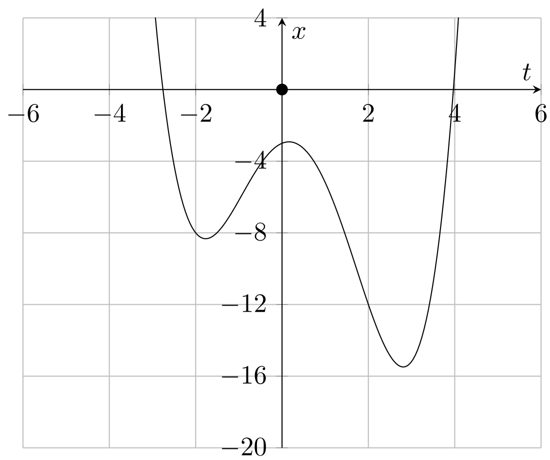
To develop an application of time-shift analysis, consider a trajectory characterized by coefficients

$$x_0 = -3 \qquad v_0 = 1 \qquad \frac{a_0}{2} = -3$$
$$\frac{j_0}{3!} = \frac{-1}{2} \qquad \frac{k}{4!} = \frac{5}{16} \; ,$$

such that

$$x(t) = -3 + t - 3t^2 - \frac{1}{2} t^3 + \frac{5}{16} t^4 \; ,$$

plotted as follows:



As written, the equation for $x(t)$ is suited such that the 'origin in time' is $t = 0$.

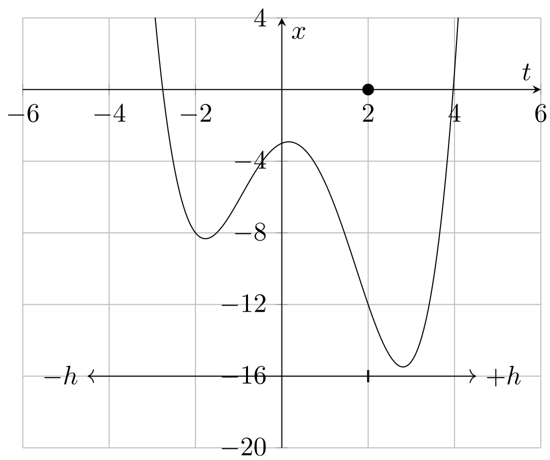
A *change of base point* is analogous to choosing a new origin, i.e. to replace the point that does $t = 0$'s job with something else. In light of time-shift analysis, this amounts to letting, say, $t_p = 2$, and then letting h do the evolution as the effective time variable. The hard work entails calculating the coefficients x_t, v_t, etc.:

$$\begin{aligned}
x_t &= \left(x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 + \frac{k}{24} t^4 \right) \Big|_2 = -12 \\
v_t &= \left(v_0 + a_0 t + \frac{j_0}{2} t^2 + \frac{k_0}{6} t^3 \right) \Big|_2 = -7 \\
a_t &= \left(a_0 + j_0 t + \frac{k_0}{2} t^2 \right) \Big|_2 = 3 \\
j_t &= \left(j_0 + k_0 t \right) \Big|_2 = 12 \\
k &= \frac{15}{2}
\end{aligned}$$

A new equation is written

$$x(t_p + h) = -12 - 7h + \frac{3}{2} h^2 + 2h^3 + \frac{5}{16} h^4 \,,$$

were the t_p-dependence is hidden in the numeric coefficients. The plot of this is in all ways identical to the original as shown:



Restoring Coefficients

For a sanity check, one may reverse-work the previous example, which is to start with the $x(h)$-equation and recover the original version $x(t)$. A brutal way to do this is to let $h = t - 2$, and then simplify.

Alternatively, use symbolic apparatus to recover the same result using inverse relations

$$\begin{aligned}
j_0 &= \left(j_t - kt \right) \Big|_2 \\
a_0 &= \left(a_t - j_t t + \frac{1}{2} k t^2 \right) \Big|_2 \\
v_0 &= \left(v_t - a_t t + \frac{1}{2} j_t t^2 - \frac{1}{6} k t^3 \right) \Big|_2 \\
x_0 &= \left(x_t - v_t t + \frac{1}{2} a_t t^2 - \frac{1}{6} j_t t^3 + \frac{1}{24} k t^4 \right) \Big|_2 \,,
\end{aligned}$$

and the original numbers pop back out.

3 Taylor Polynomial

Now we keep pushing the idea of time-shifted kinematics. Supposing we choose a fixed point in time $t = t_p$ and use h to track time, the quantity $t_p + h$ becomes the *effective* time t:

$$t = t_p + h$$

With this, a general kinematic equation for $x(t_p + h)$ can be written:

$$\begin{aligned}
x(t) = {} & x_{t_p} + v_{t_p} (t - t_p) \\
& + \frac{1}{2} a_{t_p} (t - t_p)^2 + \frac{1}{6} j_{t_p} (t - t_p)^3 + \cdots
\end{aligned}$$

Introducing a generalized notation to represent the velocity, acceleration, jerk, and so on, let us make the associations

$$\begin{aligned} x_{t_p} &\to x_{t_p}^{(0)} \\ v_{t_p} &\to x_{t_p}^{(1)} \\ a_{t_p} &\to x_{t_p}^{(2)} \\ j_{t_p} &\to x_{t_p}^{(3)} \\ k_{t_p} &\to x_{t_p}^{(4)} , \end{aligned}$$

and so on. On the left we've run out of 'named' items after snap, thus the general symbol $x_{t_p}^{(q)}$ is utilized to denote the qth coefficient.

3.1 Generalized Kinematics

In condensed form, $x(t)$ can be written in a most general way as the sum

$$x(t) = x_{t_p} + \sum_{q=1}^{n} \frac{1}{q!} x_{t_p}^{(q)} (t - t_p)^q , \tag{1.4}$$

which we'll call the *Taylor polynomial*. The upper limit n can be any number, depending on the total number of motion coefficients in play.

Generalized Coefficients

In the Taylor polynomial, note that t_p can be taken as *any* point in the domain of $x(t)$, and the equation 'adjusts' accordingly. To pay for this, for any given t_p, we have to calculate all of the 'slope terms', which looks like

$$\begin{aligned} \left(x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 + \frac{k_0}{24} t^4 + \cdots \right) \Big|_{t_p} &= x_{t_p} \\ \left(v_0 + a_0 t + \frac{j_0}{2} t^2 + \frac{k_0}{6} t^3 + \cdots \right) \Big|_{t_p} &= v_{t_p} \\ \left(a_0 + j_0 t + \frac{k_0}{2} t^2 + \cdots \right) \Big|_{t_p} &= a_{t_p} \\ \left(j_0 + k_0 t + \cdots \right) \Big|_{t_p} &= j_{t_p} , \end{aligned}$$

remembering that each of the terms x_{t_p}, v_{t_p}, etc., represent $x_{t_p}^{(0)}$, $x_{t_p}^{(1)}$, etc.

Inverse Coefficients

The inverse relations to the above, namely the structure that isolates x_0, v_0, etc., can be expressed by:

$$\begin{aligned} j_0 &= \left(j_t - k_t t + \cdots \right) \Big|_{t_p} \\ a_0 &= \left(a_t - j_t t + \frac{k_t}{2} t^2 - \cdots \right) \Big|_{t_p} \\ v_0 &= \left(v_t - a_t t + \frac{j_t}{2} t^2 - \frac{k_t}{6} t^3 + \cdots \right) \Big|_{t_p} \\ x_0 &= \left(x_t - v_t t + \frac{a_t}{2} t^2 - \frac{j_t}{6} t^3 + \frac{k_t}{24} t^4 - \cdots \right) \Big|_{t_p} \end{aligned}$$

Velocity

Consistent with the way $x(t)$ is written, we can write a similar formula for the velocity $v(t)$:

$$v(t) = v_{t_p} + \sum_{q=2}^{n} \frac{1}{(q-1)!} x_{t_p}^{(q)} (t - t_p)^{q-1}$$

By a shift of index $q - 1 = r$, this reads

$$v(t) = v_{t_p} + \sum_{r=1}^{n} \frac{1}{r!} x_{t_p}^{(r+1)} (t - t_p)^r ,$$

which can be shortened once more by making the association

$$v_{t_p}^{(r)} = x_{t_p}^{(r+1)} .$$

3.2 Slope of a Polynomial

The relationship between $x(t)$ and $v(t)$ applies, in a sense, to any polynomial. Given a polynomial $y(t)$ with arbitrary coefficients

$$y(t) = A + Bt + Ct^2 + Dt^3 + \cdots ,$$

we're still free to interpret A, B, etc., as kinematic coefficients, i.e.

$$\begin{aligned} A &= x_0 \\ B &= v_0 \\ C &= a_0/2! \\ D &= j_0/3! \\ E &= k_0/4! , \end{aligned}$$

etc., and suddenly the problem looks like kinematics again.

If the term $y(t)$ is in all respects equivalent to a position $x(t)$, then the slope of $y(t)$ must be equivalent to the velocity $v(t)$. Using the Taylor polynomial, the velocity is trivial to write:

$$v(t) = v_0 + a_0 t + \frac{1}{2!} j_0 t^2 + \frac{1}{3!} k_0 t^3 + \cdots$$

Restoring the original coefficients, we find a formula for the slope $y_t^{(1)}$ of the function $y(t)$:

$$y_t^{(1)} = B + 2Ct + 3Dt^2 + 4Et^3 + \cdots$$

3.3 Area Under a Polynomial

Recalling the uniform-acceleration regime, the plot of the velocity $v(t)$ is a straight line in time with initial value v_0.

Inevitably, one should become curious about the *area* contained above the t-axis and under $v(t)$. Doing this exercise using geometry, we find, at time t, the area A to be the sum of two parts, a rectangle and a triangle, having respective areas

$$A_{\text{rectangle}} = tv_0$$
$$A_{\text{triangle}} = \frac{1}{2}t\Delta v \ ,$$

where

$$\Delta v = v(t) - v_0 \ .$$

Taking the sum of each area, and also replacing $v(t)$ with $v_0 + at^2/2$, we find

$$A_{\text{total}} = v_0 t + \frac{1}{2}at^2 \ ,$$

which is exactly equal to the displacement $x(t) - x_0$. Evidently, for uniform acceleration at least, the area under the velocity plot equals the displacement:

$$A_{\text{total}} = x(t) - x_0$$

Riemann Sum (Optional)

Extending the idea of displacement-area-equivalence, it takes little to imagine that the displacement $x(t) - x_0$ is equal to the area under the velocity $v(t)$ plot whether or not the velocity is linear. This is typically justified using a Riemann sum, which approximates a general $v(t)$ as many conjoined straight lines such that

$$x(t) - x_0 = \lim_{N\to\infty} \sum_{q=1}^{N} v\left(t_q^*\right) \Delta t_q \ ,$$

where

$$\Delta t_q = t_q - t_{q-1} \ ,$$

and t_q^* is a value within the interval Δt_q.

In the general case, students of calculus learn to fashion the Riemann sum into a formal integral, and then the whole discussion shifts to techniques of solving integrals.

Exploiting Taylor Polynomial

Using the results painfully gained in this study by plain algebra, we can step around the calculus-based method for calculating the area under a polynomial curve. Going for the general case, suppose you're handed a polynomial with arbitrary coefficients:

$$y(t) = A + Bt + Ct^2 + Dt^3 + \cdots$$

The key is to make the association $y(t) \leftrightarrow v(t)$, which means to interpret $y(t)$ as the velocity of some so-far undetermined curve $x(t)$. The coefficients A, B, C, etc., must be put into familiar terms, where borrowing the whole kinematics apparatus, we have

$$\begin{aligned} A &= v_0 \\ B &= a_0 \\ C &= j_0/2! \\ D &= k_0/3! \ , \end{aligned}$$

and so on.

Then, without any new thinking at all, we already know what $x(t)$ should look like in terms of kinematic coefficients, which is

$$x(t) - x_0 = v_0 t + \frac{1}{2!}a_0 t^2 + \frac{1}{3!}j_0 t^3 + \frac{1}{4!}k_0 t^4 + \cdots \ .$$

Replacing the kinematic coefficients with the original unknowns, the result

$$x(t) - x_0 = At + \frac{1}{2}Bt^2 + \frac{1}{3}Ct^3 + \frac{1}{4}Dt^4 + \cdots$$

emerges, and like magic, the problem is solved.

Example 2

Calculate the area under the curve

$$y(t) = (2 + 3t)^2 \ .$$

Begin by expanding $y(t)$ to get

$$y(t) = 4 + 12t + 9t^2 \ ,$$

from which we discern

$$\begin{aligned} v_0 &= 4 \\ a_0 &= 12 \\ \frac{1}{2}j &= 9 \ . \end{aligned}$$

Assembling the quantity $x(t) - x_0$ from these coefficients, we simply write

$$x(t) - x_0 = 4t + 6t^2 + 3t^3 \ ,$$

and the problem is solved.

4 Euler's Constant

From the Taylor polynomial given as Equation (1.4), it's interesting to conceive of the situation where all 'slope terms' are the same number, i.e.

$$x_{t_p}^{(q)} = x_{t_p}^{(0)} = x_{t_p} \; ,$$

which assumes (and without loss of generality) that time is a *dimensionless* variable. This has similar consequence for x_0, v_0, a_0, and so on, for these are now identical after adjusting for units of time.

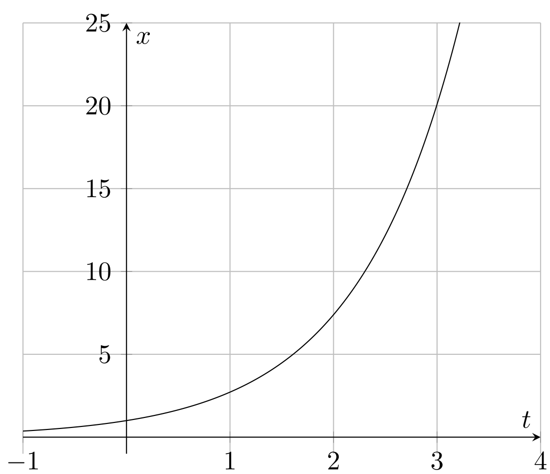
Proceeding, we have

$$x\,(t) = x_{t_p} \sum_{q=0}^{\infty} \frac{1}{q!} \left(t - t_p\right)^q \;\; ,$$

and momentarily let $x_{t_p} = 1$ and $t_p = 0$ to yield

$$x\,(t) = \sum_{q=0}^{\infty} \frac{1}{q!} t^q \; ,$$

as sketched in the following plot:



The case $t = 0$ yields a special construction

$$e = x\,(0) = \sum_{q=0}^{\infty} \frac{1}{q!} \tag{1.5}$$

named e for *Euler's constant*. With enough patience, one may estimate the value of e by brute force:

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Using Binomial Theorem

A better derivation of Euler's constant begins with the binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \; ,$$

where letting

$$\begin{aligned} x &= 1 \\ y &= 1/n \end{aligned}$$

gives

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} \; .$$

In the large-n limit, one readily finds

$$\lim_{n \to \infty} \frac{n!}{(n-k)!} \approx n^k \; ,$$

leaving

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!} \; .$$

On the right is the now-familiar Euler's constant. Evidently, e is also given by the limit on the left:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \tag{1.6}$$

To get a sense of this number, try a few increasing n-values:

$$\begin{aligned} n = 1 : \left(1 + \frac{1}{1}\right)^1 &= 2.0 \\ n = 2 : \left(1 + \frac{1}{2}\right)^2 &= 2.25 \\ n = 3 : \left(1 + \frac{1}{3}\right)^3 &\approx 2.370 \\ n = 10 : \left(1 + \frac{1}{10}\right)^{10} &\approx 2.5937 \\ n = 100 : \left(1 + \frac{1}{100}\right)^{100} &\approx 2.7048 \\ n = 1000000 : \left(1 + \frac{1}{1000000}\right)^{1000000} &\approx 2.71828 \end{aligned}$$

Evidently the Euler constant with $n \to \infty$ is settling to

$$e \approx 2.7182818284590\ldots \; .$$

4.1 Exponential Growth

Let us quickly re-derive Equation (1.6) but instead let $y = t/n$ as it occurs in the binomial theorem. In this variation we find

$$\lim_{n \to \infty} \left(1 + \frac{t}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \; ,$$

where the right side is the same $x\,(t)$ graphed above.

Meanwhile, directly from Equation (1.6) one may establish

$$e^t = \lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n .$$

Eliminating the common term, we discover the formula for natural exponential growth:

$$e^t = \sum_{q=0}^{\infty}\frac{1}{q!}t^q \tag{1.7}$$

Restored Units

To restore time as a quantity of proper dimension, start again with Equation (1.4) and assume that the ratio of ascending coefficients is a constant α:

$$x_{t_p} = x_{t_p}^{(0)} = \alpha^q x_{t_p}^{(q)}$$

Proceeding as before, a factor of α joins the t-variable

$$x(t) = x_{t_p}\sum_{q=0}^{\infty}\frac{1}{q!}\alpha^q\left(t-t_p\right)^q ,$$

and this propagates down to:

$$e^{\alpha t} = \lim_{n\to\infty}\left(1+\frac{\alpha t}{n}\right)^n \tag{1.8}$$

Note that α has inverse time units.

Exponential Growth Curves

From the above we export the final result:

$$e^{\alpha t} = \sum_{q=0}^{\infty}\frac{(\alpha t)^q}{q!} \tag{1.9}$$

Setting $\alpha \to -\alpha$, we have another relation to handle backward evolution in time:

$$e^{-\alpha t} = \sum_{q=0}^{\infty}\frac{(-\alpha t)^q}{q!} \tag{1.10}$$

Also, one may introduce a shift of variables such that the above becomes:

$$e^{\alpha(t-t_0)} = \sum_{q=0}^{\infty}\frac{(\alpha\left(t-t_0\right))^q}{q!} \tag{1.11}$$

4.2 Hyperbolic Curves

Two noteworthy combinations of the the Euler exponential equations can be constructed, namely the *hyperbolic cosine* and the *hyperbolic sine*, given by, respectively:

$$\cosh(\alpha t) = \frac{e^{\alpha t}+e^{-\alpha t}}{2} \tag{1.12}$$

$$\sinh(\alpha t) = \frac{e^{\alpha t}-e^{-\alpha t}}{2} \tag{1.13}$$

Simultaneous to these we can get the originals back by taking the sum and difference of each:

$$e^{\alpha t} = \cosh(\alpha t) + \sinh(\alpha t)$$
$$e^{-\alpha t} = \cosh(\alpha t) - \sinh(\alpha t)$$

Furthermore, given the infinite expansion of e^t, the hyperbolic functions can be written in open form:

$$\cosh(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots \tag{1.14}$$

$$\sinh(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots \tag{1.15}$$

It's straightforward to show from the above that the pair of hyperbolic functions obey, for a dimensionless variable t,

$$\left(\cosh(t)\right)^2 - \left(\sinh(t)\right)^2 = 1 . \tag{1.16}$$

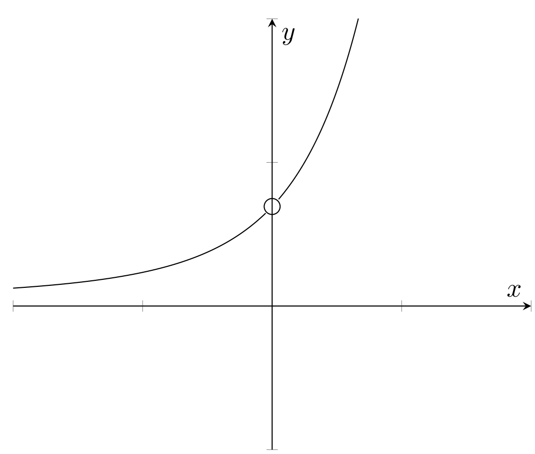
This is somewhat like the standard identity from trigonometry if it weren't for the negative sign. In fact, a whole slew of 'hyperbolic trigonometry' identities can be derived that are analogous to the 'ordinary' trig identities.

4.3 Natural Logarithm

Consider the curious quantity

$$y(x) = \lim_{x\to 0}\frac{n^x-1}{x} , \tag{1.17}$$

plotted as follows:



From the plot of Equation (1.17), we see the point at $x = 0$ is probably a removeable singularity, which is to say we can come up with an answer for $y(0)$ given how y behaves in that neighborhood.

Proceed by solving for n to write

$$n = \lim_{x \to 0} (1 + xy)^{1/x} \ ,$$

which is starting to look like the derivation of Euler's constant. By Equation (1.8), the right side evaluates to e^y, telling us

$$y(0) = \log_e(n) \ ,$$

and for a final answer we take:

$$\ln(n) = \lim_{x \to 0} \frac{n^x - 1}{x}$$

The missing point in the plot of $y(x)$ is the natural log of n.

5 Periodic Curve

Consider the generalized (infinite) Taylor polynomial given as Equation (1.4) with $t_p = 0$:

$$x(t) = \sum_{q=0}^{\infty} \frac{1}{q!} x_0^{(q)} t^q$$

Next, consider a shift of variables $t \to t + w$ such that

$$x(t + w) = \sum_{q=0}^{\infty} \frac{1}{q!} x_0^{(q)} (t + w)^q \ ,$$

which, up to w replacing h to avoid confusion, resembles the setup for time-shifted kinematics that led to Equation (1.4) originally.

For a familiar but nontrivial exercise, one can expand the right side in powers of w to write a generalization of Equation (1.3), particularly

$$x(t + w) = \sum_{q=0}^{\infty} \frac{1}{q!} x_t^{(q)} w^q \ . \tag{1.18}$$

A similarly-familiar exercise involves solving for the coefficients $x_t^{(q)}$, which in this case turn out as

$$\begin{aligned}
x_0^{(0)} + x_0^{(1)} t + x_0^{(2)} \frac{t^2}{2} + x_0^{(3)} \frac{t^3}{6} + \cdots &= x_t^{(0)} \\
x_0^{(1)} + x_0^{(2)} t + x_0^{(3)} \frac{t^2}{2} + x_0^{(4)} \frac{t^3}{6} + \cdots &= x_t^{(1)} \\
x_0^{(2)} + x_0^{(3)} t + x_0^{(4)} \frac{t^2}{2} + x_0^{(5)} \frac{t^3}{6} + \cdots &= x_t^{(2)} \ ,
\end{aligned}$$

and so on, which, just to remind, are the same as x_t, v_t, a_t, etc in kinematics-style notation.

From the above list, multiply each equation by ascending powers of w such that each has the same units of time, and then sum all terms vertically to get a curious identity:

$$\begin{aligned}
x_t^{(0)} + w x_t^{(1)} + w^2 x_t^{(2)} + \cdots = & \\
& + x_0^{(0)} + w x_0^{(1)} + w^2 x_0^{(2)} + \cdots \\
& + t \left(x_0^{(1)} + w x_0^{(2)} + w^2 x_0^{(3)} + \cdots \right) \\
& + \frac{t^2}{2} \left(x_0^{(2)} + w x_0^{(3)} + w^2 x_0^{(4)} + \cdots \right) \\
& + \frac{t^3}{6} \left(x_0^{(3)} + w x_0^{(4)} + w^2 x_0^{(5)} + \cdots \right) + \cdots
\end{aligned}$$

Periodicity Condition

Suppose, for all times t in the domain of $x(t)$, the property

$$x(t + w) = x(t)$$

always holds, called the *periodicity condition*. Immediately true, too, is the stronger statement for all orders of slope terms:

$$x_{t+w}^{(q)} = x_t^{(q)}$$

All we need is the special case of the periodicity condition

$$x(w) = x(0) = x_0$$

and subsequently

$$x_w^{(q)} = x_0^{(q)} \ .$$

To proceed, take that messy identity we wrote above and set $t = w$. The zero-order terms cancel right away due to periodicity:

$$\begin{aligned}
0 = w & \left(x_0^{(1)} + w x_0^{(2)} + w^2 x_0^{(3)} + \cdots \right) \\
& + \frac{w^2}{2} \left(x_0^{(2)} + w x_0^{(3)} + w^2 x_0^{(4)} + \cdots \right) \\
& + \frac{w^3}{6} \left(x_0^{(3)} + w x_0^{(4)} + w^2 x_0^{(5)} + \cdots \right) + \cdots \ ,
\end{aligned}$$

For this to be true in the general case, it *must* be that the parenthesized terms cannot all be positive, and cannot all be negative, else divergence would occur. Whatever the above is trying to say, let us call it the *periodicity constraint*. After a bit of algebra, it's possible to cook the periodicity constraint down to a double sum

$$0 = \sum_{k=1}^{\infty} x_0^{(k)} w^k J_k \ ,$$

where

$$J_k = \sum_{j=1}^{k} \frac{1}{j!}$$

for brevity.

To anticipate the next move, break the outer k-sum into two parts: one sum for even k, denoted k_e, and another sum for odd k, denoted k_o:

$$0 = \left(\sum_{\text{even } k}^{\infty} x_0^{(k_e)} w^{k_e} J_e \right) + \left(\sum_{\text{odd } k}^{\infty} x_0^{(k_o)} w^{k_o} J_o \right)$$

Now, the periodicity condition must also work when w is swapped with $-w$, or any integer multiple of w for that matter. Going with $w \to -w$, we see that the 'even' sum on the left would be completely unchanged by this, whereas the 'odd' sum on the right would gain a global minus sign. In other words, using generic labels, we have a situation with

$$\begin{aligned} \text{Even} + \text{Odd} &= 0 \\ \text{Even} - \text{Odd} &= 0 \,, \end{aligned}$$

which is only true if

$$\text{Even} = \text{Odd} = 0 \,.$$

Each parenthesized sum above, 'odd' and 'even', must separately equal zero.

5.1 Cosine and Sine

Separated into even and odd terms, the periodicity constraint encourages, but does not outright demand, that the $x_0^{(k)}$-terms have alternating signs and all the same magnitude. To be definitive, let us have

$$\begin{aligned} 1 &= x_0^{(0)} = x_0^{(4)} = x_0^{(8)} = \cdots \\ -1 &= x_0^{(2)} = x_0^{(6)} = x_0^{(10)} = \cdots \end{aligned}$$

for the even terms, and

$$\begin{aligned} 1 &= x_0^{(1)} = x_0^{(5)} = x_0^{(9)} = \cdots \\ -1 &= x_0^{(3)} = x_0^{(7)} = x_0^{(11)} = \cdots \end{aligned}$$

for the odd terms.

These results let us write two cases for the resulting $x(t)$, namely

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \tag{1.19}$$

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \,, \tag{1.20}$$

where of course, the variable t can be replaced by the combination αt as done previously.

Partial Sum Plots

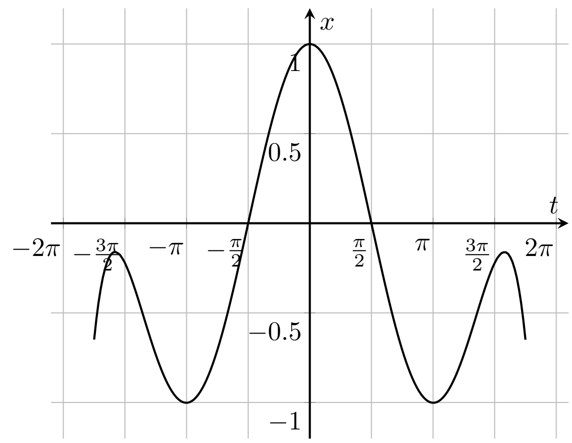


Figure 1.1: $x \approx \cos(t)$ to six terms.

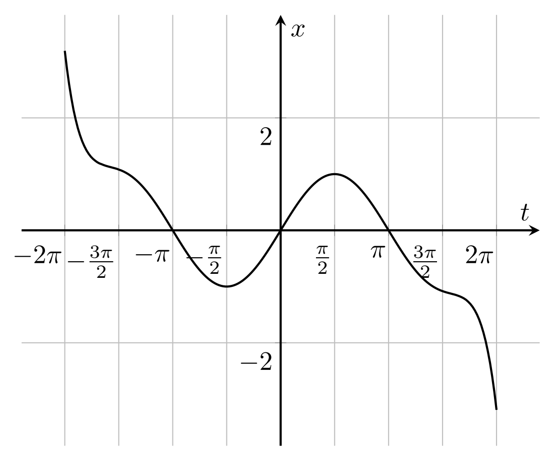


Figure 1.2: $x \approx \sin(t)$ to six terms.

6 Laws of Motion

Consider a pair of polynomials, one called $U(x)$ depending on $x(t)$, and the other called $T(v)$ depending on $v(t)$. These symbols may ring familiar as energy terms, which will indeed turn out true. As Taylor polynomials, the formulae for U and T are:

$$U(x) = U_{x_p} + \sum_{q=1}^{n} \frac{1}{q!} U_{x_p}^{(q)} (x - x_p)^q$$

$$T(v) = T_{v_p} + \sum_{q=1}^{n} \frac{1}{q!} T_{v_p}^{(q)} (v - v_p)^q$$

Conservation of Energy

Now suppose we are interested in the sum of T and U, a quantity labled E such that

$$E = T(v) + U(x) \ .$$

To make things interesting, suppose E is a constant in time, which would mean

$$T(v) + U(x) = T_{v_p} + U_{x_p} \ .$$

By doing this, we have just enforced a powerful notion called *conservation of energy.*

With the above, take the sum of the T- and U-equations to get

$$0 = \sum_{q=1}^{n} \frac{1}{q!} U_{x_p}^{(q)} \Delta x^q + \sum_{q=1}^{n} \frac{1}{q!} T_{v_p}^{(q)} \Delta v^q \ ,$$

where:

$$\Delta x = x - x_p$$
$$\Delta v = v - v_p$$

Writing out the first term in each sum and rearranging a bit gives

$$0 = \left(U_{x_p}^{(1)} \Delta x + T_{v_p}^{(1)} \Delta v \right) + \sum_{q=2}^{n} \frac{1}{q!} \left(U_{x_p}^{(q)} \Delta x^q + T_{v_p}^{(q)} \Delta v^q \right) \ .$$

First-order Equations

Now we explore the regime where both Δx and Δv are 'small intervals' such that higher powers of these quantities tend to diminish. From this we may ignore the remaining summation and keep the low-order terms already plucked out. If x and v are to be related by kinematics, it should follow that

$$\Delta x \approx v_p \Delta t$$

must hold for a similarly-small interval

$$\Delta t = t - t_p \ .$$

Boiling all this down, we transform the above down to

$$0 = U_{x_p}^{(1)} v_p \Delta t + T_{v_p}^{(1)} \Delta v \ .$$

From the first-order energy equation we may write

$$U_{x_p}^{(1)} v_p = -T_{v_p}^{(1)} \frac{\Delta v}{\Delta t} \ ,$$

which is suggestive of two proportionality relationships

$$U_{x_p}^{(1)} \propto \frac{\Delta v}{\Delta t}$$
$$T_{v_p}^{(1)} \propto v_p \ .$$

Mass

Introducing a proportionality constant m while maintaining the negative sign between the two terms, we conclude from the above that:

$$-U_{x_p}^{(1)} = m \frac{\Delta v}{\Delta t} \quad (1.21)$$
$$T_{v_p}^{(1)} = m v_p \quad (1.22)$$

In order for the quantities E, T, U to have units of energy, the constant m can only have units of mass.

Newton's Second Law

Equation (1.21) is a special case of *Newton's second law*, which is concisely written:

$$-U_{x_p}^{(1)} = m x_{t_p}^{(2)}$$

In general, the left side represents the *force*, denoted F. In the general case, *force is mass times acceleration*:

$$F = m \frac{\Delta v}{\Delta t}$$

Potential Energy

The relationship

$$F = -U_{x_p}^{(1)}$$

is a special case of energy-conserving systems, where $U(x)$ is the *potential energy* of the system:

$$U(x) = \text{potential energy}$$

Kinetic Energy

The second result $T_{v_p}^{(1)} = m v_p$ relates the linear momentum to the slope of $T(v)$, which we identify as the *kinetic energy.* Knowing the slope of $T(v_p)$ is simply $m v_p$, it's easy to see that the kinetic energy is generally given by

$$T(v) = \frac{1}{2} m v^2 \ .$$

Working the same result in the other direction, we further deduce

$$T_{v_p}^{(2)} = m \ ,$$

and all higher $T_{v_p}^{(q)}$ are zero.

Total Energy

To summarize, we have that the total energy of a body in a so-far unspecified potential is constant:

$$E = \frac{1}{2}mv^2 + U_{x_p} + \sum_{q=1}^{n} \frac{1}{q!} U_{x_p}^{(q)} \left(x - x_p\right)^q$$

6.1 Mechanical Equilibrium

Since $U(x)$ is arbitrarily-shaped, one can imagine locating a special x_p that corresponds to an extreme of U, i.e. a local peak or a valley in its profile. In such a case, the slope of U is zero at that point

$$U_{x_p}^{(1)} = 0 \; ,$$

corresponding to *mechanical equilibrium.*

Small Oscillations

Small displacements from x_p are characterized by $x - x_p$ being a small quantity. As before, we argue that higher-order terms in the above sum are negligible, however truncating the series too soon leaves a tautology supporting no motion at all:

$$E = \cancel{\frac{1}{2}mv^2} + U_{x_p} + \cancel{\sum_{q=1}^{n} \frac{1}{q!} U_{x_p}^{(q)} \left(x - x_p\right)^q}$$

In light of $U_{x_p}^{(1)}$ being zero, the lowest nonzero term in the sum corresponds to $q = 2$, thus we have, for small displacements from x_p:

$$E - U_{x_p} = \frac{1}{2}mv^2 + \frac{1}{2}U_{x_p}^{(2)} \left(x - x_p\right)^2$$

The sign on $U_{x_p}^{(2)}$ determines the stability of motion around x_p. For $U_{x_p}^{(2)} < 0$, displacement from x_p causes v to grow extremely quickly and the approximation breaks down.

Hooke's Law

When x_p corresponds to an extreme point with $U_{x_p}^{(2)} > 0$, the system exhibits small oscillations centered on x_p. The potential energy term

$$U(x) = \frac{1}{2}U_{x_p}^{(2)} \left(x - x_p\right)^2$$

can be treated as arising by a spring force centered centered at x_p

$$f(x) = -U_{x_p}^{(2)} \left(x - x_p\right) \; ,$$

with $U_{x_p}^{(2)}$ being the effective spring constant. Using Newton's second law on this situation gives an equation for the subsequent motion:

$$m x_{t_p}^{(2)} = -U_{x_p}^{(2)} \left(x - x_p\right)$$

This particular form is also called *Hooke's law* for springs.

6.2 Freefall in Gravity

Using energy considerations, we can make sense of the standard kinematic identity

$$v^2 = v_0^2 + 2a\Delta x \; .$$

Supposing the motion represented is for a body of mass m, multiply through by $m/2$ and also expand $\Delta x = x - x_0$ to get, after simplifying:

$$\frac{1}{2}mv_0^2 - max_0 = \frac{1}{2}mv^2 - max$$

By letting $a = -g$ for freefall acceleration, we conclude that the gravitational potential energy for a body near Earth's surface is given by

$$U_{\text{grav}}(x) = mgx \; ,$$

where x is measured vertically from the ground.

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