Riemannian Geometry

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Chapter 1

Riemannian Geometry

Introduction

Ordinary calculus yields a plethora of useful results for science and engineering: areas, arc lengths, volumes, trajectories, etc. Most of this is possible without ever having to concern whether the space on which calculations take place may somehow affect the results, yet careful consideration shows there is indeed more to the story of calculus.

A popular illustration of this has us consider a colony of ants unknowingly living on a large beach ball. Curious members might trace out a triangle on the surface, only to discover the interior angles sum to greater than 180°. If similar ants lived on a horseback saddle and attempted to trace a triangle, they would discover the sum of interior angles being less than 180°. Moreover, each group of ants would notice that parallel lines generally fail to remain parallel when extended.

Without ever 'lifting off' the surface, we see it's still possible to deduce that something isn't quite 'flat' about a particular surface. We formally define *curved space* or a *curved manifold* as any locally flat and differentiable surface where Euclidean geometry doesn't work.

The precise framework in which curved manifolds are handled depends on the application. Here, we develop a major branch of calculus on manifolds called *Riemannian geometry*.

1 Index Notation and Tensors

Recall that any (ordinary) vector needs just one index to register its components, and a matrix needs two indices (with vectors as a special case). Going beyond this, it's conceivable to work with mathematical objects that have *any* number of indices, called *tensors*, of which vectors and matrices are special cases.

Let us trade the arrow symbol (or any other formatting) from vectors and matrices in favor of *index notation* for tensors. The number and placement of indices determines the *type* of tensor. For instance, two ways to express a vector x are:

$$x^\mu = \left(x^1, x^2, x^3, \ldots, x^N\right) \qquad \text{type } (1,0) \text{ tensor}$$

$$x_\mu = \left(x_1, x_2, x_3, \ldots, x_N\right) \qquad \text{type } (0,1) \text{ tensor}$$

In analogy to column- and row-vectors, x^μ is called *contravariant*, and x_μ is called *covariant*. Note these are *not* the same: the up- or down-placement of the index matters.

A two-index tensor Λ has three possible types $(2,0)$, $(1,1)$, $(0,2)$, represented by $\Lambda^{\mu\nu}$, Λ^{ν}_{μ}, $\Lambda_{\mu\nu}$, respectively.

1.1 Tensor Symmetry

A tensor (of any type) is *symmetric* in a pair of indices if it obeys

$$A^{\mu\nu} = A^{\nu\mu} \ ,$$

and is *antisymmetric* if it obeys

$$A^{\mu\nu} = -A^{\nu\mu} .$$

A general tensor may be expressed in terms of symmetric and antisymmetric parts

$$A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) + \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right) ,$$

where introducing the condensed notation

$$A^{(\mu\nu)} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) \qquad\qquad A^{[\mu\nu]} = \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right) ,$$

it follows that

$$A^{\mu\nu} = A^{(\mu\nu)} + A^{[\mu\nu]} .$$

1.2 Einstein Summation Convention

From elementary linear algebra, recall that a matrix acting on an appropriately-sized vector will yield a new vector, i.e., the $A\vec{x} = \vec{b}$ calculation. In tensor language, this operation reads:

$$\Lambda^{\nu}_{\mu} x^{\mu} = x^{\nu}$$

The above statement must be thoroughly unpacked. The right side is a list of vector components referenced by ν. On the left, the μ-index appears in both the up- and down-positions, which means it is summed over and eliminated. This hidden sum maneuver is called the *Einstein summation convention*. Explicitly, the above really means

$$\Lambda^{\nu}_{1} x^{1} + \Lambda^{\nu}_{2} x^{2} + \cdots + \Lambda^{\nu}_{n} x^{n} = x^{\nu} ,$$

which is equivalent to the 'ordinary' action of a matrix acting on a vector. Repeated indices should only appear once in the up-position and once in the down-position, with no exceptions. All indices must still balance before and after a summation.

1.3 Contraction

The act of equating an up-index and a down-index is called *contraction*, and triggers a sum over that index. For example, consider a type $(1,1)$ tensor product $x^{\mu} x_{\nu}$. Setting $\mu = \nu$ implies:

$$x^{\mu} x_{\mu} = x^{1} x_{1} + x^{2} x_{2} + \cdots + x^{N} x_{N} = S^{2}$$

The scalar result is a real or complex tensor of type $(0,0)$ loosely represented as S^2, formally called the *norm*. For ordinary vectors, this is equivalent to the dot product.

Problem 1

Which of the following two-index tensors represents the trace of a matrix?

$$\Lambda^{a}_{b} \qquad \Lambda^{\nu\nu} \qquad \Lambda^{\mu\mu} \qquad \Lambda^{b}_{b} \qquad \Lambda^{\mu}_{\nu} \qquad \Lambda^{12}$$

Solution 1

$$\Lambda^{b}_{b} = \Lambda^{1}_{1} + \Lambda^{2}_{2} + \cdots = \mathrm{Tr}\left(\Lambda\right)$$

1.4 Metric Tensor

Calculus on manifolds concerns with the differential line element $d\vec{S}$ at position $\vec{r}$ on the surface. In Cartesian space, the Pythagorean theorem tells us $dS^2 = dx^2 + dy^2$, whereas for polar coordinates we have $dS^2 = dr^2 + r^2 d\theta^2$, and so on. In general, a differential interval in N dimensions is given by

$$dx^\mu dx_\nu = dx^1 dx_1 + dx^2 dx_2 + \cdots + dx^N dx_N = dS^2 \; .$$

A natural question is, which new tensor would come into play in order for the line element to be proportional to $dx^\mu dx^\nu$? This motivates defining the *metric* tensor, generally written $g_{\mu\nu}$, and arises via:

$$dS^2 = dx^\mu dx_\nu = g_{\mu\nu} dx^\mu dx^\nu$$

The metric generally functions to lower the index of another tensor, as for our case

$$g_{\mu\nu} dx^\mu = dx_\nu \; .$$

Conversely, the *inverse metric* tensor $g^{\mu\nu}$ is used to raise an index according to

$$g^{\mu\nu} dx_\nu = dx^\mu \; .$$

It follows that the metric can raise or lower any index on most objects having at least one index:

$$g_{\mu\nu} A^{\mu\alpha} = A^\alpha_\nu \qquad\qquad g^{\mu\nu} A_{\mu_\beta} = A^\nu_\beta$$

<u>Problem 2</u>

Show that a contraction between metric tensors yields a delta function in the remaining indices:

$$g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$$

<u>Solution 2</u>

Calculating $dS^2 = dx^\beta dx_\beta = g^{\beta\gamma} dx_\gamma g_{\alpha\beta} dx^\alpha$, we find $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$.

<u>Problem 3</u>

Consider two tensors $X^{\mu\nu}$ and V^μ given by

$$X^{\mu\nu} = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix} \qquad\qquad V^\mu = (-1, 2, 0, -2) \; ,$$

respectively. Using the metric

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; ,$$

find the following quantities (i) X^μ_ν, (ii) X^ν_μ, (iii) $X^{(\mu\nu)}$, (iv) $X_{[\mu\nu]}$, (v) X^λ_λ, (vi) $V^\mu V_\mu$, (vii) $V_\mu X^{\mu\nu}$.

<u>Solution 3</u>

$$X^\mu_\nu = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix} \qquad X^\nu_\mu = \begin{bmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix}$$

$$X^{(\mu\nu)} = \begin{bmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{bmatrix} \qquad X_{[\mu\nu]} = \begin{bmatrix} 0 & -1/2 & 0 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 0 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{bmatrix}$$

$$X^\lambda_\lambda = -4 \qquad V^\mu V_\mu = 7 \qquad V_\mu X^{\mu\nu} = (4, -2, 5, 7)$$

1.5 Coordinate Transformation

As part of a the formal definition of a tensor, let us demand that under a general set of coordinate transformations, a tensor must obey an analog to the $A\vec{x} = \vec{b}$ calculation from linear algebra. In the most general case possible we would have a tensor A of type (N, M) undergoing $N + M$ coordinate changes:

$$A^{\mu'_1 \cdots \mu'_N}_{\nu'_1 \cdots \nu'_M} = \frac{\partial q^{\mu'_1}}{\partial q^{\mu_1}} \cdots \frac{\partial q^{\mu'_N}}{\partial q^{\mu_N}} \frac{\partial q^{\nu_1}}{\partial q^{\nu'_1}} \cdots \frac{\partial q^{\nu_M}}{\partial q^{\nu'_M}} A^{\mu_1 \cdots \mu_N}_{\nu_1 \cdots \nu_M}$$

Strictly, any object not obeying the above is not a tensor. For the simple case of one-index vectors V^μ and V_μ, the transformation law reads

$$V^{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu \qquad\qquad V_{\mu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} V_\mu \, .$$

2 Basis Vectors

Let $\vec{S}$ denote the position vector on a manifold parameterized by generalized coordinates q^1, q^2, q^3, ..., q^N, where N is the number of dimensions on the manifold. (Note that the physical units of any given q^i are not limited to length.) Denoting the differential line element vector as $d\vec{S}$, the chain rule dictates

$$d\vec{S} = \frac{\partial \vec{S}}{\partial q^1}\, dq^1 + \frac{\partial \vec{S}}{\partial q^2}\, dq^2 + \cdots + \frac{\partial \vec{S}}{\partial q^N}\, dq^N = \vec{a}_{(\mu)}\, dq^\mu ,$$

thus partial derivate terms are interpreted as (non-normalized) basis vectors $\vec{a}_{(\mu)}$ such that

$$\vec{a}_{(\mu)} = \frac{\partial \vec{S}}{\partial q^\mu} .$$

It follows that any tangent vector V on a manifold must admit an expansion in terms of basis vectors according to

$$\vec{V} = V^\mu \vec{a}_{(\mu)} ,$$

or without vector symbols at all,

$$V = V^\mu \frac{\partial}{\partial q^\mu} .$$

Note the right side is not equivalent to the normalized expression

$$\vec{V} = v^\mu \hat{a}_{(\mu)} .$$

The square of the differential line element dS^2 is given by $d\vec{S} \cdot d\vec{S}$, or

$$dS^2 = \left(\vec{a}_{(\mu)} \cdot \vec{a}^{(\nu)}\right) dq^\mu dq_\nu = \left(\vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)}\right) dq^\mu dq^\nu ,$$

simultaneously implying:

$$\vec{a}_{(\mu)} \cdot \vec{a}^{(\nu)} = \delta^\nu_\mu \qquad \vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)} = g_{\mu\nu} \qquad \vec{a}^{(\mu)} \cdot \vec{a}^{(\nu)} = g^{\mu\nu}$$

Problem 1

Consider two vectors U and V. Use the metric tensor to prove $U^\alpha V_\alpha = U_\beta V^\beta$.

Solution 1

$$U^\alpha V_\alpha = U_\beta g^{\alpha\beta} V^\gamma g_{\gamma\alpha} = U_\beta V^\gamma \left(g^{\alpha\beta} g_{\gamma\alpha}\right) = U_\beta V^\gamma \delta^\beta_\gamma = U_\beta V^\beta$$

2.1 Divergence

The *divergence* of a vector field, also known as a contravariant $(1, 0)$ tensor $A^\mu(q^\mu)$, is a contraction across the derivative of each component

$$\frac{\partial A^\mu}{\partial q^\mu} = \partial_\mu A^\mu ,$$

which resolves to a scalar. Let $\mu = \{1, 2, 3\}$ to reproduce the three-dimensional case.

2.2 Gradient

The *gradient* of a scalar field $f(x^\mu)$ or $f(q^\mu)$ is a vector of partial derivatives with respect to each variable. In a normalized basis, this is

$$\vec{\nabla} f = \hat{a}^{(\mu)} \frac{\partial f}{\partial x^\mu} ,$$

or in the generalized representation:

$$\vec{\nabla} f = \hat{a}^{(\mu)} \frac{\partial f}{\partial x^\mu} = \hat{a}^{(\mu)} \left| \vec{a}^{(\mu)} \right| \frac{\partial f}{\partial q^\mu} = \vec{a}^{(\mu)} \frac{\partial f}{\partial q^\mu}$$

By switching to vector-free notation, note we always assume non-normalized basis vectors:

$$\partial_\mu f = \frac{\partial f}{\partial q^\mu}$$

Observe that the gradient operation converts a scalar $(0,0)$ tensor into a covariant $(0,1)$ vector. We may instead transform into a contravariant $(1,0)$ vector with the up-index gradient operator:

$$\partial^\mu f = g^{\mu\nu} \partial_\nu f$$

Problem 2

Calculate the gradient of the function

$$f(x,y) = \ln \sqrt{x^2 + y^2}$$

and convert the result to polar coordinates obeying

$$x = r\cos\theta \qquad\qquad y = r\sin\theta\ .$$

Solution 2

$$\partial_\mu f = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \qquad V_x = \frac{x}{x^2+y^2} \qquad V_y = \frac{y}{x^2+y^2}$$

$$V_r = \frac{\partial x}{\partial r} V_x + \frac{\partial y}{\partial r} V_y = \frac{1}{r} \qquad\qquad V_\theta = \frac{\partial x}{\partial \theta} V_x + \frac{\partial y}{\partial \theta} V_y = 0$$

From vector calculus, we know that the gradient operation converts a scalar into a vector (a one-index tensor). Unfortunately higher-order derivatives don't generally result in tensors at face value, as illustrated by calculating two gradients $\partial_\mu \partial_\nu f$ with respect to a primed coordinate system as

$$\partial_{\mu'} \partial_{\nu'} f = \frac{\partial q^\alpha}{\partial q^{\mu'}} \frac{\partial}{\partial q^\alpha} \left(\frac{\partial q^\beta}{\partial q^{\nu'}} \frac{\partial f}{\partial q^\beta} \right) = \frac{\partial q^\alpha}{\partial q^{\mu'}} \frac{\partial q^\beta}{\partial q^{\nu'}} \partial_\alpha \partial_\beta f + \frac{\partial q^\alpha}{\partial q^{\mu'}} \frac{\partial^2 q^\beta}{\partial q^\alpha \partial^{\nu'}} \frac{\partial f}{\partial q^\beta}\ ,$$

which clearly violates the tensor transformation rule. (The last term shouldn't be there.)

Problem 3

Consider the covariant vector field $V_\mu(q^\mu)$. Under general coordinate transformations $q^\mu \rightarrow q^{\mu'}$, show that the quantity $\partial\left[{}_\alpha V_\beta\right]$ transforms as a type $(0,2)$ tensor.

Solution 3

$$\begin{aligned}
\partial'\left[{}_\alpha V'_\beta\right] &= \frac{1}{2}\left(\partial'_\alpha V'_\beta - \partial'_\beta V'_\alpha\right) = \frac{1}{2}\left(\frac{\partial}{\partial q^{\alpha'}} V'_\beta - \frac{\partial}{\partial q^{\beta'}} V'_\alpha\right) \\
&= \frac{1}{2}\left(\frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial}{\partial q^\mu}\left(\frac{\partial q^\nu}{\partial q^{\beta'}} V_\nu\right) - \frac{\partial q^\mu}{\partial q^{\beta'}} \frac{\partial}{\partial q^\mu}\left(\frac{\partial q^\nu}{\partial q^{\alpha'}} V_\nu\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial}{\partial q^{\beta'}}\left(\frac{\partial q^\nu}{\partial q^\mu}\right) - \frac{\partial q^\mu}{\partial q^{\beta'}} \frac{\partial}{\partial q^{\alpha'}}\left(\frac{\partial q^\nu}{\partial q^\mu}\right)\right) V_\nu \\
&\quad + \frac{1}{2}\left(\frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial q^\nu}{\partial q^{\beta'}} - \frac{\partial q^\mu}{\partial q^{\beta'}} \frac{\partial q^\nu}{\partial q^{\alpha'}}\right) \partial_\mu V_\nu = \frac{\partial q^\mu}{\partial q^{[\alpha']}} \frac{\partial q^\nu}{\partial q^{[\beta']}} \partial_\mu V_\nu
\end{aligned}$$

2.3 Curl

The vector product or *cross product* between two vectors is generalized to N dimensions using the *Levi-Civita* symbol ϵ_{ijk}. If the indices i, j, k are an even permutation of the sequence $1, 2, 3$, then $\epsilon = 1$. For odd permutations, $\epsilon = -1$, and for any two equal indices, $\epsilon = 0$. Note ϵ_{ijk} is *not* a tensor and ignores the up- or down-placement of indices. To generalize to higher dimensions, add more indices next to i, j, k.

For a $3D$ example, the cross product reads

$$(\vec{u} \times \vec{v})^i = \epsilon_{ijk} u^j v^k \ ,$$

and the curl of a vector is

$$\left(\vec{\nabla} \times \vec{v}\right)^i = \epsilon^{ijk} \frac{\partial}{\partial x^j} \left(\vec{v}\right)^k \ ,$$

or

$$\vec{\nabla} \times \vec{v} = \hat{a}^{(j)} \frac{\partial}{\partial x^j} \times \left(v^i \hat{a}_{(i)}\right) \ .$$

3 Covariant Derivative

Let us continue considering an N-dimensional manifold S mapped by generalized coordinates q^i, having line element

$$d\vec{S} = \frac{\partial \vec{S}}{\partial q^1}\, dq^1 + \frac{\partial \vec{S}}{\partial q^2}\, dq^2 + \cdots + \frac{\partial \vec{S}}{\partial q^N}\, dq^N = \vec{a}_{(\mu)}\, dq^\mu \; ,$$

and corresponding metric

$$g_{\mu\nu} = \vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)} \; .$$

The fun stops relatively early, however. Right away we find that the divergence of a rank $(1,0)$ tensor V^β is not a tensor. Take two representations of V as observed in two coordinate systems

$$V^{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu \; ,$$

and differentiate with respect to $q^{\nu'}$ to find

$$\partial_{\nu'} V^{\mu'} = \frac{\partial}{\partial q^{\nu'}} V^{\mu'} = \frac{\partial}{\partial q^{\nu'}} \left(\frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu \right) = \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \partial_\nu V^\mu + \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial^2 q^{\mu'}}{\partial q^\nu \partial q^\mu} V^\mu \; ,$$

which would have transformed as a tensor if it weren't for that second term.

3.1 Motivation

The fact that a tangent vector's divergence is not a tensor motivates the *covariant derivative* operator D_ν to act on V^μ and force the result to be a tensor:

$$D_{\nu'} V^{\mu'} = \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} D_\nu V^\mu$$

The operator D_ν must include a new term that can subtract off any non-tensorial components by construction. We then propose

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \; ,$$

where the factors $\Gamma^\mu_{\nu\beta}$ are called *Christoffel symbols*. By testing coordinate transformations (see below), it's readily shown that Γ cannot be a tensor. To prove this, we calculate $D_{\nu'} V^{\mu'}$ to find

$$\begin{aligned} D_{\nu'} V^{\mu'} &= \partial_{\nu'} V^{\mu'} + (\Gamma')^{\mu'}_{\nu'\lambda'} V^{\lambda'} \\ &= \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \partial_\nu V^\mu + \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial^2 q^{\mu'}}{\partial q^\nu \partial q^\lambda} V^\lambda + (\Gamma')^{\mu'}_{\nu'\lambda'} \frac{\partial q^{\lambda'}}{\partial q^\lambda} V^\lambda \\ &= \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \left(\partial_\nu V^\mu + \frac{\partial q^\mu}{\partial q^\alpha} \frac{\partial^2 q^\alpha}{\partial q^\nu \partial q^\lambda} V^\lambda + (\Gamma')^\alpha_{\beta\gamma} \frac{\partial q^\beta}{\partial q^\nu} \frac{\partial q^\mu}{\partial q^\alpha} \frac{\partial q^\gamma}{\partial q^\lambda} V^\lambda \right) \; . \end{aligned}$$

The quantity in parentheses must resolve to $\partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda$ by the tensor transformation law, revealing the restriction on Γ to be

$$(\Gamma')^\alpha_{\beta\gamma} = \frac{\partial q^\nu}{\partial q^\beta} \frac{\partial q^\alpha}{\partial q^\mu} \frac{\partial q^\lambda}{\partial q^\gamma} \Gamma^\mu_{\nu\lambda} - \frac{\partial q^\nu}{\partial q^\beta} \frac{\partial q^\lambda}{\partial q^\gamma} \frac{\partial^2 q^\alpha}{\partial q^\nu \partial q^\lambda} \; ,$$

which is unambiguously not a tensor (the second term gives it away).

3.2 Vector Derivative

To reinforce the motivation for the covariant derivative, we write the divergence of a vector $\vec{V}$, namely

$$\partial_\nu \left(V^\mu \, \vec{a}_{(\mu)} \right) = \partial_\nu \left(V^\mu \right) \vec{a}_{(\mu)} + V^\mu \partial_\nu \left(\vec{a}_{(\mu)} \right) \; ,$$

and observe that the derivative of $\vec{a}_{(\mu)}$ involves a second derivative of the position vector $\vec{r}$, analogous to an acceleration-like term that points toward the center of curvature, which may be well-off the manifold itself. We correct for this by replacing the last term with

$$V^\mu \partial_\nu \left(\vec{a}_{(\mu)}\right) = \Gamma^\mu_{\nu\beta} V^\beta \vec{a}_{(\mu)} \, .$$

Problem 1

Solve for the components of $\Gamma^\mu_{\nu\beta}$ in terms of position $\vec{S}$ on the manifold and the basis vectors $\vec{a}^{(\mu)}$ and interpret the result. That is, eliminate any reference to V in the above.

Solution 1

$$\begin{aligned}
\Gamma^\mu_{\nu\beta} V^\beta \vec{a}_{(\mu)} &= V^\mu \partial_\nu \left(\vec{a}_{(\mu)}\right) \\
\Gamma^\mu_{\nu\beta} V^\beta \cancel{\left(\vec{a}^{(\gamma)} \cdot \vec{a}_{(\mu)}\right)} \delta^\gamma_\mu &= \vec{a}^{(\gamma)} \cdot V^\mu \partial_\nu \left(\vec{a}_{(\mu)}\right) \\
\Gamma^\gamma_{\nu\beta} \cancel{V^\beta} &= \vec{a}^{(\gamma)} \cdot \cancel{V^\beta} \partial_\nu \left(\vec{a}_{(\beta)}\right) \\
\Gamma^\gamma_{\mu\nu} &= \vec{a}^{(\gamma)} \cdot \frac{\partial}{\partial q^\mu} \vec{a}_{(\nu)} \\
\Gamma^\gamma_{\mu\nu} &= \vec{a}^{(\gamma)} \cdot \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu} \vec{S}
\end{aligned}$$

From ordinary calculus, recall that derivatives of $\vec{S}$ with respect to q^μ and q^ν commute, implying a symmetry in the lower index of Γ:

$$\Gamma^\gamma_{\mu\nu} = \Gamma^\gamma_{\nu\mu}$$

Problem 2

Find a formula for $D_\nu V^\mu$ purely in terms of partial derivatives and basis vectors. (No Christoffel symbols.)

Solution 2

$$\begin{aligned}
\partial_\nu \left(V^\mu \, \vec{a}_{(\mu)}\right) &= \partial_\nu \left(V^\mu\right) \vec{a}_{(\mu)} + \Gamma^\mu_{\nu\beta} V^\beta \vec{a}_{(\mu)} \\
\vec{a}^{(\mu)} \cdot \partial_\nu \left(V^\mu \, \vec{a}_{(\mu)}\right) &= \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\beta} V^\beta\right) \vec{a}^{(\mu)} \cdot \vec{a}_{(\mu)} \\
\vec{a}^{(\mu)} \cdot \partial_\nu \left(V^\mu \, \vec{a}_{(\mu)}\right) &= D_\nu V^\mu
\end{aligned}$$

Problem 3

Check if the covariant derivative follows similar rules to ordinary calculus for two tensors U and V.

Solution 3

$$\begin{aligned}
D_\alpha \left(U^\mu + V^\nu\right) &= D_\alpha U^\mu + D_\alpha V^\nu \\
D_\alpha \left(U^\mu V^\nu\right) &= \left(D_\alpha U^\nu\right) V^\nu + U^\mu \left(D_\alpha V^\nu\right) \\
D_\mu \left(A^\alpha_{\alpha\nu}\right) &= \left(DA\right)^\alpha_{\mu\alpha\nu}
\end{aligned}$$

3.3 Scalar Derivative

A scalar field is represented by a type $(0,0)$ tensor $f = U_\alpha V^\alpha$. Applying the covariant derivative to such a field, we have

$$D_\nu f = \left(D_\nu U_\alpha\right) V^\alpha + U_\alpha \left(D_\nu V^\alpha\right) \, .$$

To handle the covariant derivative on a down-index, momentarily introduce a modified Christoffel symbol $\tilde{\Gamma}$ as follows:

$$\begin{aligned} D_\nu f &= \left(\partial_\nu U_\alpha + \tilde{\Gamma}^\beta_{\nu\alpha} U_\beta\right) V^\alpha + U_\alpha \left(\partial_\nu V^\alpha + \Gamma^\alpha_{\nu\beta} V^\beta\right) \\ &= V^\alpha \partial_\nu U_\alpha + U_\alpha \partial_\nu V^\alpha + \tilde{\Gamma}^\beta_{\nu\alpha} U_\beta V^\alpha + \Gamma^\alpha_{\nu\beta} U_\alpha V^\beta \\ &= \partial_\nu f + U_\beta V^\alpha \left(\tilde{\Gamma}^\beta_{\nu\alpha} + \Gamma^\beta_{\nu\alpha}\right) \end{aligned}$$

Of course, the derivative of a scalar field can not be subject to the same distortions as a tensor field. Simultaneously conclude that

$$D_\nu f = \partial_\nu f \qquad\qquad \tilde{\Gamma}^\beta_{\nu\alpha} = -\Gamma^\beta_{\nu\alpha} \,,$$

and as a corollary, the covariant derivative on a down-index reads:

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\beta_{\nu\mu} V_\beta$$

3.4 Tensor Derivative

We have seen that the covariant derivative on an up-index introduces a factor of Γ, and on a down-index we get $-\Gamma$. This pattern extends generally as illustrated:

$$\begin{aligned} D_\alpha T^{\mu\nu} &= \partial_\alpha T^{\mu\nu} + \Gamma^\mu_{\alpha\beta} T^{\beta\nu} + \Gamma^\nu_{\alpha\rho} T^{\mu\rho} \\ D_\alpha U_{\mu\nu} &= \partial_\alpha U_{\mu\nu} - \Gamma^\rho_{\alpha\mu} U_{\rho\nu} - \Gamma^\gamma_{\alpha\nu} U_{\mu\gamma} \\ D_\alpha V^\mu_\nu &= \partial_\alpha V^\mu_\nu + \Gamma^\mu_{\alpha\gamma} V^\gamma_\nu - \Gamma^\beta_{\alpha\nu} V^\mu_\beta \,, \end{aligned}$$

You are encouraged to check that the V^μ_ν-equation with $\mu = \nu$ is consistent with the covariant derivative of a scalar field.

Problem 4

Show that if $A_{\mu\nu} = -A_{\nu\mu}$ is an antisymmetric type $(0, 2)$ tensor, all connection coefficients cancel out of

$$D\left[{}_\mu A_{\nu\rho}\right] = \partial\left[{}_\mu A_{\nu\rho}\right] \,.$$

Solution 4

$$\begin{aligned} D\left[{}_\mu A_{\nu\rho}\right] &= \frac{1}{3!}\left(D_\mu A_{\nu\rho} + D_\rho A_{\mu\nu} + D_\nu A_{\rho\mu} - D_\mu A_{\rho\nu} - D_\rho A_{\nu\mu} - D_\nu A_{\mu\rho}\right) \\ &= \frac{1}{3}\left(D_\mu A_{\nu\rho} + D_\rho A_{\mu\nu} + D_\nu A_{\rho\mu}\right) \\ &= \partial_\mu A_{\nu\rho} - \Gamma^\gamma_{\mu\nu} A_{\gamma\rho} - \Gamma^\gamma_{\mu\rho} A_{\nu\gamma} \\ &\quad + \partial_\rho A_{\mu\nu} - \Gamma^\gamma_{\rho\mu} A_{\gamma\nu} - \Gamma^\gamma_{\rho\nu} A_{\mu\gamma} \\ &\quad + \partial_\nu A_{\rho\mu} - \Gamma^\gamma_{\nu\rho} A_{\gamma\mu} - \Gamma^\gamma_{\nu\mu} A_{\rho\gamma} \\ &= \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu} \\ &= \partial\left[{}_\mu A_{\nu\rho}\right] \end{aligned}$$

3.5 Torsion Tensor

The *torsion*, defined as $T^\mu_{\nu\gamma} = \Gamma^\mu_{\nu\gamma} - \Gamma^\mu_{\gamma\nu}$, qualifies as a tensor. To prove this, we calculate

$$\begin{aligned} T'^{\mu'}_{\nu'\gamma'} &= \Gamma'^{\mu'}_{\nu'\gamma'} - \Gamma'^{\mu'}_{\gamma'\nu'} \\ &= \frac{\partial q^{\mu'}}{\partial q^\beta}\frac{\partial q^\alpha}{\partial q^{\nu'}}\frac{\partial q^\rho}{\partial q^{\gamma'}}\Gamma^\beta_{\alpha\rho} - \frac{\partial q^{\mu'}}{\partial q^\beta}\frac{\partial q^\rho}{\partial q^{\gamma'}}\frac{\partial q^\alpha}{\partial q^{\nu'}}\Gamma^\beta_{\rho\alpha} \\ &= \frac{\partial q^{\mu'}}{\partial q^\beta}\frac{\partial q^\alpha}{\partial q^{\nu'}}\frac{\partial q^\rho}{\partial q^{\gamma'}} T^\beta_{\alpha\rho} \,, \end{aligned}$$

which adheres to the transformation law for tensors. In most subjects, particularly general relativity, the torsion is assumed to be zero unless heavily disclaimed.

If the above is violated, the reason would trace back to non-commuting dot products between basis vectors on the manifold. This notion isn't realistic (i.e. not needed to derive any known useful physics), although efforts have been made using torsion to understand quantum spin.

3.6 Metric Compatibility

The *metric compatibility* condition states that the covariant derivative of the metric is automatically zero at all points on the manifold:

$$D_\alpha g_{\mu\nu} = 0$$

Problem 5

Show that the inverse relation

$$D_\alpha g^{\mu\nu} = 0$$

also holds by considering the product $D_\alpha \left(g_{\mu\nu} g^{\gamma\nu}\right)$.

Solution 5

$$\begin{aligned} D_\alpha \left(g_{\mu\nu} g^{\gamma\nu}\right) &= g_{\mu\nu} D_\alpha g^{\gamma\nu} + g^{\gamma\nu} D_\alpha g_{\mu\nu} \\ D_\alpha \left(\delta^\gamma_\mu\right) &= g_{\mu\nu} D_\alpha g^{\gamma\nu} + g^{\gamma\nu} \cdot 0 \\ g^{\mu\rho} D_\alpha \left(\delta^\gamma_\mu\right) &= g^{\mu\rho} g_{\mu\nu} D_\alpha g^{\gamma\nu} \\ 0 &= D_\alpha g^{\gamma\rho} \end{aligned}$$

3.7 Connection Coefficients

The Christoffel symbols Γ have another label called *connection coefficients*. Using the metric compatibility relation $D_\alpha g_{\mu\nu} = 0$ along with the symmetry requirement $\Gamma^\mu_{\nu\gamma} = \Gamma^\mu_{\gamma\nu}$, we may solve for each component of Γ.

Begin by explicitly writing the compatibility relation in terms of Γ, and then write two permutations of the same equation in the indices α, μ, and ν:

$$\begin{aligned} 0 &= \partial_\alpha g_{\mu\nu} - \Gamma^\gamma_{\alpha\mu} g_{\gamma\nu} - \Gamma^\gamma_{\alpha\nu} g_{\mu\gamma} \\ 0 &= \partial_\nu g_{\alpha\mu} - \Gamma^\gamma_{\nu\alpha} g_{\gamma\mu} - \Gamma^\gamma_{\nu\mu} g_{\alpha\gamma} \\ 0 &= \partial_\mu g_{\nu\alpha} - \Gamma^\gamma_{\mu\nu} g_{\gamma\alpha} - \Gamma^\gamma_{\mu\alpha} g_{\nu\gamma} \end{aligned}$$

Subtracting the second two equations from the first, find

$$0 = \partial_\alpha g_{\mu\nu} - \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\nu\alpha} + 2\Gamma^\gamma_{\mu\nu} g_{\alpha\gamma} \ ,$$

which has only one instance of Γ. Isolating this, we have

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) \ .$$

Problem 6

Using the rise-over-run notion of the derivative, find an expression for $\partial_\gamma \Gamma^\mu_{\nu\beta}$ based at a specific point $q^\mu(p)$ on the manifold.

Solution 6

Start with

$$\begin{aligned} \frac{\Gamma^\mu_{\nu\beta}\left(q^\mu\left(p'\right)\right) - \Gamma^\mu_{\nu\beta}\left(q^\mu\left(p\right)\right)}{q^\gamma\left(p'\right) - q^\gamma\left(p\right)} &= \partial_\gamma \Gamma^\mu_{\nu\beta} \\ \Gamma^\mu_{\nu\beta}\left(q^\mu\left(p'\right)\right) - \Gamma^\mu_{\nu\beta}\left(q^\mu\left(p\right)\right) &= \left(q^\gamma\left(p'\right) - q^\gamma\left(p\right)\right) \partial_\gamma \Gamma^\mu_{\nu\beta} \ , \end{aligned}$$

where if we define shorthand such shat

$$\Gamma^{\mu}_{\nu\beta}\left(q^{\mu}\left(p\right)\right)=\Gamma^{\mu}_{\nu\beta}\left(p\right) \qquad q^{\gamma}\left(p'\right)-q^{\gamma}\left(p\right)=b^{\gamma}_{\left(p'p\right)}\ ,$$

the tighter expression reads:

$$\Gamma^{\mu}_{\nu\beta}\left(p'\right)-\Gamma^{\mu}_{\nu\beta}\left(p\right)=b^{\gamma}_{\left(p'p\right)}\partial_{\gamma}\Gamma^{\mu}_{\nu\beta}$$

3.8 Vector Calculus Operators

The covariant derivative is a starting point for several useful identities. First multiply through by $\vec{a}^{(\nu)}\cdot\vec{a}_{(\mu)}$ to write the divergence on a manifold:

$$\vec{\nabla}\cdot\vec{V}=\left(\vec{a}^{(\nu)}\cdot\vec{a}_{(\mu)}\right)D_{\nu}V^{\mu}=D_{\mu}V^{\mu}$$

Multiply instead by $\vec{a}^{(\nu)}\times\vec{a}_{(\mu)}$ to get the generalized curl:

$$\vec{\nabla}\times\vec{V}=\left(\vec{a}^{(\nu)}\times\vec{a}_{(\mu)}\right)D_{\nu}V^{\mu}$$

The Laplacian operator $\vec{\nabla}\cdot\vec{\nabla}=\nabla^2=\Delta$ involves two derivatives of a scalar field f. The 'inner' derivative is easy because we found $D_{\alpha}f=\partial_{\alpha}f$. To continue, assume that vector V above is really a gradient of some scalar function f, as in $V^{\mu}=D^{\mu}f$. The Laplacian operator on a manifold is therefore expressed by any of:

$$\Delta f=\nabla^2 f=D_{\mu}D^{\mu}f=g^{\mu\nu}D_{\mu}D_{\nu}f=\left(\vec{a}^{(\mu)}\cdot\vec{a}^{(\nu)}\right)D_{\mu}D_{\nu}f$$

4 Flat Manifolds

A *flat manifold* is defined as any space where Euclidean geometry works: triangles have π internal radians, a circle obeys $A = \pi r^2$, and so on. It should be emphasized that simply because a coordinate system involves 'curvy' coordinates (i.e. anything but Cartesian), the manifold may still be flat. Saving the problem of objective curvature detection for the next section, we pause here to write the properties of flat manifolds.

As usual, we work with an N-dimensional manifold S mapped by generalized coordinates q^i, having line element

$$d\vec{S} = \frac{\partial \vec{S}}{\partial q^1}\, dq^1 + \frac{\partial \vec{S}}{\partial q^2}\, dq^2 + \cdots + \frac{\partial \vec{S}}{\partial q^N}\, dq^N = \vec{a}_{(\mu)}\, dq^\mu \ ,$$

and corresponding metric

$$g_{\mu\nu} = \vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)} \ .$$

4.1 Cartesian Coordinates

A three-dimensional flat space is easily represented by the Cartesian position vector

$$\vec{S} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} \ .$$

In this coordinate system, the line element is

$$d\vec{S} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} \ ,$$

with corresponding interval

$$dS^2 = dx^2 + dy^2 + dz^2 \ .$$

The Cartesian metric for flat space has its own symbol η whose components are trivial to read from $d\vec{S}$ or dS^2:

$$\eta_{\mu\mu} = 1 \qquad\qquad \eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu$$

In block (not matrix!) form, the Cartesian metric reads

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

Needless to mention that all connection coefficients Γ, which depend on derivatives of η, are zero in Cartesian coordinates.

4.2 Cylindrical Coordinates

Starting with Cartesian coordinates, we choose a polar representation of the xy-plane and leave the z-component unchanged to get cylindrical coordinates

$$\vec{S} = r\,\hat{r} + z\,\hat{k} \ ,$$

where:

$$r = \sqrt{x^2 + y^2} \qquad\qquad \hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad\qquad \frac{d\hat{r}}{d\theta} = \hat{\theta}$$

Our first reflex should be to derive the differential line element $d\vec{S}$ and the interval dS^2 for this system, resulting in

$$d\vec{S} = dr\,\hat{r} + r d\theta\,\hat{\theta} + dz\,\hat{k} \qquad\qquad dS^2 = dr^2 + r^2 d\theta^2 + dz^2 \ .$$

Basis vectors for this system are inferred from $d\vec{S}$, namely

$$\vec{a}_{(r)} = \hat{r} \qquad\qquad \vec{a}_{(\theta)} = r\,\hat{\theta} \qquad\qquad \vec{a}_{(z)} = \hat{k} \ ,$$

and using $\vec{a}_{(\mu)} \cdot \vec{a}^{(\nu)} = \delta^\nu_\mu$, we also have

$$\vec{a}^{(r)} = \hat{r} \qquad \vec{a}^{(\theta)} = r^{-1}\,\hat{\theta} \qquad \vec{a}^{(z)} = \hat{k}\,.$$

Components of the metric are inferred from dS^2, or equivalently, $\vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)} = g_{\mu\nu}$, which turn out to be

$$g_{rr} = 1 \qquad g_{\theta\theta} = r^2 \qquad g_{zz} = 1\,,$$

or with raised indices,

$$g^{rr} = 1 \qquad g^{\theta\theta} = r^{-2} \qquad g^{zz} = 1\,.$$

Of course, we could have used the tensor transformation law

$$g_{\mu'\nu'} = \frac{\partial q^\alpha}{\partial q^{\mu'}} \frac{\partial q^\beta}{\partial q^{\nu'}} \eta_{\alpha\beta}$$

to derive the components of $g_{\mu\nu}$ from the Cartesian metric. In the above, the primed indices refer to parameters r, θ, z, whereas the unprimed indices are for x, y, and z.

If the components of a vector $\vec{V} = V^\mu \vec{a}_{(\mu)} = v^\mu \hat{a}_{(\mu)}$ are handed to you in the form V^r, V^θ, V^z, use the metric to lower the index via $V_\mu = g_{\mu\nu} V^\nu$:

$$V_r = V^r \qquad V_\theta = r^2\, V^\theta \qquad V_z = V^z$$

To isolate v^μ, calculate $\vec{V} \cdot \hat{a}_{(\mu)}$ to get

$$v^r = V^r \qquad v^\theta = r\, V^\theta \qquad v^z = V^z\,.$$

Connection coefficients can be cumbersome to calculate blindly, however most components of Γ resolve to zero for common coordinate systems, with cylindrical coordinates mapping flat space being the first almost-trivial cases. Using either of

$$\Gamma^\gamma_{\mu\nu} = \vec{a}^{(\gamma)} \cdot \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q^\nu} \vec{S} \qquad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right)\,,$$

your effort should distill to

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -r\,.$$

Problem 1

Using index notation, write the gradient of a scalar function f in flat space mapped by cylindrical coordinates. Also find the divergence and the curl of a vector $\vec{V} = V^\mu \vec{a}_{(\mu)} = v^\mu \hat{a}_{(\mu)}$ in the same system. Finish off by finding the Laplacian operator.

Solution 1

$$\begin{aligned}
\vec{\nabla} f &= \vec{a}^{(\mu)} \partial_\mu f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{\partial f}{\partial z}\,\hat{z} \\
\vec{\nabla} \cdot \vec{V} &= \partial_r V^r + \partial_\theta V^\theta + \partial_z V^z + \Gamma^\theta_{\theta r} V^r = \frac{1}{r}\frac{\partial}{\partial r}\left(r v^r\right) + \frac{1}{r}\frac{\partial}{\partial \theta} v^\theta + \frac{\partial}{\partial z} v^z \\
\vec{\nabla} \times \vec{V} &= \left(\frac{1}{r}\frac{\partial}{\partial \theta} v^z - \frac{\partial}{\partial z} v^\theta \right) \hat{r} + \left(\frac{\partial}{\partial z} v^r - \frac{\partial}{\partial r} v^z \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r}\left(r v^\theta\right) - \frac{\partial}{\partial \theta} v^r \right) \hat{z} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

4.3 Spherical Coordinates

A three-dimensional flat space mapped by spherical coordinates is parameterized in terms of two angles and one radius as

$$\vec{S} = r\,\hat{r} = r\sin\theta\cos\phi\,\hat{\imath} + r\sin\theta\sin\phi\,\hat{\jmath} + r\cos\theta\,\hat{k}\;,$$

where

$$r = \sqrt{x^2+y^2+z^2}\;.$$

Normalized basis vectors are calculated from $\vec{S}$ according to:

$$\hat{r} = \frac{\partial\vec{S}}{\partial r}\cdot\left|\frac{\partial\vec{S}}{\partial r}\right|^{-1} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \frac{\partial\vec{S}}{\partial \theta}\cdot\left|\frac{\partial\vec{S}}{\partial \theta}\right|^{-1} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = \frac{\partial\vec{S}}{\partial \phi}\cdot\left|\frac{\partial\vec{S}}{\partial \phi}\right|^{-1} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

From $\vec{S}$, we also calculate the differential line element $d\vec{S}$ and the interval dS^2 to get

$$d\vec{S} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi} \qquad\qquad dS^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\;,$$

telling us the basis vectors for this system are

$$\vec{a}_{(r)} = \hat{r} \qquad \vec{a}_{(\theta)} = r\,\hat{\theta} \qquad \vec{a}_{(\phi)} = r\sin\theta\,\hat{\phi}\;,$$

and $\vec{a}_{(\mu)}\cdot\vec{a}^{(\nu)} = \delta^{\nu}_{\mu}$ gives

$$\vec{a}^{(r)} = \hat{r} \qquad \vec{a}^{(\theta)} = r^{-1}\,\hat{\theta} \qquad \vec{a}^{(\phi)} = (r\sin\theta)^{-1}\;\hat{\phi}\;.$$

Components of the metric are inferred from dS^2, or equivalently, $\vec{a}_{(\mu)}\cdot\vec{a}_{(\nu)} = g_{\mu\nu}$, which turn out to be

$$g_{rr} = 1 \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2\sin^2\theta\;,$$

or with raised indices,

$$g^{rr} = 1 \qquad g^{\theta\theta} = r^{-2} \qquad g^{\phi\phi} = \left(r^2\sin^2\theta\right)^{-1}\;.$$

Of course, we could have used the tensor transformation law

$$g_{\mu'\nu'} = \frac{\partial q^{\alpha}}{\partial q^{\mu'}}\frac{\partial q^{\beta}}{\partial q^{\nu'}}\eta_{\alpha\beta}$$

to derive the components of $g_{\mu\nu}$ from the Cartesian metric. In the above, the primed indices refer to parameters r, θ, ϕ, whereas the unprimed indices are for x, y, and z.

If the components of a vector $\vec{V} = V^{\mu}\vec{a}_{(\mu)} = v^{\mu}\hat{a}_{(\mu)}$ are handed to you in the form V^r, V^{θ}, V^{ϕ}, use the metric to lower the index via $V_{\mu} = g_{\mu\nu}V^{\nu}$:

$$V_r = V^r \qquad V_{\theta} = r^2\,V^{\theta} \qquad V_{\phi} = r^2\sin^2\theta\,V^{\phi}$$

To isolate v^{μ}, calculate $\vec{V}\cdot\hat{a}_{(\mu)}$ to get

$$v^r = V^r \qquad v^{\theta} = r\,V^{\theta} \qquad v^{\phi} = r\sin\theta\,V^{\phi}\;.$$

Problem 2

Calculate all nonzero connection coefficients in flat space mapped by spherical coordinates.

Solution 2

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\phi}_{r\phi} = \frac{1}{r} \qquad \Gamma^{r}_{\theta\theta} = -r \qquad \Gamma^{r}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

Problem 3

Using index notation, write the gradient of a scalar function f in flat space mapped by spherical coordinates. Also find the divergence and the curl of a vector $\vec{V} = V^\mu \vec{a}_{(\mu)} = v^\mu \hat{a}_{(\mu)}$ in the same system. Finish off by finding the Laplacian operator.

Solution 3

$$\vec{\nabla} f = \vec{a}^{(\mu)} \partial_\mu f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\phi}$$

$$\begin{aligned}
\vec{\nabla}\cdot\vec{V} &= \partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi + \frac{2V^r}{r} + V^\theta\cot\theta \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v^r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v^\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v^\phi
\end{aligned}$$

$$\begin{aligned}
\left(\vec{\nabla}\times\vec{V}\right)_r &= \sin\theta\,\partial_\theta V^\phi + 2\cos\theta\,V^\phi - \frac{1}{\sin\theta}\partial_\phi V^\theta \\
&= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta v^\phi\right) - \frac{\partial}{\partial\phi}v^\theta\right) \\
\left(\vec{\nabla}\times\vec{V}\right)_\theta &= \frac{1}{r\sin\theta}\partial_\phi V^r - 2\sin\theta\,V^\phi - r\sin\theta\left(\partial_r V^\phi\right) \\
&= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}v^r - \frac{\partial}{\partial r}\left(r v^\phi\right)\right) \\
\left(\vec{\nabla}\times\vec{V}\right)_\phi &= r\partial_r V^\theta + 2V^\theta - \frac{1}{r}\partial_\theta V^r \\
&= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r v^\theta\right) - \frac{\partial}{\partial\theta}v^r\right)
\end{aligned}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

4.4 Parabolic Coordinates

Starting with two-dimensional Cartesian coordinates, we choose a parabolic representation of the xy-plane

$$\vec{S} = x\,\hat{\imath} + y\,\hat{\jmath}\,,$$

where the parabolic coordinates u, v are defined by

$$x = uv \qquad y = \frac{1}{2}\left(u^2 - v^2\right)\,.$$

Problem 4

Find the equation of a circle of radius r centered at the origin.

Solution 4

$$\begin{aligned} r^2 = x^2 + y^2 &= u^2 v^2 + \frac{1}{4}\left(u^4 + v^4 - 2u^2 v^2\right) \\ &= \frac{u^2 v^2}{2} + \frac{u^2 + v^2}{4} = \frac{1}{4}\left(u^2 + v^2\right)^2 \\ 2r &= u^2 + v^2 \end{aligned}$$

Normalized basis vectors are calculated from $\vec{S}$ according to

$$\hat{u} = \frac{\partial \vec{S}}{\partial u} \cdot \left|\frac{\partial \vec{S}}{\partial u}\right|^{-1} = \frac{v\,\hat{i} + u\,\hat{j}}{\sqrt{u^2 + v^2}} \qquad \hat{v} = \frac{\partial \vec{S}}{\partial v} \cdot \left|\frac{\partial \vec{S}}{\partial v}\right|^{-1} = \frac{u\,\hat{i} - v\,\hat{j}}{\sqrt{u^2 + v^2}} ,$$

where solving for $\hat{i}$, $\hat{j}$ respectively, we have

$$\hat{i} = \frac{v\,\hat{u} + u\,\hat{v}}{\sqrt{u^2 + v^2}} \qquad \hat{j} = \frac{u\,\hat{u} - v\,\hat{v}}{\sqrt{u^2 + v^2}} ,$$

and therefore $\vec{S}$ becomes

$$\vec{S} = \frac{\sqrt{u^2 + v^2}}{2}\left(u\,\hat{u} + v\,\hat{v}\right) .$$

The next job is to calculate the line element $d\vec{S}$, which entails first discovering

$$d\hat{u} = \frac{udv - vdu}{u^2 + v^2}\,\hat{v} \qquad d\hat{v} = \frac{vdu - udv}{u^2 + v^2}\,\hat{u} ,$$

and by careful substitution into

$$d\vec{S} = d\left(\frac{\sqrt{u^2 + v^2}}{2}\right)\left(u\,\hat{u} + v\,\hat{v}\right) + \frac{\sqrt{u^2 + v^2}}{2}\,d\left(u\,\hat{u} + v\,\hat{v}\right) ,$$

the line element becomes

$$d\vec{S} = \sqrt{u^2 + v^2}\,\left(du\,\hat{u} + dv\,\hat{v}\right) ,$$

where clearly the interval dS^2 resolves to

$$dS^2 = \left(u^2 + v^2\right)\left(du^2 + dv^2\right) .$$

Of course, dS^2 can be calculated *much* more easily while avoiding any mention of $\hat{u}$, $\hat{v}$ by starting with $dx^2 + dy^2$, but the path we chose is more instructive.

From $d\vec{S}$ we infer the basis vectors for this system obeying $\vec{a}_{(\mu)} \cdot \vec{a}^{(\nu)} = \delta^{\nu}_{\mu}$:

$$\vec{a}_{(u)} = \sqrt{u^2 + v^2}\,\hat{u} \qquad \vec{a}_{(v)} = \sqrt{u^2 + v^2}\,\hat{v}$$

$$\vec{a}^{(u)} = \frac{\hat{u}}{\sqrt{u^2 + v^2}} \qquad \vec{a}^{(v)} = \frac{\hat{v}}{\sqrt{u^2 + v^2}}$$

Components of the metric are inferred from dS^2, or equivalently, $\vec{a}_{(\mu)} \cdot \vec{a}_{(\nu)} = g_{\mu\nu}$, which turn out to be

$$g_{uu} = u^2 + v^2 \qquad g_{vv} = u^2 + v^2 ,$$

or with raised indices,

$$g^{uu} = \frac{1}{u^2 + v^2} \qquad g^{vv} = \frac{1}{u^2 + v^2} .$$

Of course, we could have used the tensor transformation law

$$g_{\mu'\nu'} = \frac{\partial q^{\alpha}}{\partial q^{\mu'}} \frac{\partial q^{\beta}}{\partial q^{\nu'}} \eta_{\alpha\beta}$$

to derive the components of $g_{\mu\nu}$ from the Cartesian metric. In the above, the primed indices refer to parameters u, v, whereas the unprimed indices are for x and y.

If the components of a vector $\vec{W} = W^{\mu}\vec{a}_{(\mu)} = w^{\mu}\hat{a}_{(\mu)}$ are handed to you in the form W^u, W^v, use the metric to lower the index via $W_{\mu} = g_{\mu\nu}W^{\nu}$:

$$W_u = \left(u^2 + v^2\right) W^u \qquad\qquad W_v = \left(u^2 + v^2\right) W^v$$

To isolate w^{μ}, calculate $\vec{W} \cdot \hat{a}_{(\mu)}$ to get

$$w^u = \sqrt{u^2 + v^2}\, W^u \qquad\qquad w^v = \sqrt{u^2 + v^2}\, W^v \, .$$

Problem 5

Calculate all nonzero connection coefficients in two-dimensional flat space mapped by parabolic coordinates.

Solution 5

$$\Gamma^u_{uu} = \Gamma^v_{uv} = \frac{u}{u^2 + v^2} \qquad\qquad \Gamma^u_{uv} = \Gamma^v_{vv} = \frac{v}{u^2 + v^2}$$

$$\Gamma^u_{vv} = \frac{-u}{u^2 + v^2} \qquad\qquad \Gamma^v_{uu} = \frac{-v}{u^2 + v^2}$$

Problem 6

Using index notation, write the gradient of a scalar function f in flat space mapped by parabolic coordinates. Also find the divergence and the curl of a vector $\vec{W} = W^{\mu}\vec{a}_{(\mu)} = w^{\mu}\hat{a}_{(\mu)}$ in the same system. Finish off by finding the Laplacian operator.

Solution 6

$$\vec{\nabla} f = \vec{a}^{(\mu)}\partial_{\mu} f = \frac{1}{\sqrt{u^2 + v^2}} \left(\frac{\partial f}{\partial u}\, \hat{u} + \frac{\partial f}{\partial v}\, \hat{v} \right)$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{W} &= D_{\mu}W^{\mu} = g^{\alpha\mu} D_{\mu} W_{\alpha} = \frac{1}{u^2 + v^2} \left(\partial_u W_u + \partial_v W_v \right) \\ &= \frac{1}{u^2 + v^2} \left(\partial_u \left(\sqrt{u^2 + v^2}\, w^u \right) + \partial_v \left(\sqrt{u^2 + v^2}\, w^v \right) \right) \end{aligned}$$

$$\begin{aligned} \left| \vec{\nabla} \times \vec{W} \right| &= \left| \vec{a}^{(\nu)} \times \vec{a}_{(\mu)} \right| D_{\nu} W^{\mu} = D_u W^v - D_v W^u \\ &= \partial_v W^u - \partial_u W^v - \frac{u + v}{u^2 + v^2} \left(W^u + W^v \right) \end{aligned}$$

$$\nabla^2 f = \frac{1}{u^2 + v^2} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right)$$

Problem 7

Consider flat space mapped by three-dimensional parabolic coordinates:

$$x = uv\cos\phi \qquad\qquad y = uv\sin\phi \qquad\qquad z = \frac{1}{2}\left(u^2 - v^2\right)$$

Show that surfaces of constant u and v are confocal paraboloids that open upward and downward, respectively.

Solution 7

$$x^2 + y^2 = u^2 v^2 \qquad 2z = u^2 - v^2$$

$$z_u = \frac{1}{2}\left(\frac{x^2+y^2}{u^2} - v^2\right) \qquad z_v = \frac{1}{2}\left(u^2 - \frac{x^2+y^2}{v^2}\right)$$

Problem 8

Determine all components of the metric in flat space mapped by three-dimensional parabolic coordinates.

Solution 8

$$g_{uu} = \frac{\partial q^x}{\partial q^u}\frac{\partial q^x}{\partial q^u}\eta_{xx} + \frac{\partial q^y}{\partial q^u}\frac{\partial q^y}{\partial q^u}\eta_{yy} + \frac{\partial q^z}{\partial q^u}\frac{\partial q^z}{\partial q^u}\eta_{zz} = u^2 + v^2$$

$$g_{vv} = u^2 + v^2 \qquad g_{\phi\phi} = u^2 v^2$$

Problem 9

Find the Laplacian operator in flat space mapped by three-dimensional parabolic coordinates.

Solution 9

$$\nabla^2 f = \frac{1}{u^2+v^2}\left(\frac{1}{u}\,\partial_u\left(u\,\partial_u f\right) + \frac{1}{v}\,\partial_v\left(v\,\partial_v f\right)\right) + \frac{1}{u^2 v^2}\,\partial_{\phi\phi} f$$

4.5 Hyperspherical Coordinates

It's totally reasonable to conceive of a four-dimensional flat space permitting a metric that extends from the three-dimensional case, namely

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

In so-called hyperspherical coordinates, the position vector needs a radius and three angles as follows:

$$x = r\cos\psi \qquad y = r\sin\psi\cos\theta$$

$$z = r\sin\psi\sin\theta\cos\phi \qquad w = r\sin\psi\sin\theta\sin\phi$$

To calculate the interval

$$dS^2 = dx^2 + dy^2 + dz^2 + dw^2 ,$$

one may engage in the tedious task of computing the sum of the squares of each differential, however the tensor transformation law

$$g_{\mu'\nu'} = \frac{\partial q^\alpha}{\partial q^{\mu'}}\frac{\partial q^\beta}{\partial q^{\nu'}}\eta_{\alpha\beta}$$

is much more elegant (yet almost as tedious). Carrying this out, we find

$$g_{rr} = 1 \qquad g_{\psi\psi} = r^2$$

$$g_{\theta\theta} = r^2\sin^2\psi \qquad g_{\phi\phi} = r^2\sin^2\psi\sin^2\theta .$$

Then, the interval is

$$dS^2 = g_{\mu\nu}dq^\mu dq^\nu = dr^2 + r^2\,d\psi^2 + r^2\sin^2\psi\,d\theta^2 + r^2\sin^2\psi\sin^2\theta\,d\phi^2 .$$

Of course, one may lock the radius r such that $dr = 0$, reducing the interval to

$$dS^2 = r^2\left(d\psi^2 + \sin^2\psi\,d\theta^2 + \sin^2\psi\sin^2\theta\,d\phi^2\right) ,$$

which is a three-dimensional manifold. It should be noted that such a three-sphere should not be assumed flat space.

5 Affine Parameter

The coordinates $q^\mu(\lambda)$ on a manifold may be parameterized in terms of (nearly) any variable λ without 'physical' consequence. Some choices of λ however are more natural than others, with the most natural parameterization being arc length on the manifold, already named dS. Any term that is proportional to the arc length

$$dS \propto d\lambda$$

is called an *affine parameter*.

5.1 Proper Time

Borrowing notation from special relativity, we shall relate dS to $c\,d\tau$, where c is the speed of light on the manifold, and τ is called the *proper time*. One notational annoyance we must adopt is a minus sign connecting the proper time to the arc length as

$$dS^2 = -c^2\,d\tau^2\ ,$$

in accordance with conventions from special relativity. (Some texts have all signs in the metric flipped so as to avoid introducing a minus sign in this moment.)

On a manifold, the derivative of the position vector q^μ with respect to the proper time τ is called the *proper velocity* vector, denoted U^μ. A second derivative gives the *proper acceleration*:

$$U^\mu = \frac{dq^\mu}{d\tau} \qquad\qquad \frac{d}{d\tau}U^\mu = \frac{d^2q^\mu}{d\tau^2}$$

The differential interval dS^2 in terms of proper time and property velocity is

$$dS^2 = dq_\mu dq^\mu = g_{\mu\nu}dq^\mu dq^\nu = g_{\mu\nu}U^\mu U^\nu\,d\tau^2\ ,$$

telling us the norm of the proper velocity vector is a constant called an *invariant*:

$$U^\mu U_\mu = g_{\mu\nu}U^\mu U^\nu = -c^2$$

We may also solve for $d\tau$ in terms of generalized coordinates:

$$d\tau = \frac{1}{c}\sqrt{-g_{\mu\nu}dq^\mu dq^\nu}$$

5.2 Minkowski Space

Let us include time t as the zeroth component of the Cartesian position vector $q^1 = x$, $q^2 = y$, $q^3 = z$, such that $q^0 = ct$, where c is a normalization constant interpreted as the speed of light on the manifold. The flat metric tensor $\eta_{\mu\nu}$ is known to have zero mixed terms with any $\eta_{\lambda\lambda} = 1$ if $\lambda \geq 1$. By tradition from special relativity, the time component of the metric η_{00} includes a minus sign, namely

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ .$$

The Minkowski interval dS^2 resolves to

$$dS^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2 = -c^2\,dt^2 + d\vec{x}^2\ .$$

Applying $dS^2 = -c^2\,d\tau^2$ to the Minkowski interval, we may solve for $dt/d\tau$ in terms of $d\vec{x}^2/dt^2 = v^2$,

$$-c^2\,d\tau^2 = -c^2\,dt^2 + d\vec{x}^2 \qquad\rightarrow\qquad c^2\frac{d\tau^2}{dt^2} = c^2 - v^2$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma \,,$$

resulting in the famed 'gamma factor' that captures time dilation. In light of the gamma factor, the position vector q^μ and proper velocity $dq^\mu/d\tau$ are

$$q^\mu = (ct, \vec{x}) \qquad\qquad U^\mu = (\gamma c, \gamma \vec{v}) \ ,$$

where the relation $U^\mu U_\mu = -c^2$ is readily satisfied.

Going a step further, let us suppose that the position vector q^μ traces the path of a particle of mass m. Multiplying m into the proper velocity vector gives us the *proper momentum*

$$P^\mu = mU^\mu = m\left(\gamma c, \gamma \vec{v}\right) = \left(\gamma m c, \vec{p}\right) \ ,$$

whose norm is $P^\mu P_\mu = -m^2 c^2$. Using this, we find

$$\begin{aligned} -m^2 c^2 &= -m^2 \gamma^2 c^2 + m^2 v^2 \\ \left(\gamma m c^2\right)^2 &= p^2 c^2 + m^2 c^4 \\ \gamma m c^2 = E &= \sqrt{p^2 c^2 + m^2 c^4} \,. \end{aligned}$$

For reasons that are especially obvious from special relativity, identify the left side of the result as the total energy E of the particle. While at rest ($v = 0$, $p = 0$, $\gamma = 1$), the energy famously reduces to

$$E_0 = mc^2 \,.$$

On the other hand, we may still talk about the energy and momentum of a massless particle moving at speed c, also known as a *photon*, whose energy is the product

$$E_\gamma = pc \,.$$

The proper momentum may be written

$$P^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

to cover each case.

Multiply the mass m into the proper acceleration to write the *proper force*

$$F^\mu = m\frac{d}{d\tau}U^\mu = \frac{d}{d\tau}P^\mu \,,$$

where a contraction with U_μ resolves to zero, as

$$F^\mu U_\mu = mU_\mu \frac{d}{d\tau}U^\mu = \frac{m}{2}\frac{d}{d\tau}\left(U^\mu U_\mu\right) = \frac{m}{2}\frac{d}{d\tau}\left(-c^2\right) = 0 \,.$$

Unpacking the definition of F^μ, the proper force looks like:

$$F^\mu = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right) = \gamma\left(\frac{\vec{F}\cdot\vec{v}}{c}, \vec{F}\right)$$

6 Geodesic Equation

The notion of *geodesics* enters the game to play among among tensors, the metric, and the covariant derivative in order to properly understand geometry in curved space. Specifically, the geodesic equation on a manifold is the generalization of the zero-acceleration case in flat space.

6.1 Parallel Transport

The Christoffel symbols Γ arose from the need to ensure that the covariant derivative always delivers a tensor. Now we reinforce their existence by employing a less formal but more physically concrete analysis called *parallel transport*, which entails keeping the components of a constant vector V^μ fixed while undergoing a change in base point q^μ.

To demonstrate parallel transport and its implications, consider a flat two-dimensional manifold mapped by polar coordinates via

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}\ .$$

A vector V obeys the contraction

$$|V|^2 = g_{\mu\nu}V^\mu V^\nu = g_{rr}\left(V^r\right)^2 + g_{\theta\theta}\left(V^\theta\right)^2 = g_{\mu\nu}V^\mu V^\nu = \left(V^r\right)^2 + r^2\left(V^\theta\right)^2\ ,$$

where $g_{rr} = 1$ and $g_{\theta\theta} = r^2$ have been used. It follows that

$$V^r = |V|\cos\phi \qquad\qquad V^\theta = |V|\,\frac{1}{r}\sin\phi\ ,$$

where ϕ is the angle made between $\vec{V}$ and its base point position vector $\vec{r}$.

The base point of V may be parallel-transported in two independent ways, namely along the pure r-direction, or purely in θ, resulting in a vector $\tilde{V}$ with the same magnitude. For small displacements in r and θ respectively, we write

$$\tilde{V}^r = \left|\tilde{V}\right|\cos\phi \qquad\qquad \tilde{V}^\theta = \left|\tilde{V}\right|\frac{1}{r+\Delta r}\sin\phi$$

$$\tilde{V}^r = \left|\tilde{V}\right|\cos\left(\phi - \Delta\theta\right) \qquad\qquad \tilde{V}^\theta = \left|\tilde{V}\right|\frac{1}{r}\sin\left(\phi - \Delta\theta\right)$$

From the above equations containing a $\sin\phi$ term, we may eliminate said term and both magnitudes $\left|\tilde{V}\right|$ and $|V|$ to solve for $\tilde{V}^\theta$ in terms of V^θ:

$$\tilde{V}^\theta = V^\theta \cancel{\left|\tilde{V}\right|}\frac{1}{\cancel{|V|}}\frac{r}{r+\Delta r} \approx V^\theta - V^\theta\frac{\Delta r}{r}$$

Repeating a similar exercise to solve for $\tilde{V}^r$, we have

$$\tilde{V}^r = \left|\tilde{V}\right|\left(\cos\phi\cos\Delta\theta + \sin\phi\sin\Delta\theta\right) \approx V^r + V^\theta r\,\Delta\theta$$

Observe that the components of the parallel-transported vector adhere to a pattern, namely

$$\tilde{V}^\mu \approx V^\mu - (?)^\mu_{\nu\beta}\,V^\nu\,dq^\beta\ ,$$

where the $(?)^\mu_{\nu\beta}$ object acts suspiciously like a Christoffel symbol. By comparison to $\Gamma^\mu_{\nu\beta}$ in $2D$ polar coordinates, it happens that the factors $1/r$ and $-r$ are precisely equal to $\Gamma^\theta_{r\theta}$ and $\Gamma^r_{\theta\theta}$. In summary, we uncover a Calculus 101-eqsue derivative formula

$$\tilde{V}^\mu \approx V^\mu + \Gamma^\mu_{\nu\beta}V^\nu\,dq^\beta\ ,$$

reaffirming that the Christoffel symbols track the 'slope' in the coordinate system on which V is placed.

6.2 Derivation by Parallel Transport

Recalling the star conclusion from parallel transport analysis, note that the proper velocity vector U must obey

$$\tilde{U}^\mu \approx U^\mu + \Gamma^\mu_{\nu\beta} U^\nu \, dq^\beta$$

under a small change of base point. If we rearrange and divide through by the proper time to write

$$0 \approx \frac{U^\mu - \tilde{U}^\mu}{d\tau} + \Gamma^\mu_{\nu\beta} U^\nu U^\beta \, ,$$

this limits to the *geodesic equation*:

$$0 = \frac{d}{d\tau} U^\mu + \Gamma^\mu_{\nu\beta} U^\nu U^\beta \qquad\qquad 0 = \frac{d^2 q^\mu}{d\tau^2} + \Gamma^\mu_{\nu\beta} \frac{dq^\nu}{d\tau} \frac{dq^\beta}{d\tau}$$

6.3 Derivation by Covariant Derivative

A stronger argument for the geodesic equation arises by setting the covariant derivative of a parallel-transported vector $V^\mu(\lambda)$ to zero. Doing so, we find

$$0 = D_\lambda V^\mu = \frac{D}{d\lambda} V^\mu = \frac{dq^\nu}{d\lambda} D_\nu V^\mu = V^\nu D_\nu V^\mu = V^\nu \partial_\nu V^\mu + \Gamma^\mu_{\nu\beta} V^\nu V^\beta \, ,$$

which resolves to the previous result for the special case $V^\mu(\lambda) = U^\mu(\tau)$. Note for scalar fields f, $D_\lambda V^\mu$ is the dot product between the tangent vector and the gradient (directional derivative).

To discover a restriction on the parameter λ, continue assuming the form $V^\mu = dq^\mu / d\lambda$ and apply the chain rule:

$$\begin{aligned}
\frac{d^2 q^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\beta} \frac{dq^\mu}{d\lambda} \frac{dq^\beta}{d\lambda} &= \frac{d\tau}{d\lambda} \frac{d}{d\tau} \left(\frac{d\tau}{d\lambda} \frac{dq^\mu}{d\tau} \right) + \left(\frac{d\tau}{d\lambda} \right)^2 \Gamma^\mu_{\nu\beta} \frac{dq^\mu}{d\tau} \frac{dq^\beta}{d\tau} \\
&= \frac{d\tau}{d\lambda} \frac{dq^\mu}{d\tau} \frac{d}{d\tau} \left(\frac{d\tau}{d\lambda} \right) + \left(\frac{d\tau}{d\lambda} \right)^2 \cancel{\left(\frac{d^2 q^\mu}{d\tau^2} + \Gamma^\mu_{\nu\beta} \frac{dq^\mu}{d\tau} \frac{dq^\beta}{d\tau} \right)} \\
&= -\frac{dq^\mu}{d\lambda} \left(\frac{d\lambda}{d\tau} \right)^{-2} \left(\frac{d^2 \lambda}{d\tau^2} \right) + 0 \\
&= -V^\mu \frac{\left(d^2 \lambda / d\tau^2 \right)}{\left(d\lambda / d\tau \right)^2}
\end{aligned}$$

To get zero on the right side, the second derivative of λ must vanish, thus any $\lambda \propto \tau$ is an affine parameter that satisfies the geodesic equation.

6.4 Derivation by Variations

Between two points τ_i and τ_f, the integral of $c\,d\tau$, i.e. the *action* is extremized:

$$S = \int_{\tau_i}^{\tau_f} c \, d\tau = \int_{\tau_i}^{\tau_f} \sqrt{-g_{\mu\nu} \frac{dq^\mu}{d\tau} \frac{dq^\nu}{d\tau}} \, d\tau$$

Next introduce small variations in generalized coordinates and in the metric as

$$q^\gamma \rightarrow q^\gamma + \delta q^\gamma \qquad\qquad g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta q^\gamma \frac{\partial}{\partial q^\gamma} g_{\mu\nu} \, ,$$

where each δ-term is considered small, and the product of two or more such terms is negligible. On the left side of the equation we have $S \rightarrow S + \delta S$. Note all variations are zero at the endpoints τ_1 and τ_2.

The square root term inside the integral is approximately

$$\sqrt{-g_{\mu\nu}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau}} \approx \sqrt{c^2 - \delta q^\alpha \frac{\partial g_{\mu\nu}}{\partial q^\alpha}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} - g_{\mu\nu}\frac{d\delta q^\mu}{d\tau}\frac{dq^\nu}{d\tau} - g_{\mu\nu}\frac{dq^\mu}{d\tau}\frac{d\delta q^\nu}{d\tau}}\ ,$$

and since c^2 is greater than all terms under the square root, we use the approximation

$$\sqrt{c^2 - A} \approx c - \frac{A}{2c}$$

and cancel a factor of S from each side of the equation to get

$$\delta S = \frac{1}{2c}\int_{\tau_i}^{\tau_f} d\tau \left(\delta q^\alpha \frac{\partial g_{\mu\nu}}{\partial q^\alpha}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} + g_{\mu\nu}\frac{d\delta q^\mu}{d\tau}\frac{dq^\nu}{d\tau} + g_{\mu\nu}\frac{dq^\mu}{d\tau}\frac{d\delta q^\nu}{d\tau}\right)\ .$$

For notational convenience, respectively label the first, second, and third terms as δS_1, δS_2, and δS_3.

Integrate δS_3 by parts, i.e., $\int u\, dv = uv\big| - \int v\, du$ by letting:

$$u = g_{\mu\nu}\frac{dq^\mu}{d\tau} \qquad du = \frac{dg_{\mu\nu}}{d\tau}\frac{dq^\mu}{d\tau} + g_{\mu\nu}\frac{d^2q^\mu}{d\tau^2}$$

$$dv = \frac{d\delta q^\nu}{d\tau}d\tau \qquad v = \delta q^\nu$$

Since the boundary term is zero by construction, δS_3 evaluates to

$$\delta S_3 = -\frac{1}{2c}\int_{\tau_i}^{\tau_f} d\tau\, (\delta q^\alpha)\left(\frac{dg_{\alpha\mu}}{dq^\nu}\frac{dq^\nu}{d\tau}\frac{dq^\mu}{d\tau} + g_{\mu\alpha}\frac{d^2q^\mu}{d\tau^2}\right)\ .$$

Repeat the calculation for δS_2 to get

$$\delta S_2 = -\frac{1}{2c}\int_{\tau_i}^{\tau_f} d\tau\, (\delta q^\alpha)\left(\frac{dg_{\nu\alpha}}{dq^\mu}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau} + g_{\mu\alpha}\frac{d^2q^\mu}{d\tau^2}\right)\ ,$$

to show that the total δS is

$$\delta S = \frac{1}{c}\int_{\tau_i}^{\tau_f} d\tau\, (\delta q^\alpha)\left[g_{\mu\alpha}\frac{d^2q^\mu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^\alpha} - \frac{\partial g_{\nu\alpha}}{\partial q^\mu} - \frac{\partial g_{\alpha\mu}}{\partial q^\nu}\right)\left(\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau}\right)\right]\ .$$

Finally we invoke the argument that δS goes to zero for the 'true' path of motion on the manifold, thus the quantity in square brackets equals zero. Contracting the equation with $g^{\rho\alpha}$ gives

$$0 = \frac{d^2q^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\alpha}\left(\frac{\partial g_{\mu\nu}}{\partial q^\alpha} - \frac{\partial g_{\nu\alpha}}{\partial q^\mu} - \frac{\partial g_{\alpha\mu}}{\partial q^\nu}\right)\left(\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau}\right)\ .$$

Of course, the terms involving the metric combine to equal the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ (surely you spotted it). The result is the geodesic equation:

$$0 = \frac{d^2q^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau}$$

6.5 Null Geodesics

Recall that the differential interval on a manifold can be written

$$\frac{dS^2}{d\tau^2} = g_{\mu\nu}U^\mu U^\nu = U^\mu U_\mu = -c^2\ .$$

It turns out that nontrivial results can follow from setting $dS^2 = d\tau^2 = 0$, called a *null geodesic*:

$$0 = g_{\mu\nu}U^\mu U^\nu = U^\mu U_\mu$$

6.6 Problems

Parallel Transport

Problem 1

Consider a vector field V^μ that has unit length and points in the positive x direction everywhere. Find an expression for V in two-dimensional plane polar coordinates, and then show that its covariant derivative is zero.

Solution 1

$$V^r = \frac{\partial q^r}{\partial x} V^x + \frac{\partial q^r}{\partial y} \cancel{V^y} = \cos\theta$$

$$V^\theta = \frac{\partial q^\theta}{\partial x} V^x + \frac{\partial q^\theta}{\partial y} \cancel{V^y} = \frac{-1}{\sqrt{1 - x^2/r^2}} \frac{d}{dx} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) = \frac{-\sin\theta}{r}$$

$$D_r V^r = D_r V^\theta = D_\theta V^r = D_\theta V^\theta = 0$$

Problem 2

Consider a spherical shell that embeds a circle C at constant latitude θ_0. At $\phi = 0$ the vector V^μ has components $V^\theta = 0$, $V^\phi = 1$. Compute the components of V^μ as a function of ϕ as it is parallel-transported around C.

Solution 2

$$0 = \frac{dV^\mu}{d\phi} + \frac{dq^\alpha}{d\phi} \Gamma^\mu_{\alpha\nu} V^\nu \qquad \alpha = \{\theta, \phi\}$$

$$0 = \frac{dV^\theta}{d\phi} - V^\phi \sin\theta \cos\theta \qquad 0 = \frac{dV^\phi}{d\phi} + V^\theta \cot\theta$$

$$0 = \frac{d^2 V^\theta}{d\phi^2} + \cos^2\theta_0 \, V^\theta \qquad 0 = \frac{d^2 V^\phi}{d\phi^2} + \cos^2\theta_0 \, V^\phi$$

$$V^\theta = \sin\theta_0 \cdot \sin\left[\phi \cdot \cos\theta_0\right] \qquad V^\phi = \cos\left[\phi \cdot \cos\theta_0\right]$$

Geodesics on a Plane

Geodesics in the Cartesian plane are expressed by

$$x(\lambda) = a\lambda + x_0 \qquad y(\lambda) = b\lambda + y_0 \,,$$

where λ is a dimensionless parameter. In $2D$ plane polar coordinates, variables (x, y) relate to (r, θ) by

$$x = r\cos\theta \qquad y = r\sin\theta \,.$$

Thus, straight lines in polar coordinates look like

$$r(\lambda) = \sqrt{(a\lambda + x_0)^2 + (b\lambda + y_0)^2} \qquad \theta(\lambda) = \tan^{-1}\left(\frac{b\lambda + y_0}{a\lambda + x_0}\right) \,.$$

Using what we know about $\Gamma^\mu_{\nu\rho}$ in two-dimensional polar coordinates, the geodesic equation yields two bits of information:

$$\frac{d^2 r}{d\lambda^2} - r\left(\frac{d\theta}{d\lambda}\right)^2 = 0 \qquad \frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} = 0$$

Problem 3

Show that $r(\lambda)$ and $\theta(\lambda)$ given above are solutions to the geodesic equation for $2D$ plane polar coordinates.

Solution 3

$$r' = \frac{a\left(a\lambda + x_0\right) + b\left(b\lambda + y_0\right)}{r} \qquad \theta' = \frac{bx_0 - ay_0}{r^2}$$

$$r'' = \frac{a^2 + b^2}{r} - \frac{\left(r'\right)^2}{r} \qquad \theta'' = \frac{-2r'\theta'}{r}$$

$$r'' - r\left(\theta'\right)^2 = \frac{r^2\left(a^2 + b^2\right) - \left(a\left(a\lambda + x_0\right) + b\left(b\lambda + y_0\right)\right)^2}{r^3} - \frac{a^2y_0^2 + b^2x_0^2 - 2abx_0y_0}{r^3} = 0$$

Geodesics on a Two-Sphere

Problem 4

On the surface of a spherical shell, prove that any nontrivial path with constant θ is only geodesic on the equator.

Solution 4

$$\frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0 \qquad \text{Only true if } \theta = \frac{\pi}{2}\,.$$

Problem 5

On the surface of a spherical shell, prove that any path with constant ϕ (longitude) is a geodesic.

Solution 5

A vector $\vec{v}$ that traces a circular arc on the shell at constant ϕ is

$$\vec{v} = r\sin\theta\,\hat{r} + r\cos\theta\,\hat{\theta}\,,$$

where in contravariant form, we have

$$V^r = v^r = r\sin\theta \qquad V^\theta = \frac{1}{r}v^\theta = \cos\theta\,.$$

Applying the geodesic equation to each component, we discover

$$\frac{dV^r}{d\theta} + \Gamma^r_{\theta\theta}V^\theta = 0 \qquad \frac{dV^\theta}{d\theta} + \Gamma^\theta_{\theta r}V^r = 0\,.$$

Geodesics in Minkowski Space

The geodesic equation in Minkowski space reduces to

$$\frac{\partial^2 q^\mu}{\partial\tau^2} = 0\,,$$

where

$$q^\mu = \left(ct, \vec{x}\right)\,.$$

As per usual in classical mechanics, integrate twice with respect to (proper) time to gain and equation for $q^\mu\left(\tau\right)$:

$$q^\mu\left(\tau\right) = q_0^\mu + U_0^\mu\tau$$

Of course, the normalization condition

$$-c^2 = \eta_{\mu\nu}U_0^\mu U_0^\nu$$

restricts the U_0^0 component as

$$U_0^0 = \pm\sqrt{c^2 + \vec{U}_0\cdot\vec{U}_0}\,.$$

Problem 6

Show that a particle would require $v = c$ to follow null geodesics in Minkowski space.

Solution 6

$$0 = \eta_{\mu\nu} U^\mu U^\nu = -\gamma^2 c^2 + \gamma^2 c^2 = 0$$

7 Curvature

7.1 Riemann Curvature Tensor

So far, we have no tool to discern if a manifold is truly curved versus truly flat. For instance you might discover that the Earth is at least cylindrical by walking consistently westward, but can we do something that does not involve walking around the whole surface? That answer, of course, is yes.

The new idea is to parallel-transport a vector V on a given manifold in a closed circuit without walking around the whole manifold. If the orientation of the vector is unchanged after a trip around the circuit, the space is measurably flat. On the other hand, distortion in V can only be caused by curved space.

7.2 Parallel Transport Analysis

Our starting place is the geodesic equation in the form

$$0 = dV^\rho + \Gamma^\rho_{\mu\nu} dq^\mu V^\nu \; .$$

Abbreviating $V^\rho \left(q^\mu \left(j\right)\right)$ as $V^\rho \left(j\right)$, we write a general first-order expansion for the vector V evaluated at any point:

$$\begin{aligned} V^\rho \left(j'\right) &= V^\rho \left(j\right) + \int_j^{j'} \frac{dV^\rho}{dq^\mu} \, dq^\mu \\ &= V^\rho \left(j\right) + \int_j^{j'} \left(-\Gamma^\rho_{\mu\nu} V^\nu\right) \, dq^\mu \end{aligned}$$

Implementing the above result on the parallelogram (a, b), $(a + \delta a, b)$, $(a + \delta a, b + \delta b)$, $(a, b + \delta b)$, we have:

$$\begin{aligned} V^\rho \left(1\right) &= V^\rho \left(0\right) + \int_a^{a+\delta a} \left(-\Gamma^\rho_{\mu\nu} \left(b\right) V^\nu \left(b\right)\right) \, dq^\mu \\ V^\rho \left(2\right) &= V^\rho \left(1\right) + \int_b^{b+\delta b} \left(-\Gamma^\rho_{\mu\nu} \left(a+\delta a\right) V^\nu \left(a+\delta a\right)\right) \, dq^\mu \\ V^\rho \left(3\right) &= V^\rho \left(2\right) + \int_{a+\delta a}^{a} \left(-\Gamma^\rho_{\mu\nu} \left(b+\delta b\right) V^\nu \left(b+\delta b\right)\right) \, dq^\mu \\ V^\rho \left(0'\right) &= V^\rho \left(3\right) + \int_{b+\delta b}^{b} \left(-\Gamma^\rho_{\mu\nu} \left(a\right) V^\nu \left(a\right)\right) \, dq^\mu \end{aligned}$$

The total change in V is the sum of the four paths, i.e. $\delta V^\rho = V^\rho \left(0'\right) - V^\rho \left(0\right)$:

$$\begin{aligned} \delta V^\rho = &\int_a^{a+\delta a} \left(\Gamma^\rho_{\mu\nu} \left(b+\delta b\right) V^\nu \left(b+\delta b\right) - \Gamma^\rho_{\mu\nu} \left(b\right) V^\nu \left(b\right)\right) \, dq^\mu \\ &+ \int_b^{b+\delta b} \left(\Gamma^\rho_{\mu\nu} \left(a\right) V^\nu \left(a\right) - \Gamma^\rho_{\mu\nu} \left(a+\delta a\right) V^\nu \left(a+\delta a\right)\right) \, dq^\mu \; . \end{aligned}$$

We next evaluate δV^ρ to lowest order to get

$$\begin{aligned} \delta V^\rho &\approx \int_a^{a+\delta a} \delta b^\gamma \frac{\partial}{\partial q^\gamma} \left(\Gamma^\rho_{\mu\nu} V^\nu\right) \, dq^\mu - \int_b^{b+\delta b} \delta a^\gamma \frac{\partial}{\partial q^\gamma} \left(\Gamma^\rho_{\mu\nu} V^\nu\right) \, dq^\mu \\ &\approx \int_a^{a+\delta a} \delta b^\gamma \frac{\partial}{\partial q^\gamma} \left(\Gamma^\rho_{\mu\nu} V^\nu\right) \, dq^\mu - \int_b^{b+\delta b} \delta a^\mu \frac{\partial}{\partial q^\mu} \left(\Gamma^\rho_{\gamma\nu} V^\nu\right) \, dq^\gamma \\ &\approx \delta a^\mu \delta b^\gamma \frac{\partial}{\partial q^\gamma} \left(\Gamma^\rho_{\mu\nu} V^\nu\right) - \delta a^\mu \delta b^\gamma \frac{\partial}{\partial q^\mu} \left(\Gamma^\rho_{\gamma\nu} V^\nu\right) \\ &\approx \delta a^\mu \delta b^\gamma \left(\partial_\gamma \Gamma^\rho_{\mu\nu} V^\nu + \Gamma^\rho_{\mu\nu} \partial_\gamma V^\nu - \partial_\mu \Gamma^\rho_{\gamma\nu} V^\nu - \Gamma^\rho_{\gamma\nu} \partial_\mu V^\nu\right) \; . \end{aligned}$$

Replacing the derivatives of V, the total change in V becomes

$$\begin{aligned}\delta V^\rho &\approx \delta a^\mu \delta b^\gamma \left(\partial_\gamma \Gamma^\rho_{\mu\sigma} V^\sigma + \Gamma^\rho_{\mu\nu}\left(-\Gamma^\nu_{\gamma\sigma} V^\sigma\right) - \partial_\mu \Gamma^\rho_{\gamma\sigma} V^\sigma - \Gamma^\rho_{\gamma\nu}\left(-\Gamma^\nu_{\mu\sigma} V^\sigma\right)\right) \\ &\approx \delta a^\mu \delta b^\gamma V^\sigma \left(\partial_\gamma \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\nu}\Gamma^\nu_{\gamma\sigma} - \partial_\mu \Gamma^\rho_{\gamma\sigma} + \Gamma^\rho_{\gamma\nu}\Gamma^\nu_{\mu\sigma}\right) \ .\end{aligned}$$

In flat space, the parenthesized quantity evaluates to zero. Deduce that any information on curved space is stored in the *Riemann curvature tensor*:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

7.3 Geodesic Deviation Analysis

Consider a family of geodesics characterized by $q^\rho(\tau, s)$, where τ is an affine parameter and s is a perpendicular arc length parameter. Define the (small) *deviation vector* S^ρ such that

$$q^\rho \to q^\rho + S^\rho \ ,$$

where S^ρ is perpendicular to the (tangent) four-velocity $U^\rho = dq^\rho/d\tau$ at any point q^ρ. The geodesic equation under deviations $q^\rho \to q^\rho + S^\rho$ appears as

$$\frac{d^2\left(q^\rho + S^\rho\right)}{d\tau^2} + \Gamma^\rho_{\mu\nu}\left(q^\rho + S^\rho\right)\frac{d\left(q^\mu + S^\mu\right)}{d\tau}\frac{d\left(q^\nu + S^\nu\right)}{d\tau} = 0 \ .$$

Subtracting off the un-deviated geodesic equation ($S^\rho = 0$), the above boils down to

$$\frac{d^2 S^\rho}{d\tau^2} + 2\Gamma^\rho_{\mu\nu} U^\mu \frac{dS^\nu}{d\tau} + U^\mu U^\nu S^\sigma \partial_\sigma \Gamma^\rho_{\mu\nu} = 0 \ ,$$

where the chain rule and geodesic equation interpret the first term:

$$\frac{d^2 S^\rho}{d\tau^2} = \frac{d}{d\tau}\left(U^\mu \partial_\mu S^\rho\right) = -\Gamma^\gamma_{\mu\nu} U^\mu U^\nu \partial_\gamma S^\rho + U^\mu U^\nu \partial_\nu \partial_\mu S^\rho \ .$$

To proceed, define the *relative velocity* and *relative acceleration* as

$$V^\rho = \frac{d}{d\tau} S^\rho = U^\sigma D_\sigma S^\rho \qquad\qquad A^\rho = \frac{d}{d\tau} V^\rho \ ,$$

and substitute V into A to write

$$A^\rho = \left(U^\lambda D_\lambda U^\sigma\right) D_\sigma S^\rho + U^\sigma U^\mu D_\mu D_\sigma S^\rho \ ,$$

and since the parenthesized quantity is identically the geodesic equation, the first term vanishes. Computing out the second term, we have

$$\begin{aligned}A^\rho &= U^\sigma U^\mu \left(\partial_\mu \left(D_\sigma S^\rho\right) + \Gamma^\rho_{\mu\gamma} D_\sigma S^\gamma - \Gamma^\gamma_{\mu\sigma} D_\gamma S^\rho\right) \\ &= U^\sigma U^\mu \left(\partial_\mu \partial_\sigma S^\rho + \partial_\mu \left(\Gamma^\rho_{\sigma\gamma} S^\gamma\right) + \Gamma^\rho_{\mu\gamma} D_\sigma S^\gamma - \Gamma^\gamma_{\mu\sigma} D_\gamma S^\rho\right) \\ &= U^\mu U^\sigma S^\nu \left(-\partial_\nu \Gamma^\rho_{\mu\sigma} + \partial_\mu \Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\mu\gamma}\Gamma^\gamma_{\sigma\nu} - \Gamma^\rho_{\gamma\nu}\Gamma^\gamma_{\mu\sigma}\right) \\ &= U^\mu U^\sigma S^\nu R^\rho_{\sigma\mu\nu} \ ,\end{aligned}$$

which is nicely represented in terms of the Riemann curvature tensor, therefore this analysis is alternative derivation of $R^\rho_{\sigma\mu\nu}$. Keep in mind A^ρ is not an equation of motion, rather it tracks adjacent geodesics as a bundle.

7.4 Second Derivative Analysis

A delicate 'probe' for curvature on a manifold checks for second-order changes in a vector whose derivatives are computed in alternating order. That is, by calculating $D_\nu D_\mu V_\sigma - D_\mu D_\nu V_\sigma$, any nonzero result indicates curved space. Carrying this out for the first term, we find

$$\begin{aligned} D_\nu D_\mu V_\sigma &= \partial_\nu \left(D_\mu V_\sigma\right) - \Gamma^\rho_{\nu\mu} \left(D_\rho V_\sigma\right) - \Gamma^\rho_{\nu\sigma} \left(D_\mu V_\rho\right) \\ &= \partial_\nu \left(\partial_\mu V_\sigma - \Gamma^\rho_{\mu\sigma} V_\rho\right) - \Gamma^\rho_{\nu\mu} \left(\partial_\rho V_\sigma - \Gamma^\lambda_{\rho\sigma} V_\lambda\right) - \Gamma^\rho_{\nu\sigma} \left(\partial_\mu V_\rho - \Gamma^\lambda_{\mu\rho} V_\lambda\right) \ . \end{aligned}$$

The second term has the first two indices swapped, so we have

$$D_\mu D_\nu V_\sigma = \partial_\mu \left(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma} V_\rho\right) - \Gamma^\rho_{\mu\nu} \left(\partial_\rho V_\sigma - \Gamma^\lambda_{\rho\sigma} V_\lambda\right) - \Gamma^\rho_{\mu\sigma} \left(\partial_\nu V_\rho - \Gamma^\lambda_{\nu\rho} V_\lambda\right) \ ,$$

where taking the difference $\Delta V_{\nu\mu\sigma} = D_\nu D_\mu V_\sigma - D_\mu D_\nu V_\sigma$ gives

$$\begin{aligned} \Delta V_{\nu\mu\sigma} &= \partial_\mu \left(\Gamma^\rho_{\nu\sigma} V_\rho\right) - \partial_\nu \left(\Gamma^\rho_{\mu\sigma} V_\rho\right) + \Gamma^\rho_{\mu\sigma} \left(\partial_\nu V_\rho - \Gamma^\lambda_{\nu\rho} V_\lambda\right) - \Gamma^\rho_{\nu\sigma} \left(\partial_\mu V_\rho - \Gamma^\lambda_{\mu\rho} V_\lambda\right) \\ \Delta V_{\nu\mu\sigma} &= \partial_\mu \left(\Gamma^\rho_{\nu\sigma}\right) V_\rho - \partial_\nu \left(\Gamma^\rho_{\mu\sigma}\right) V_\rho + \Gamma^\rho_{\mu\sigma} \left(-\Gamma^\lambda_{\nu\rho} V_\lambda\right) - \Gamma^\rho_{\nu\sigma} \left(-\Gamma^\lambda_{\mu\rho} V_\lambda\right) \\ \Delta V_{\nu\mu\sigma} &= \left(\partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}\right) V_\rho \ , \end{aligned}$$

and the parenthesized term is none other than the Riemann curvature tensor. In summary, we have found

$$D_\nu D_\mu V_\sigma - D_\mu D_\nu V_\sigma = R^\rho_{\sigma\mu\nu} V_\rho \ .$$

A vector in flat space cannot have two separate answers for the second derivate when taken in swapped directions, but curved space evidently permits it.

7.5 Properties of the Riemann Curvature Tensor

As a four-index object, the Riemann curvature tensor in general has $4^4 = 256$ individual components. Due to certain symmetries in $R^\rho_{\sigma\mu\nu}$ though, the number of independent components is reduced significantly. Following are four properties you are encouraged to verify:

- $R_{\alpha\beta\gamma\rho}$ is antisymmetric in the last two indices:

$$R_{\alpha\beta\gamma\rho} = -R_{\alpha\beta\rho\gamma}$$

- $R_{\alpha\beta\gamma\rho}$ is antisymmetric in the first two indices:

$$R_{\alpha\beta\gamma\rho} = -R_{\beta\alpha\gamma\rho}$$

- $R_{\alpha\beta\gamma\rho}$ is symmetric when exchanging pairs of indices:

$$R_{\alpha\beta\gamma\rho} = R_{\gamma\rho\alpha\beta}$$

- The Riemann curvature tensor obeys a cyclic algebraic identity:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\gamma\beta\delta} = 0$$

- Derivatives of the Riemann curvature tensor obey the *Bianchi* identity:

$$D_\alpha R_{\beta\gamma\delta\epsilon} + D_\beta R_{\gamma\alpha\delta\epsilon} + D_\gamma R_{\alpha\beta\delta\epsilon} = 0$$

Using the above symmetries, it can be shown that the number of independent components in the Riemann curvature tensor goes from d^4 to

$$N = \frac{d^2 \left(d^2 - 1\right)}{12} \ ,$$

where d is the number of dimensions. For example, $d = 1$ can't possibly involve curvature, and we correspondingly find $R_{1111} = 0$. In two dimensions, there is only one independent component of the Riemann tensor, namely $R_{0101} \neq 0$.

Problem 1

Calculate the $R^\theta_{\phi\theta\phi}$-component of the Riemann curvature tensor for a three-dimensional flat space mapped by spherical coordinates.

Solution 1

$$
\begin{aligned}
R^\theta_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi} \\
&= \partial_\theta \Gamma^\theta_{\phi\phi} - \cancel{\partial_\phi \Gamma^\theta_{\theta\phi}} + \Gamma^\theta_{\theta r}\Gamma^r_{\phi\phi} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi} \\
&= \partial_\theta \left(-\sin\theta\cos\theta\right) - \frac{1}{r}\left(r\sin^2\theta\right) - \left(-\sin\theta\cos\theta\right)\cot\theta \\
&= -\cos^2\theta + \sin^2\theta - \sin^2\theta + \cos^2\theta = 0
\end{aligned}
$$

Problem 2

Show that a spherical shell of fixed radius is a curved space by calculating $R^\theta_{\phi\theta\phi}$. Also calculate $R^\phi_{\theta\phi\theta}$.

Solution 2

$$
\begin{aligned}
R^\theta_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi} \\
&= \partial_\theta \Gamma^\theta_{\phi\phi} - \cancel{\partial_\phi \Gamma^\theta_{\theta\phi}} + \cancel{\Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi}} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi} \\
&= \partial_\theta \left(-\sin\theta\cos\theta\right) - \left(-\sin\theta\cos\theta\right)\cot\theta \\
&= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta \\
R^\phi_{\theta\phi\theta} &= \cancel{\partial_\phi \Gamma^\phi_{\theta\theta}} - \partial_\theta \Gamma^\phi_{\phi\theta} + \cancel{\Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{\theta\theta}} - \Gamma^\phi_{\theta\lambda}\Gamma^\lambda_{\phi\theta} \\
&= -\partial_\theta \cot\theta - \cot^2\theta = 1
\end{aligned}
$$

7.6 Ricci Tensor and Ricci Scalar

Despite its symmetries, $R^\rho_{\mu\sigma\nu}$ is still an unruly type $(1, 3)$ tensor. A contraction over the top and bottom-middle indices yields a more accessible object called the *Ricci tensor*

$$
R_{\mu\nu} = R^\rho_{\mu\rho\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \,,
$$

which tracks the growth rate of volume elements on a manifold.

Curvature information can be projected into a scalar by contracting the Ricci tensor indices via

$$
g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu = R \,,
$$

which resolves to the *Ricci scalar*. Note that R is generally to equivalent to the 'mean' curvature or the Gaussian curvature on the manifold. (In two dimensions, it turns out that R is twice the Gaussian curvature.)

Problem 3

Calculate the necessary components of the Ricci tensor to find the Ricci scalar for a spherical shell of fixed radius.

Solution 3

$$\begin{aligned}
R_{\theta\theta} &= g^{\alpha\beta} R_{\alpha\theta\beta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = g^{\phi\phi}\left(g_{\phi\alpha} R^{\alpha}_{\theta\phi\theta}\right) = g^{\phi\phi}\left(g_{\phi\phi} R^{\phi}_{\theta\phi\theta}\right) = \left(g^{\phi\phi} g_{\phi\phi}\right) R^{\phi}_{\theta\phi\theta} \\
&= 1 \\
R_{\phi\phi} &= g^{\alpha\beta} R_{\alpha\phi\beta\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = g^{\theta\theta}\left(g_{\theta\alpha} R^{\alpha}_{\phi\theta\phi}\right) = g^{\theta\theta}\left(g_{\theta\theta} R^{\theta}_{\phi\theta\phi}\right) = \left(g^{\theta\theta} g_{\theta\theta}\right) R^{\theta}_{\phi\theta\phi} \\
&= \sin^2\theta \\
R_{\theta\phi} &= R_{\phi\theta} = 0 \\
R &= g^{\mu\nu} R_{\mu\nu} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{1}{r^2} + \frac{\cancel{\sin^2\theta}}{r^2 \cancel{\sin^2\theta}} = \frac{2}{r^2}
\end{aligned}$$

Problem 4

Calculate the necessary components of the Ricci tensor to find the Ricci scalar for a spherical shell of variable radius.

Solution 4

$$R_{rr} = R_{\theta\theta} = R_{\phi\phi} = R_{r\theta} = R_{\theta r} = R_{r\phi} = R_{\phi r} = R_{\theta\phi} = R_{\phi\theta} = 0$$

7.7 Einstein Tensor

The next available question is, which jumbling of R-objects has zero covariant derivative? Preemptively calling the result $G^{\mu\nu}$, begin by requiring

$$D_\nu G^{\nu\mu} = 0 \qquad\qquad D_\nu G^{\nu}_{\mu} = 0\,.$$

To proceed, raise the fourth index across the Bianchi identity by multiplying through by $g^{\delta\rho}$ to get

$$D_\alpha R^{\rho}_{\mu\nu\epsilon} + D_\mu R^{\rho}_{\nu\alpha\epsilon} + D_\nu R^{\rho}_{\alpha\mu\epsilon} = 0\,,$$

and also raise the last index by multiplying through by $g^{\alpha\epsilon}$, causing a contraction on α:

$$D_\alpha R^{\rho\alpha}_{\mu\nu} + D_\mu R^{\rho\alpha}_{\nu\alpha} + D_\nu R^{\rho\alpha}_{\alpha\mu} = 0$$

Let $\nu = \rho$ to invoke yet another contraction, and also exploit the symmetry properties of the Riemann curvature tensor to adjust index position and minus signs, eventually landing at

$$-D_\nu R^{\nu\alpha}_{\mu\alpha} + D_\mu R^{\nu\alpha}_{\nu\alpha} - D_\nu R^{\nu\alpha}_{\mu\alpha} = 0\,.$$

Observe the first and third terms are equal, and the second term is the divergence of the Ricci scalar. Seeking a total derivative, we write

$$D_\nu R^{\nu}_{\mu} - \frac{1}{2} D_\mu R = D_\nu R^{\nu}_{\mu} - \frac{1}{2} g^{\nu}_{\mu} D_\nu R = D_\nu \left(R^{\nu}_{\mu} - \frac{1}{2} g^{\nu}_{\mu} R\right) = 0\,,$$

where the parenthesized term is the $(1,1)$ form of the *Einstein tensor*. Shuffling indices a little, the take-away result is that

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

has zero derivative on a manifold.

8 Weak Curvature

8.1 Perturbed Flat Metric

A manifold that is essentially flat while exhibiting slight curvature admits a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; ,$$

where all components of h are much less than η. The up-index version of the metric is not simply $\eta^{\mu\nu} + h^{\mu\nu}$, instead the best we can write is

$$g^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu}$$

for some nontrivial tensor H.

To gain some traction, insert both versions of g into the delta function:

$$\begin{aligned}
\delta^\mu_\sigma &= g^{\mu\rho} g_{\rho\sigma} \\
\cancel{\delta^\mu_\sigma} &= \cancel{\delta^\mu_\sigma} + \eta^{\mu\rho} h_{\rho\sigma} + H^{\mu\rho}\eta_{\rho\sigma} + H^{\mu\rho} h_{\rho\sigma} \\
0 &= g^{\mu\rho} h_{\rho\sigma} + H^{\mu\rho}\eta_{\rho\sigma} \\
0 &= \eta^{\sigma\nu} g^{\mu\rho} h_{\rho\sigma} + H^{\mu\rho}\cancel{\eta_{\rho\sigma}\eta^{\sigma\nu}}\delta^\nu_\rho \\
H^{\mu\nu} &= -g^{\mu\rho}\eta^{\sigma\nu} h_{\rho\sigma}
\end{aligned}$$

Thus, the expression for $g^{\mu\nu}$ reads

$$g^{\mu\nu} = \eta^{\mu\nu} - g^{\mu\rho}\eta^{\sigma\nu} h_{\rho\sigma} \; ,$$

which of course contains a factor of $g^{\mu\rho}$, so we may re-insert the expression containing H:

$$\begin{aligned}
g^{\mu\nu} &= \eta^{\mu\nu} - \left(\eta^{\mu\rho} + H^{\mu\rho}\right)\eta^{\sigma\nu} h_{\rho\sigma} \\
&= \eta^{\mu\nu} - \left(\eta^{\mu\rho} - g^{\mu\alpha}\eta^{\sigma\rho} h_{\alpha\sigma}\right)\eta^{\sigma\nu} h_{\rho\sigma} \\
&= \eta^{\mu\nu} - h^{\mu\nu} + \left(\eta^{\mu\alpha} + H^{\mu\alpha}\right) h^\rho_\alpha h^\nu_\rho \\
&= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho} h^\nu_\rho + H^{\mu\alpha} h^\rho_\alpha h^\nu_\rho \\
&= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho} h^\nu_\rho - h^{\mu\alpha} h^\rho_\alpha h^\nu_\rho - H^{\mu\beta} h^\alpha_\beta h^\rho_\alpha h^\nu_\rho \\
&= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho} h^\nu_\rho - h^{\mu\alpha} h^\rho_\alpha h^\nu_\rho + h^{\mu\beta} h^\alpha_\beta h^\rho_\alpha h^\nu_\rho - O\left(5\right)
\end{aligned}$$

The same result can be derived by expanding g and H as a power series in a parameter λ that we set to one at the end:

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \lambda h_{\mu\nu} \\
H^{\mu\nu} &= \lambda H^{\mu\nu}_{(1)} + \lambda^2 H^{\mu\nu}_{(2)} + \lambda^3 H^{\mu\nu}_{(3)} + \cdots
\end{aligned}$$

Setting up a similar calculation as before, we have

$$\begin{aligned}
\delta^\mu_\sigma &= g_{\nu\sigma} g^{\mu\nu} \\
\delta^\mu_\sigma &= \left(\eta_{\nu\sigma} + \lambda h_{\nu\sigma}\right)\left(\eta^{\mu\nu} + \lambda H^{\mu\nu}_{(1)} + \lambda^2 H^{\mu\nu}_{(2)} + \lambda^3 H^{\mu\nu}_{(3)} + \cdots\right) \\
\cancel{\delta^\mu_\sigma} &= \cancel{\delta^\mu_\sigma} + \lambda\left(h^\mu_\sigma + \eta_{\nu\sigma} H^{\mu\nu}_{(1)}\right) + \lambda^2\left(h_{\nu\sigma} H^{\mu\nu}_{(1)} + \eta_{\nu\sigma} H^{\mu\nu}_{(2)}\right) \\
&\quad + \lambda^3\left(h_{\nu\sigma} H^{\mu\nu}_{(2)} + \eta_{\nu\sigma} H^{\mu\nu}_{(3)}\right) + \lambda^4\left(O\left(4\right)\right) \; ,
\end{aligned}$$

where each parenthesized term is independently zero. Solving for each $H_{(k)}$ in order delivers the same expansion coefficients:

$$H^{\mu\nu}_{(1)} = -h^{\mu\nu} \qquad\qquad H^{\mu\nu}_{(2)} = -H^{\mu\rho}_{(1)} h^\nu_\rho = h^{\mu\rho} h^\nu_\rho$$

$$H^{\mu\nu}_{(3)} = -H^{\mu\rho}_{(2)} h^\nu_\rho = -h^{\mu\alpha} h^\rho_\alpha h^\nu_\rho \qquad\qquad \text{etc.}$$

8.2 Acceleration from Curvature

Consider a perturbed Minkowski space admitting a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; .$$

In the low-velocity *non-relativistic* limit, the proper velocity is approximately

$$U^{\mu} \approx (c, 0, 0, 0) \; ,$$

which embeds the approximation $d\tau \approx dt$. Meanwhile, the geodesic equation tells us

$$\frac{d^2 q^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00} U^0 U^0 = 0 \; ,$$

where Γ^{μ}_{00} is given by (retaining only first-order terms in h)

$$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_{\nu} g_{00} \approx -\frac{1}{2} \eta^{\mu\nu} \partial_{\nu} h_{00} \; .$$

Putting this all together, we find

$$\frac{d^2 q^{\mu}}{dt^2} = \frac{c^2}{2} \eta^{\mu\nu} \partial_{\nu} h_{00} \; ,$$

where the $\mu = 0$ channel tells us that h_{00} has no time derivative. In vector form, the remaining spatial components are

$$\frac{d^2 \vec{q}}{dt^2} = \frac{c^2}{2} \vec{\nabla} h_{00} \; .$$

This was done with no mention of external forces, yet an acceleration still arises due solely to the curvature on the manifold. If this smells like gravity to you, you're on the right track! More generally, the above can be written

$$h_{00} = -\frac{2}{c^2} V\left(\vec{x}\right) \; ,$$

where $V\left(\vec{x}\right)$ is the gravitational potential. Finally, note that the 00-component of the $(0, 2)$ Ricci tensor is readily shown to obey, to first order,

$$R_{00} = \partial_{\mu} \Gamma^{\mu}_{00} = -\frac{1}{2} \partial^{\mu} \partial_{\mu} h_{00} = -\frac{1}{2} \nabla^2 h_{00} \; .$$

9 Curved Manifolds

9.1 Two-Sphere

We have seen that embedding a sphere $x^2 + y^2 + z^2 = r^2$ in three-dimensional space while fixing the radius results in a two-dimensional curved manifold that is 'unaware' of a third dimension. This is captured in non-vanishing components of the Riemann curvature tensor $R^{\rho}_{\mu\sigma\nu}$, which gives way to the Ricci scalar, namely $R = 2/r^2$ for a two-sphere. The square of the line element (a.k.a. the interval) falls out of flat three-dimensional spherical coordinates by setting $dr = 0$:

$$dS^2 = r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

Using dS^2, it's trivial to explore lines of constant ϕ or constant θ. For a line connecting the two poles of the sphere, we set $d\phi = 0$ and let $0 \leq \theta \leq \pi$, as in

$$S_{\phi_0} = \int dS = r \int_0^{\pi} d\theta = \pi r \; .$$

Meanwhile, on the equator we have $\theta = \pi/2$ and $0 \leq \phi < 2\pi$:

$$S_{\theta_0} = \int dS = r \cancel{\sin\left(\frac{\pi}{2}\right)} \int_0^{2\pi} d\phi = 2\pi r$$

Problem 1

Borrowing results from flat three-dimensional spherical coordinates, fix the radius r and write down the metric, all surviving connection coefficients, and the Laplacian operator in two-dimensional spherical space.

Solution 1

$$g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

9.2 Non-Round Two-Sphere

If we maintain the axial symmetry of the two-sphere while generalizing the profile from trigonometric functions $\sin\theta$, $\cos\theta$ to a more general pair $f(\theta)$, $g(\theta)$, we begin with

$$x = r f(\theta)\cos\phi \qquad y = r f(\theta)\sin\phi \qquad z = r g(\theta) \; .$$

The interval $dS^2 = dx^2 + dy^2 + dz^2$ becomes

$$dS^2 = r^2 \left(\left(\frac{df}{d\theta}\right)^2 + \left(\frac{dg}{d\theta}\right)^2 \right) d\theta^2 + r^2 f(\theta)^2 \; d\phi^2 \; .$$

In the same way that $\sin\theta$, $\cos\theta$ are related by trigonometric identities, $f(\theta)$, $g(\theta)$ shall be defined to obey

$$\left(\frac{df}{d\theta}\right)^2 + \left(\frac{dg}{d\theta}\right)^2 = 1 \; ,$$

such that the interval reduces to

$$dS^2 = r^2 \left(d\theta^2 + f(\theta)^2 \; d\phi^2 \right) \; .$$

Lines of constant ϕ that connect the two poles of the sphere still obey $d\phi = 0$ and $0 \leq \theta \leq \pi$, so we again have

$$S_{\phi_0} = \int dS = r \int_0^{\pi} d\theta = \pi r \; .$$

For a trip around the equator, we set $z = 0$, $\theta = \pi/2$, and let $0 \leq \phi < 2\pi$ to integrate the arc length:

$$S_{\theta_0} = \int dS = rf\left(\frac{\pi}{2}\right)\int_0^{2\pi} d\phi = 2\pi rf\left(\frac{\pi}{2}\right)$$

Problem 2

The Earth is not a perfectly round sphere. Instead, the polar radius, measured to be $r_{\phi_0} = 6357\,\text{km}$, is slightly less than the equatorial radius, measured as $r_{\theta_0} = 6378\,\text{km}$. Supposing the surface of the earth is modeled by

$$f(\theta) = \sin\theta\left(1 + \epsilon\sin^2\theta\right) ,$$

what values of r and ϵ would best represent the data?

Solution 2

$$\begin{aligned}
\cancel{2\pi}\,(6357\,\text{km}) &= \cancel{2\pi} r \\
\cancel{2\pi}\,(6378\,\text{km}) &= \cancel{2\pi} r \cancel{f\left(\frac{\pi}{2}\right)}(1+\epsilon) \\
1+\epsilon &= \frac{6378}{6357} \\
\epsilon &= \frac{6378}{6357} - 1 \approx 0.0033 \\
r &= 6357\,\text{km}
\end{aligned}$$

9.3 Three-Sphere

A hypersphere of four dimensions with fixed radius becomes a curved three-dimensional manifold. It can be shown that such a manifold has an interval (a slight variation from the one we found previously)

$$dS^2 = d\theta^2 + \sin^2\theta\, d\psi^2 + \cos^2\theta\, d\phi^2 ,$$

with corresponding metric components

$$g_{\theta\theta} = 1 \qquad g_{\psi\psi} = \sin^2\theta \qquad g_{\phi\phi} = \cos^2\theta .$$

Using our established methods, the surviving connection coefficients are

$$\Gamma^{\theta}_{\psi\psi} = -\sin\theta\cos\theta \qquad \Gamma^{\theta}_{\phi\phi} = \sin\theta\cos\theta$$

$$\Gamma^{\psi}_{\theta\psi} = \Gamma^{\psi}_{\psi\theta} = \cot\theta \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = -\cot\theta .$$

After some careful algebra, the components of the Ricci tensor turn out to be

$$R_{\theta\theta} = 2 \qquad R_{\psi\psi} = 2\sin^2\theta \qquad R_{\phi\phi} = 2\cos^2\theta ,$$

or more compactly,

$$R_{\mu\nu} = 2g_{\mu\nu} .$$

The Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = 2g^{\mu\nu}g_{\mu\nu} = 2\cdot 3 = 6 .$$

9.4 Hyperbolic Coordinates

A two-dimensional space with negative curvature can be studied by considering a three-dimensional space space parameterized by a radius and two angles

$$x = r \sinh\psi \cos\phi \qquad y = r \sinh\psi \sin\phi \qquad z = r\cosh\psi$$

with

$$\vec{S} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\,.$$

Note first that $x^2 + y^2 + z^2$ is not equivalent to r^2. Computing this out, we find

$$\sqrt{x^2 + y^2 + z^2} = r\,\sqrt{\cosh^2\psi + \sinh^2\psi}\,,$$

which of course reduces to r for spherical coordinates. Proceed by finding basis vectors from $\vec{S}$ according to:

$$\begin{aligned}
\hat{r} &= \frac{\partial \vec{S}}{\partial r} \cdot \left|\frac{\partial \vec{S}}{\partial r}\right|^{-1} = \frac{\sinh\psi\cos\phi\,\hat{\imath} + \sinh\psi\sin\phi\,\hat{\jmath} + \cosh\psi\,\hat{k}}{\sqrt{\cosh^2\psi + \sinh^2\psi}} \\
\hat{\psi} &= \frac{\partial \vec{S}}{\partial \psi} \cdot \left|\frac{\partial \vec{S}}{\partial \psi}\right|^{-1} = \frac{\cosh\psi\cos\phi\,\hat{\imath} + \cosh\psi\sin\phi\,\hat{\jmath} + \sinh\psi\,\hat{k}}{\sqrt{\cosh^2\psi + \sinh^2\psi}} \\
\hat{\phi} &= \frac{\partial \vec{S}}{\partial \phi} \cdot \left|\frac{\partial \vec{S}}{\partial \phi}\right|^{-1} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}
\end{aligned}$$

Solving for $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, we find

$$\begin{aligned}
\hat{\imath} &= \sqrt{\cosh^2\psi + \sinh^2\psi}\,\left(-\sinh\psi\cos\phi\,\hat{r} + \cosh\psi\cos\phi\,\hat{\psi}\right) - \sin\phi\,\hat{\phi} \\
\hat{\jmath} &= \sqrt{\cosh^2\psi + \sinh^2\psi}\,\left(-\sinh\psi\sin\phi\,\hat{r} + \cosh\psi\sin\phi\,\hat{\psi}\right) + \cos\phi\,\hat{\phi} \\
\hat{k} &= \sqrt{\cosh^2\psi + \sinh^2\psi}\,\left(\cosh\psi\,\hat{r} - \sinh\psi\,\hat{\psi}\right)\,.
\end{aligned}$$

Inserting $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ into $\vec{S}$, the $\hat{\psi}$ and $\hat{\phi}$ components cancel out, leaving us with

$$\vec{S} = r\,\sqrt{\cosh^2\psi + \sinh^2\psi}\,\hat{r}\,.$$

Letting $H = \sqrt{\cosh^2\psi + \sinh^2\psi}$, the differential line element is

$$d\vec{S} = \hat{r}\,H\,dr + \hat{r}\,r\,dH + rH\,d\hat{r}\,,$$

where

$$dH = \frac{2\sinh\psi\cosh\psi}{H}\,d\psi$$

and

$$d\hat{r} = -\frac{2\sinh\psi\cosh\psi}{H}\,d\psi\,\hat{r} + \frac{\sinh\psi}{H}\,d\phi\,\hat{\phi} + \frac{1}{H} d\psi\,\hat{\psi}\,,$$

which boil $d\vec{S}$ down to

$$d\vec{S} = \;H\,dr\,\hat{r} + r\sinh\psi\,d\phi\,\hat{\phi} + r\,d\psi\,\hat{\psi}\,.$$

The square of the differential line element is the distance interval, which resolves to

$$dS^2 = H^2\,dr^2 + r^2\sinh^2\psi\,d\phi^2 + r^2\,d\psi^2\,.$$

Next, fix the radius to $r = a$ such that $dr = 0$. The non-vanishing connection coefficients resemble those from spherical coordinates with hyperbolic trigonometric functions:

$$\Gamma^{\psi}_{\phi\phi} = -\sinh\psi\cosh\psi \qquad \Gamma^{\phi}_{\phi\psi} = \coth\psi$$

The essential component of the Riemann curvature tensor is $R_{\psi\phi\psi\phi}$, which comes out to

$$R_{\psi\phi\psi\phi} = -a^2 \sinh^2 \psi \, .$$

The nonzero components of the Ricci tensor are $R_{\psi\psi}$, $R_{\phi\phi}$, which are

$$R_{\psi\psi} = -1 \qquad\qquad R_{\psi\psi} = -\sinh^2 \psi \, .$$

Finally, the Ricci scalar for this system turns out to be

$$\begin{aligned} R &= g^{\psi\psi} R_{\psi\psi} + g^{\phi\phi} R_{\phi\phi} \\ &= \frac{-2}{a^2} \, , \end{aligned}$$

affirming the negative curvature of hyperbolic space.

10 Symmetry and Invariance

10.1 Killing's Equation

A key identity arises by analyzing change in the metric along vector field lines on a manifold. For instance, the metric is homogeneous throughout the Cartesian coordinate system permitting any translation, whereas the metric in cylindrical coordinates varies with the radius r, so only certain translations leave the metric unchanged. Here we develop a formal way to discuss what the notion 'metric is unchanged' actually means.

Consider a manifold that is host to a vector field K^μ. At a point p_0, take in infinitesimal step along a contour of K^μ onto the point p_1 such that

$$q^\mu\left(p_1\right) = q^\mu\left(p_0\right) + \epsilon K^\mu \; ,$$

where ϵ is a small parameter. The new location $q^\mu\left(p_1\right)$ is interpreted as a change in coordinates from $\tilde{q}$ to q such that

$$\tilde{q}^\mu\left(p_0\right) = q^\mu\left(p_1\right) \; ,$$

meaning p_1 becomes the new base point in the shifted system. It follows that the change in the metric $g_{\mu\nu}$ obeys

$$\Delta g_{\mu\nu} = g_{\mu\nu}\left(q\left(p_1\right)\right) - \tilde{g}_{\mu\nu}\left(\tilde{q}\left(p_0\right)\right) \; ,$$

where $\tilde{g}_{\mu\nu}$ is the metric re-calculated in the shifted coordinate system.

The first term on the right side is rather easy, as to first order in ϵ, we have

$$g_{\mu\nu}\left(q\left(p_1\right)\right) \approx g_{\mu\nu}\left(q\left(p_0\right)\right) + \epsilon K^\rho \partial_\rho g_{\mu\nu}\left(q\left(p_0\right)\right) \; .$$

Unpacking the second term requires the tensor transformation law

$$\tilde{g}_{\mu\nu}\left(q\left(p_0\right)\right) = \frac{\partial q^\alpha}{\partial \tilde{q}^\mu}\frac{\partial q^\beta}{\partial \tilde{q}^\nu} g_{\mu\nu}\left(q\left(p_0\right)\right) \; ,$$

where the derivative terms are handled by writing q^μ in terms of $\tilde{q}^\mu$, namely

$$q^\mu\left(p_0\right) = \tilde{q}^\mu\left(p_0\right) - \epsilon K^\mu \; ,$$

allowing the calculation

$$\frac{\partial q^\alpha}{\partial \tilde{q}^\mu} = \delta^\alpha_\mu - \epsilon \partial_\mu K^\alpha \; .$$

So far, the shifted metric $\tilde{g}$ reads, to first order,

$$\begin{aligned}
\tilde{g}_{\mu\nu}\left(q\left(p_0\right)\right) &= \left(\delta^\alpha_\mu - \epsilon\partial_\mu K^\alpha\right)\left(\delta^\beta_\nu - \epsilon\partial_\nu K^\beta\right) g_{\alpha\beta}\left(q\left(p_0\right)\right) \\
&= \delta^\alpha_\mu \delta^\beta_\nu g_{\alpha\beta}\left(q\left(p_0\right)\right) - \epsilon\left(\delta^\alpha_\mu g_{\alpha\beta}\partial_\nu K^\beta + \delta^\beta_\nu g_{\alpha\beta}\partial_\mu K^\alpha\right) + \cancel{O\left(\epsilon^2\right)} \\
&= g_{\mu\nu}\left(q\left(p_0\right)\right) - \epsilon\left(g_{\mu\beta}\partial_\nu K^\beta + g_{\alpha\nu}\partial_\mu K^\alpha\right) \; .
\end{aligned}$$

The calculation for $\Delta g_{\mu\nu}$ simplifies as

$$\begin{aligned}
\Delta g_{\mu\nu} &= \epsilon\left(\cancel{g_{\mu\nu}\left(q\left(p_0\right)\right)} + K^\rho\partial_\rho g_{\mu\nu} - \cancel{g_{\mu\nu}\left(q\left(p_0\right)\right)} + g_{\mu\beta}\partial_\nu K^\beta + g_{\alpha\nu}\partial_\mu K^\alpha\right) \\
&= \epsilon\left(K^\rho\partial_\rho g_{\mu\nu} + g_{\mu\beta}\partial_\nu K^\beta + g_{\alpha\nu}\partial_\mu K^\alpha\right) \; .
\end{aligned}$$

To handle the partial derivative of the metric, invoke the identity

$$0 = \partial_\rho g_{\mu\nu} - \Gamma^\alpha_{\rho\mu} g_{\alpha\nu} - \Gamma^\beta_{\rho\nu} g_{\mu\beta} \; ,$$

and collect like terms to get

$$\Delta g_{\mu\nu} = \epsilon\left(g_{\mu\beta}\left(\partial_\nu K^\beta + \Gamma^\beta_{\rho\nu} K^\rho\right) + g_{\alpha\nu}\left(\partial_\mu K^\alpha + \Gamma^\alpha_{\rho\mu} K^\rho\right)\right) \; ,$$

where the nested terms are tightly expressed as covariant derivatives:

$$\Delta g_{\mu\nu} = \epsilon\left(g_{\mu\beta} D_\nu K^\beta + g_{\alpha\nu} D_\mu K^\alpha\right) = \epsilon\left(D_\nu K_\mu + D_\mu K_\nu\right) \; ,$$

as the metric compatibility condition allows us to lower the indices on K through the covariant derivative. The isometric condition $\Delta g_{\mu\nu} = 0$ leads us to Killing's equation, where K_μ is called a *Killing vector*, named after Wilhelm Killing:

$$0 = D_\nu K_\mu + D_\mu K_\nu$$

10.2 Invariant Quantities

Consider a quantity

$$Q = K_\mu U^\mu \ ,$$

where U^μ is the proper velocity (tangent vector on the manifold) $dq^\mu/d\tau$, and K_μ is some vector field. Taking a full derivative of Q and applying the chain rule, we find

$$\begin{aligned}\frac{dQ}{d\tau} &= \frac{dq^\lambda}{d\tau} D_\lambda \left(K_\mu U^\mu\right) \\ &= U^\lambda U^\mu D_\lambda K_\mu + \left(U^\lambda D_\lambda U^\mu\right) K_\mu \ ,\end{aligned}$$

where the parenthesized is identically the geodesic equation and is thus zero. By symmetry in the indices λ, μ, we are left with

$$\begin{aligned}\frac{dQ}{d\tau} &= \frac{1}{2}\left(U^\lambda U^\mu D_\lambda K_\mu + U^\mu U^\lambda D_\mu K_\lambda\right) \\ &= \frac{1}{2} U^\lambda U^\mu \left(D_\lambda K_\mu + D_\mu K_\lambda\right) \ ,\end{aligned}$$

where the parenthesized quantity is identically Killing's equation, resolving to zero, provided that K_μ is a Killing vector. This notion readily generalizes to tell us each linearly independent Killing vector K_j on a manifold implies an invariant quantity Q_j.

It's possible to show that the maximum number of Killing vectors on a manifold of dimension d is equal to

$$N = d + \frac{1}{2} d\left(d-1\right) = \frac{1}{2} d\left(d+1\right) \ ,$$

meaning there are d translational symmetries and $d\left(d-1\right)/2$ rotations. If time is included as a parameter, then any constant-velocity shift is also a symmetry called a *boost*. One special case is the *Poincare* group, consisting of all symmetries allowed in Minkowski space having $d = 1 + 3$, with $N = 10$ members:

$$4 \text{ Translations} + 3 \text{ Rotations} + 3 \text{ Boosts} = \text{Poincare Group}$$

Furthermore, it can be shown curved manifolds carrying maximum symmetry have constant scalar (Ricci) curvature R such that the Riemann tensor may be written

$$R_{\alpha\beta\mu\nu} = \frac{R}{d\left(d-1\right)} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\beta\mu} g_{\alpha\nu}\right) \ .$$

10.3 Conformal Killing Equation

A slightly weaker condition than Killing's equation $D\mu K_\nu + D_\nu K_\mu = 0$ is called the *conformal killing equation*, namely

$$D_\mu K_\nu + D_\nu K_\mu = f\left(q^\rho\right) g_{\mu\nu} \ .$$

We may show that a quantity Q is still invariant under the conformal killing equation for the case of null geodesics:

$$\frac{dQ}{d\tau} = \frac{1}{2} U^\lambda U^\mu \left(D_\lambda K_\mu + D_\mu K_\lambda\right) = \frac{1}{2} U^\lambda U^\mu f\left(q^\rho\right) g_{\lambda\mu} = \frac{1}{2} f\left(q^\rho\right) \cancel{U^\mu U_\mu} = 0$$

<u>Problem 1</u>

Show that the vector field

$$V = q^\mu \frac{\partial}{\partial q^\mu}$$

is a conformal Killing vector of Minkowski space.

<u>Solution 1</u>

$$\vec{V} = q^\mu \, \vec{a}_{(\mu)} \qquad\qquad V^\mu = q^\mu = \left(ct, x, y, z\right)$$

$$\begin{aligned} D_\mu V_\nu + D_\nu V_\mu &= \frac{\partial}{\partial q^\mu} V_\nu + \frac{\partial}{\partial q^\nu} V_\mu = \eta_{\lambda\nu} \frac{\partial}{\partial q^\mu} V^\lambda + \eta_{\lambda\mu} \frac{\partial}{\partial q^\nu} V^\lambda \\ &= \eta_{\lambda\nu} \frac{\partial q^\lambda}{\partial q^\mu} + \eta_{\lambda\mu} \frac{\partial q^\lambda}{\partial q^\nu} = \eta_{\lambda\nu} \delta^\lambda_\mu + \eta_{\lambda\mu} \delta^\lambda_\nu \\ &= \eta_{\mu\nu} + \eta_{\nu\mu} \\ &= 2\, \eta_{\mu\nu} \end{aligned}$$

$$f\left(q^\rho\right) = 2$$

10.4 Problems

Flat Space Isometry

Problem 2

Show that any direction q^μ is a symmetry direction in flat space mapped by Cartesian coordinates. (Check for changes in the metric under constant translations.)

Solution 2

$$V = \frac{\partial}{\partial x} \qquad V^x = 1 \qquad V^y = V^z = 0$$

$$\Delta g_{\mu\nu} = \epsilon \left(V^\rho \partial_\rho g_{\mu\nu} + g_{\mu\beta} \partial_\nu V^\beta + g_{\alpha\nu} \partial_\mu V^\alpha \right) = 0$$

Problem 3

Show that rotations in θ are symmetries of flat space mapped by Cartesian coordinates.

Solution 3

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad V^x = V_x = -y \qquad V^y = V_y = x$$

$$D_\mu V_\nu + D_\nu V_\mu = \begin{cases} \partial_x V_x + \partial_x V_x = 0 \\ \partial_y V_y + \partial_y V_y = 1 - 1 = 0 \\ \partial_x V_y + \partial_y V_x = 0 \end{cases}$$

Problem 4

Show that rotations in θ are symmetries of flat space mapped by two-dimensional polar coordinates.

Solution 4

$$V = \frac{\partial}{\partial \theta} \qquad V^r = V_r = 0 \qquad V^\theta = 1 \qquad V_\theta = r^2$$

$$D_\mu V_\nu + D_\nu V_\mu = \begin{cases} \partial_r V_r + \partial_r V_r = 0 \\ \partial_\theta V_\theta + \partial_\theta V_\theta = 0 \\ \partial_r V_\theta + \partial_\theta V_r = 2r - 2r^2/r = 0 \end{cases}$$

Minkowski Invariant

Problem 5

Show that the quantity

$$q_\mu P^\mu = -Et + \vec{p} \cdot \vec{x}$$

is invariant along geodesics in Minkowski space.

Solution 5

Taking a proper time derivative, we find

$$\frac{d}{d\tau}\left(q_\mu P^\mu\right) = \frac{dq^\lambda}{d\tau} D_\lambda \left(q_\mu P^\mu\right) = mU^\lambda D_\lambda \left(q_\mu U^\mu\right)$$
$$= mU^\lambda U^\mu D_\lambda q_\mu + m q_\mu U^\lambda D_\lambda U^\mu \, .$$

The second term contains the geodesic equation and vanishes. Due to the symmetry in λ, μ, the result can be written

$$\frac{d}{d\tau}\left(q_\mu P^\mu\right) = \frac{m}{2} U^\lambda U^\mu \left(D_\lambda q_\mu + D_\mu q_\lambda\right) \, .$$

Noting that any position q^μ qualifies as Killing vector in Minkowski space, the term in parentheses resolves to zero, finishing the calculation:

$$\frac{d}{d\tau}\left(q_\mu P^\mu\right) = 0$$

Two-Sphere Conformal Killing Vectors

Problem 6

The unit two-sphere having differential interval

$$dS^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

has three conformal Killing vectors that can be derived from the translational Killing vectors

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y} \qquad Z = \frac{\partial}{\partial z}$$

in Cartesian space. Convert X, Y, Z into spherical coordinates to write down $X^{\mu'}$ in spherical coordinates of unit radius. Check that each resulting vector satisfies the conformal Killing equation.

Solution 6

$$V^{\mu'} = \left(\partial q^{\mu'}/\partial q^\mu\right) V^\mu = \begin{cases} V^r = \sin\theta\cos\phi \, V^x + \sin\theta\sin\phi \, V^y + \cos\theta \, V^z \\ V^\theta = \left(\cos\theta\cos\phi \, V^x + \cos\theta\sin\phi \, V^y - \sin\theta \, V^z\right)/r \\ V^\phi = \left(-\sin\phi \, V^x + \cos\phi \, V^y\right)/r\sin\theta \end{cases}$$

Rearranging and adding the first two equations, we find

$$\sin\theta \, V^r + r\cos\theta \, V^\theta = \cos\phi \, V^x + \sin\phi \, V^y \, ,$$

which helps solve for V^x, V^y, V^z (noting that $V_\mu = \partial/\partial q^\mu$):

$$V^x = \sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$V^y = \sin\theta\sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$V^z = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

By setting $r = 1$, we have three conformal Killing vectors with two angular components:

$$X^\theta = X_\theta = \cos\theta\cos\phi \qquad X^\phi = -\frac{\sin\phi}{\sin\theta} \qquad X_\phi = -\sin\theta\sin\phi$$

$$Y^\theta = Y_\theta = \cos\theta\sin\phi \qquad Y^\phi = \frac{\cos\phi}{\sin\theta} \qquad Y_\phi = \sin\theta\cos\phi$$

$$Z^\theta = Z_\theta = -\sin\theta \qquad Z^\phi = Z_\phi = 0$$

The nontrivial conformal Killing equations to verify are

$$D_\mu W_\theta + D_\theta W_\mu = f\left(q^\rho\right) g_{\mu\theta} \qquad\qquad D_\mu W_\phi + D_\phi W_\mu = f\left(q^\rho\right) g_{\mu\phi}$$

for each $W = X,\, Y,\, Z$. These work out to be:

$$\begin{aligned}
2D_\theta X_\theta &= 2\left(\partial_\theta X_\theta + \cancel{\Gamma^\lambda_{\theta\theta}} X_\lambda\right) = -2\sin\theta\cos\phi\left(1^2\right) = \left(-2\sin\theta\cos\phi\right) g_{\theta\theta} \\
2D_\phi X_\phi &= 2\left(\partial_\phi X_\phi + \Gamma^\theta_{\phi\phi} X_\theta\right) = -2\sin\theta\cos\phi\left(\sin^2\theta\right) = \left(-2\sin\theta\cos\phi\right) g_{\phi\phi} \\
2D_\theta Y_\theta &= 2\left(\partial_\theta Y_\theta + \cancel{\Gamma^\lambda_{\theta\theta}} Y_\lambda\right) = -2\sin\theta\sin\phi\left(1^2\right) = \left(-2\sin\theta\sin\phi\right) g_{\theta\theta} \\
2D_\phi Y_\phi &= 2\left(\partial_\phi Y_\phi + \Gamma^\theta_{\phi\phi} Y_\theta\right) = -2\sin\theta\sin\phi\left(\sin^2\theta\right) = \left(-2\sin\theta\sin\phi\right) g_{\phi\phi} \\
2D_\theta Z_\theta &= 2\left(\partial_\theta Z_\theta + \cancel{\Gamma^\lambda_{\theta\theta}} Z_\lambda\right) = -2\cos\theta\left(1^2\right) = \left(-2\cos\theta\right) g_{\theta\theta} \\
2D_\phi Z_\phi &= 2\left(\partial_\phi Z_\phi + \Gamma^\theta_{\phi\phi} Z_\theta\right) = -2\cos\theta\left(\sin^2\theta\right) = \left(-2\cos\theta\right) g_{\phi\phi}
\end{aligned}$$