Quadratics and Cubics
MANUSCRIPT

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Chapter 1

Quadratics and Cubics

1 Quadratic Expressions

Factoring quadratic expressions is among the most tortured exercises in mathematics education. It is often the student's first encounter with something presented as not *purely* formulaic, which is to mean factoring can still go wrong - to not turn out an answer - without making a technical mistake. It's the student's first real brush with the algebraic abyss.

Un-Foiling an Equation

It's worthwhile to pause on why factoring is difficult in the first place. Factoring is introduced to the student after topics like 'distribution' and polynomial multiplication are thoroughly gained, so statements such as

$$(2x + 3)(4x - 1) = 8x^2 + 10x - 3$$

are easily understood reading left-to-right.

On the left are *four* terms grouped by signs and parentheses, but on the right, after carrying out a swift FOIL operation, are *three* terms. Despite the algebraic balance of the equation above, there's still a sense that some information is lost in going left to right.

Factoring, of course, is the reverse job of polynomial multiplication. For the example on hand, we must begin with three pieces of information, namely $8x^2$, $10x$, -3, and use this to generate the whole cluster to the left of the = sign.

As if un-scrambling eggs, it's not obvious how to un-multiply a polynomial into separate products. This topic is therefore not only difficult to learn, but inevitably difficult to teach. The student-teacher relationship won't be comparably strained until integral calculus.

1.1 Quadratic Formula

Factoring a quadratic expression $Ax^2 + Bx + C$ is equivalent to seeking the x-intercepts in the graph

$$y = Ax^2 + Bx + C \, .$$

By setting $y = 0$, we can find two locations, each represented by the same letter x (confusingly enough), where the graph touches the x-axis:

$$0 = Ax^2 + Bx + C \tag{1.1}$$

$$0 = A(x - p)^2 + Aq \tag{1.2}$$

Introducing two new variables p, q as written, the problem translates to solving for these in terms of A, B, and C. Doing this exercise, one finds

$$p = \frac{-B}{2A} \tag{1.3}$$

$$q = \frac{C}{A} - \frac{B^2}{4A^2} \,, \tag{1.4}$$
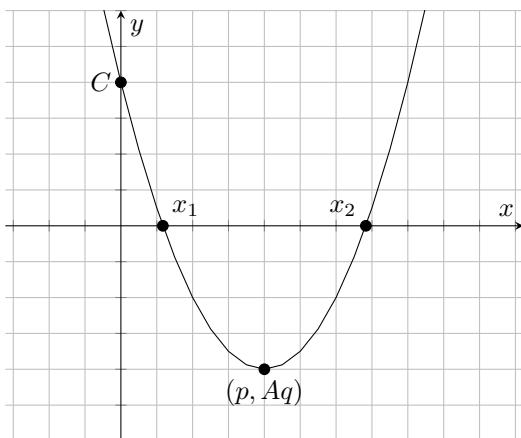
so now x may be written

$$x = p \pm \sqrt{-q} \,. \tag{1.5}$$

Replacing p, q via Equations (1.3), (1.4) gives the *quadratic formula*:

$$x = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \tag{1.6}$$

Due to the plus-or-minus ($\pm$) symbol, there are two solutions to the quadratic formula, x_1, x_2. The factored answer can be written:

$$y = A\,(x - x_1)\,(x - x_2)$$

Factoring quadratics could begin and end at the quadratic formula if mathematics were taught a certain way. By understanding the above derivation, the student hits all of the essential points required for factoring: solutions as x-intercepts, completing the square, substitution, and the quadratic formula.

Problem 1
    Derive Equations (1.3), (1.4), (1.5) from Equations (1.1), (1.2).

Product of Quadratic Solutions

A seldom-utilized identity that pertains to factoring quardatic expressions relates the product of two the

soltuions x_1, x_2. Carrying this out, one finds

$$x_1 \cdot x_2 =$$
$$\left(\frac{-B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A}\right)\left(\frac{-B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A}\right),$$

simplifying to:

$$x_1 \cdot x_2 = \frac{C}{A}$$

2   Factoring Techniques

There are at least half a dozen 'common' methods for factoring quadratic expressions, some more useful than others in certain regimes. Following is a brief survey of some of these.

2.1   Box Method

The most grotesque means of factoring is surely the *box method*, usually confined to pre-college pedagogy.
    To factor the quadratic expression

$$2x^2 - x - 6 \,,$$

first draw a box, and then put the first and third terms along the diagonal as shown:

$2x^2$	
	-6

Off to the side, multiply the first and third coefficients to get -12. List every factorization of this number:

$$-12 = \begin{cases} \pm 1 \times \mp 12 \\ \pm 2 \times \mp 6 \\ \pm 3 \times \mp 4 \end{cases}$$

The pair of factors that sums to the expression's *middle* coefficient, namely -1, is the pair we need. The box is then updated:

$2x^2$	$-4x$
$3x$	-6

Finally, look across each row and each column of the box. In the margin of each, write the greatest common factor occurring in that row or column:

	x	-2
$2x^2$	$-4x$	$2x$
$3x$	-6	3

The final result is read from the margins:

$$2x^2 - x - 6 = (x - 2)\,(2x + 3)$$

The box method is popular for numerous reasons, one being that it's easy for an instructor to prepare factoring worksheets. Correctness can be judged

based on whether the student writes the correct term in the correct box. Easy to fill in, easy to grade.

On the other hand, one immediately sees the departure from mathematical thinking inherent to the box method. Once the box is drawn, the student is asked to set aside the tools of algebraic procedure, giving way to something akin to spreadsheet heuristics.

Problem 1

Use the box method to show:

$$6x^2 + 11x + 4 = (3x + 4)(2x + 1)$$

Hint:

	$3x$	4	
$6x^2$	$8x$	$2x$
$3x$	4	1

Problem 2

Use the box method to show:

$$3x^2 - 2x - 8 = (x - 2)(3x + 4)$$

Hint:

	x	-2	
$3x^2$	$-6x$	$3x$
$4x$	-8	4

2.2 Split-Term Method

A relatively quick means of factoring a quadratic expression involves splitting the middle term.

For the trinomial

$$2x^2 + 7x + 3 \,,$$

first record the product of the two outer coefficients, namely $(2)(3) = 6$. The task becomes splitting the middle coefficient (7) into two numbers whose product is 6, which is straightforwardly done:

$$2x^2 + 7x + 3 = 2x^2 + \underbrace{6x + x}_{7x} + 3$$

We then proceed to write

$$2x^2 + 7x + 3 = 2x(x + 3) + (x + 3)$$
$$= (2x + 1)(x + 3) \,,$$

and the problem is solved after some regrouping of terms.

The split-term method works well when it's easy to divine *how* the middle term should be split. Like most common methods, this can depend on a bit of luck, and works best when the coefficients involved are tame integers.

2.3 Completing the Square

Completing the square is a robust tool for factoring quadratic expressions.

Supposing we're given

$$x^2 + 6x + 2 \,,$$

the first move is to look at the middle coefficient, namely 6, and divide that by two. This number becomes the term inside a 'square' factor as follows:

$$x^2 + 6x + 2 = (x + 3)^2 - 3^2 + 2$$

Note that a constant -3^2 is introduced to keep algebraic balance.

To proceed, seek solutions to

$$(x + 3)^2 - 3^2 + 2 = 0 \,,$$

resulting in

$$x_1 = -3 + \sqrt{7}$$
$$x_2 = -3 - \sqrt{7} \,,$$

so the final answer is written

$$x^2 + 6x + 2 = \left(x + 3 - \sqrt{7}\right)\left(x + 3 + \sqrt{7}\right) \,.$$

Problem 3

Derive the quadratic formula by completing the square on:

$$ax^2 + bx + c = 0$$

Hint:

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c$$

2.4 Algebraic Identities in Factoring

The limit case of a lucky factoring problem is one that maps perfectly onto a known algebraic identity. Committing identities to memory such as

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2 \,,$$

quick work can be made of problems that fit them:

$$x^2 + 10x + 25 = (x + 5)^2$$
$$x^2 - 6x + 9 = (x - 3)^2$$
$$9x^2 - 16 = (3x + 4)(3x - 4)$$

2.5   Normalizing a Quadratic

Factoring a quadratic expression is undoubtedly simpler when the leading coefficient A doesn't complicate things, epitomized by $A = 1$. When this isn't the case, one can always factor A from the whole expression

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right),$$

and focus on the parenthesized quantity, proceeding as if there is no leading coefficient. Of course, the coefficients B and C are then modified by A, which can make the problem much less penetrable.

2.6   Special Quadratic Coefficients

Unit Leading Coefficient

In the special case that $A = 1$, we have

$$x^2 + Bx + C = (x - x_1)(x - x_2),$$

implying

$$x_1 + x_2 = -B$$
$$x_1 \cdot x_2 = C,$$

and furthermore

$$x_1 = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right)$$
$$x_2 = \frac{1}{2}\left(-B - \sqrt{B^2 - 4C}\right).$$

Unit First and Third

In the special case that the first and third coefficients are equal to one, we have

$$x^2 + Bx + 1 = (x - x_1)(x - x_2),$$

implying

$$x_1 + x_2 = -B$$
$$x_1 \cdot x_2 = 1.$$

With $x_1 \cdot x_2 = 1$ established, we can define a variable

$$z = \frac{1}{x}$$

and seek solutions to a modified expression

$$1 + \frac{B}{z} + z^2 = 0,$$

equivalent to

$$z^2 + Bz + 1 = 0,$$

exactly what we started with, up to a change of variable. Solving one equation for x solves the modified equation (should you encounter one) for z, and vice versa.

2.7   Discriminant

One can't encounter quadratic-anything without dealing with the *discriminant*:

$$\mathcal{D} = B^2 - 4AC$$

For solutions to a quadratic equation be real-valued, the discriminant must be positive. The special case $B^2 = 4A$ corresponds to there being one solution to the quadratic, i.e. $x_1 = x_2$. When the discriminant is negative, solutions are imaginary or complex.

3   Method of Transform

Now we develop a technique for factoring quadratic expressions that is not in the common teachings.

Starting with a general quadratic expression

$$Ax^2 + Bx + C,$$

split the middle term B into the sum of two unknown variables f_1 and f_2:

$$Ax^2 + Bx + C = Ax^2 + f_1 x + f_2 x + C$$

Equivalently, we may recast the right side in factored form

$$Ax^2 + Bx + C = \frac{1}{C}(f_1 x + C)(f_2 x + C),$$

or also equivalently:

$$Ax^2 + Bx + C = \frac{1}{A}(Ax + f_1)(Ax + f_2)$$

3.1   Transform Kernel

For the above forms to remain consistent, the restriction on $f_{1,2}$ emerges:

$$f_1 + f_2 = B$$
$$f_1 \cdot f_2 = AC$$

This is the quantification of the unavoidable kernel - the 'hard part' of factoring - that requires us to dream up two numbers whose sum is B and whose product is AC.

3.2 Transformed Quadratic

As a system of two equations and two unknowns, the restriction on $f_{1,2}$ can be conjoined to a single equation which is itself quadratic

$$f^2 - Bf + AC = 0 \,,$$

and the problem is now transformed into finding solutions for f.

Interestingly, this exposes a deeper quality of factoring problems, in that the nonlinear system of equations, i.e. the kernel, can be transposed into yet another quadratic equation - yet another factoring problem. Students are quietly asked to solve this the 'hard' way without knowing it.

All we've done is transform one quadratic problem into another, but notice now that the leading coefficient is now absorbed elsewhere. For this reason, it's often easy enough to 'eyeball' the solutions for f_1 and f_2. When this isn't the case, the standard gamut of factoring techniques are all the easier to apply to the f-equation.

3.3 Completing the Rectangle

The so-called method of transform can be regarded as 'completing the rectangle'. In terms of $f_{1,2}$, note we can generally write

$$Ax^2 + Bx + C = \left(\frac{f_1}{\sqrt{C}}x + \sqrt{C} \right) \left(\frac{f_2}{\sqrt{C}}x + \sqrt{C} \right) \,,$$

or equivalently,

$$Ax^2 + Bx + C = \left(\sqrt{A}x + \frac{f_1}{\sqrt{A}} \right) \left(\sqrt{A}x + \frac{f_2}{\sqrt{A}} \right) \,,$$

and so on.

$f_1 x$	C
Ax^2	$f_2 x$

Completing the rectangle is a way to frame a factored quadratic expression as the product of two unequal 'lengths'. No matter how this is done though, one inevitably confronts the kernel represented by the transformed quadratic.

3.4 Special Cases

Unit Leading Coefficient

In the special case that $A = 1$, the method of transform returns something *almost* tautological in the sense that

$$x^2 + Bx + C$$

implies

$$f^2 - Bf + C = 0 \,.$$

The left-hand expressions are the same up to a minus sign on the active variable, so we identify

$$f = -x \,.$$

That is, if we find solutions to f, their negations are the solutions to x.

3.5 Examples: Method of Transform

Example 1
    Factor:
$$6x^2 + 11x + 4$$

Identify $B = 11$ and $AC = 24$ to write

$$f^2 - 11f + 24 = 0 \,,$$

from which we discern

$$(f - 8)(f - 3) = 0 \,.$$

With two solutions for f, proceed using the split-term method:

$$\begin{aligned} 6x^2 + 11x + 4 &= 6x^2 + 8x + 3x + 4 \\ &= 2x(3x + 4) + (3x + 4) \\ &= (2x + 1)(3x + 4) \end{aligned}$$

Example 2
    Factor:
$$3x^2 - 2x - 8$$

Identify $B = -2$ and $AC = -24$ to write

$$f^2 + 2f - 24 = 0 \,,$$

from which we discern

$$(f + 6)(f - 4) = 0 \,.$$

With two solutions for f, proceed using the split-term method:

$$\begin{aligned} 3x^2 - 2x - 8 &= 3x^2 - 6x + 4x - 8 \\ &= 3x(x - 2) + 4(x - 2) \\ &= (3x + 4)(x - 2) \end{aligned}$$

Example 3

Factor:

$$15x^2 + 14x - 8$$

Identify $B = 14$ and $AC = -120$ to write

$$f^2 - 14f - 120 = 0 \,,$$

from which we discern

$$(f + 6)(f - 20) = 0 \,.$$

With two solutions for f, proceed by completing the rectangle:

$$15x^2 + 14x - 8 = \frac{-1}{8}(-6x - 8)(20x - 8)$$
$$= (5x - 2)(3x + 4)$$

Example 4

Factor:

$$36x^2 - 121y^2$$

Identify $B = 0$ and $AC = -36 \cdot 121y^2$ to write

$$f^2 = 36 \cdot 121y^2 \,,$$

from which we discern

$$f = \pm 66y \,.$$

With two solutions for f, proceed by completing the rectangle:

$$36x^2 - 121y^2 = \frac{1}{A}(Ax + 66y)(Ax - 66y)$$
$$= (6x + 11y)(6x - 11y)$$

Example 5

Factor:

$$x^2 + 3x - 28$$

Identify $B = 3$ and $AC = -28$ to write

$$f^2 - 3f - 28 = 0 \,,$$

reminiscent of the original problem, up the the variable change $f = -x$. Proceed by standard means to find $f_1 = 4$, $f_2 = -7$. The solutions in x are the negatives of these, so we finally have

$$x^2 + 3x - 28 = (x - 4)(x + 7) \,.$$

4   Problems: Factoring Quadratics

Use any factoring method to prove the following:

Problem 1

$$4x^2 + 15x + 9 = (4x + 3)(x + 3)$$

Problem 2

$$6x^2 + 23x - 4 = (6x - 1)(x + 4)$$

Problem 3

$$10x^2 + x - 3 = (5x + 3)(2x - 1)$$

Problem 4

$$15x^2 - 7x - 4 = (3x + 1)(5x - 4)$$

Problem 5

$$4x^2 + 2x - 12 = 2(2x - 3)(x - 2)$$

Problem 6

$$8x^2 - 2xy - 3y^2 = (2x + y)(4x - 3y)$$

Problem 7

$$(x + 4)^3 - 9x - 36 = (x + 4)(x + 1)(x + 7)$$

Problem 8

$$x^2 + 2x + 1 - y^2 = (x + 1 + y)(x + 1 - y)$$

Problem 9

$$x^2 - y^2 + 12y - 36 = (x + y - 6)(x - y + 6)$$

Problem 10

$$3x^3 + 5x^2y + xy^2 - y^3 = (3x - y)(x + y)^2$$

5   Cubic Expressions

A *cubic expression* is a *third-order* polynomial:

$$Ax^3 + Bx^2 + Cx + D$$

As a generalized quadratic expression, one might wonder how a cubic expression could be factored, and whether there exists an order-three analog to the quadratic formula. As it turns out, this answer isn't so trivial to come by. Unlike quadratic expressions, whose mysteries were shooed away millennia ago, it took until the sixteenth century to crack the problem of cubics.
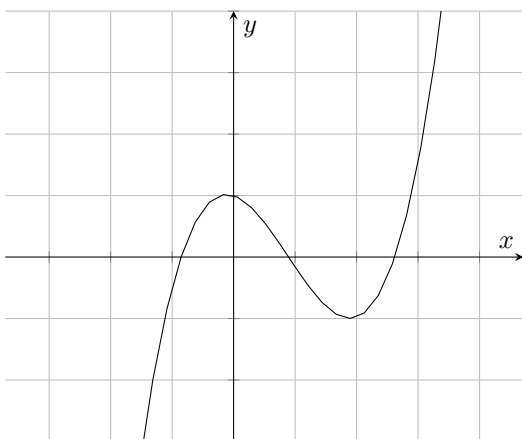
5.1 Cubic Equations

To approach this problem we first construct a *cubic equation*

$$y = Ax^3 + Bx^2 + Cx + D ,$$

qualitatively appearing as follows:

In the above, the coefficients are chosen such that

$$A = 1/2$$
$$B = -4/3$$
$$C = -1/3$$
$$D = 1$$

The leading coefficient A determines the overall steepness of the curve. For $A > 0$, the curve grows upward for increasing x, while dipping (very) negative for decreasing x. Far from the origin, this is true regardless of B, C, D, as the x^3-term grows much faster than the lower-order terms. For $A < 0$, similar comments apply.

The coefficient D plays the role of the y-intercept, controlling the vertical placement of the plot on the Cartesian plane. The coefficients B and C are responsible for the overall structure near $x = 0$. In the general case, a cubic equation flaunts two vertex points and up to three x-intercepts. The number of x-intercepts can vary depending on the values of B, C, D, but there is always at least one.

5.2 Factoring Cubic Equations

We now develop a method to factor any cubic expression, which amounts to looking for x-intercepts in the cubic equation

$$y = Ax^3 + Bx^2 + Cx + D .$$

5.3 Depressed Cubic

To begin, we employ a trick to do away with the x^2-term by making the curious substitution

$$x = z - \frac{B}{3A} .$$

Letting the algebra carry forth, we find

$$0 = z^3 + z \left(-\frac{B^2}{3A^2} + \frac{C}{A} \right)$$
$$+ 2 \left(\frac{B}{3A} \right)^3 - \frac{C}{A} \left(\frac{B}{3A} \right) + \frac{D}{A} ,$$

which is a bit ugly, but conveniently omits a z^2-term. Proceed by setting

$$b = -\frac{B^2}{3A^2} + \frac{C}{A}$$
$$-c = 2 \left(\frac{B}{3A} \right)^3 - \frac{C}{A} \left(\frac{B}{3A} \right) + \frac{D}{A}$$

to arrive at the equation of the *depressed cubic*:

$$z^3 + bz = c$$

Henceforth we'll work in the variable z, keeping in mind that we get back to x using the initial subsitution.

Geometric Form of the Cubic

Using geometry as an aid to solving cubic equations is a trick attributed to Gerolamo Cardano in 1545. Tracing Cardano's steps, begin with a cube of side p, and then introduce three planes inside the cube, parallel to the top, right, and back faces. Set each plane length q from the cube's respective sides as shown.
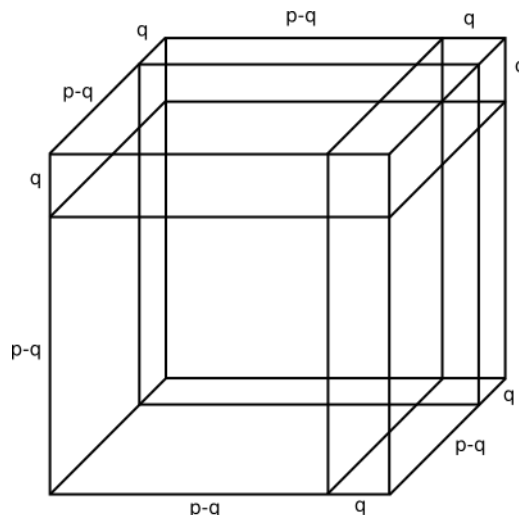
The total volume inside the total cube consists of the 'primary' cube of side $p - q$, three slabs of volume $q (p - q)^2$, three bars of volume $q^2 (p - q)$, and a

small cube of side q. Meanwhile, the total volume is simply p^3. Equating the total volumes, we write

$$p^3 = (p-q)^3 + 3q^2\,(p-q) + 3\,(p-q)^2\,q + q^3\,,$$

readily simplifying to

$$(p-q)^3 + 3pq\,(p-q) = p^3 - q^3\,.$$

Looking closely, note the above is a depressed cubic equation. Identifying

$$\begin{aligned} p - q &= z \\ 3pq &= b \\ p^3 - q^3 &= c\,, \end{aligned}$$

we recover the form $z^3 + bz = c$.

Ferro-Tartaglia Formula

Picking up from Cardano's geometric formulation of the depressed cubic problem

$$z^3 + bz = c\,,$$

eliminate q between the latter two equations to get

$$p^6 - cp^3 - \left(\frac{b}{3}\right)^3 = 0\,,$$

which is in fact a quadratic equation in the variable p^3, easily isolated by the quadratic formula:

$$p^3 = \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$$

This automatically gives us q^3, specifically

$$q^3 = p^3 - c = -\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}\,,$$

allowing a solution for z to be written:

$$z_0 = p - q$$
$$z_0 = \left(\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}\right)^{1/3}$$
$$+ \left(\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}\right)^{1/3}$$

The above is associated with Italian mathematician N. F. Tartaglia (1500-1557), although its discovery is credited to another Italian mathematician S. del Ferro (1465-1526).

For shorthand, the Ferro-Tartaglia formula is also written

$$z_0 = \left(\frac{c}{2} + q\right)^{1/3} + \left(\frac{c}{2} - q\right)^{1/3}\,,$$

where

$$q = \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}\,.$$

<u>Example 1</u>
   Find one solution to:

$$z^3 - \frac{z}{3} - \frac{2}{27} = 0$$

Step 1: Identify the above as a depressed cubic equation and pick out coefficients:

$$\begin{aligned} 3pq = b &= -\frac{1}{3} \\ p^3 - q^3 = c &= \frac{2}{27} \end{aligned}$$

Step 2: Solve for p^3, and write p and q:

$$p^3 = \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$$
$$p^3 = \frac{1}{27} + \sqrt{\left(\frac{1}{27}\right)^2 - \left(\frac{1}{9}\right)^3} = \frac{1}{27}$$
$$p = \frac{1}{3}$$
$$q = -\frac{1}{3}$$

Step 3: Write the solution for z in terms of p and q:

$$z = p - q = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

5.4   Completing the Cubic Solution

In the general case, the geometric approach to solving a depressed cubic equation

$$z^3 + bz = c$$

produces one solution, however there should exist (up to) three total solutions to the equation. Labeling the known solution as z_0, it follows that $(z - z_0)$ can be factored out of the depressed cubic equation to get

$$z^3 + bz - c = (z - z_0)\left(z^2 + z_0 z + \frac{c}{z_0}\right)\,.$$

Remaining solutions to the depressed cubic equation are given by

$$0 = z^2 + z_0 z + \frac{c}{z_0}\,,$$

an easy application of the quadratic formula.

Example 2

Use the known solution $z_0 = 2/3$ to continue factoring the expression:

$$z^3 - \frac{z}{3} - \frac{2}{27}$$

Step 1: Substitute $x_0 = 2/3$ and $c = -2/27$ into the quadratic equation for z and simplify:

$$z = \frac{2/3}{2} \left(-1 \pm \sqrt{1 - \frac{4 \cdot 2/27}{(2/3)^3}} \right) = -\frac{1}{3}$$

Step 2: Pick out any new solution(s) gained. In this case, we have two copies of the same number:

$$z_1 = z_2 = -\frac{1}{3}$$

Step 3: Write the final form:

$$z^3 - \frac{x}{3} - \frac{2}{27} = \left(z - \frac{2}{3} \right) \left(z + \frac{1}{3} \right)^2$$

Depressed Cubic in Disguise

Consider the rather exotic quantity

$$\left(7 + \sqrt{50} \right)^{1/3} + \left(7 - \sqrt{50} \right)^{1/3} ,$$

which might seem impossible to evaluate, namely because $7 - \sqrt{50}$ is surely negative, hinting of complex numbers. Proceeding with caution, let us store the whole quantity in a variable x, and then calculate x^3:

$$x^3 = \left(\left(7 + \sqrt{50} \right)^{1/3} + \left(7 - \sqrt{50} \right)^{1/3} \right)^3$$

$$= 14 + 3 \left(\left(7 + \sqrt{50} \right)^{1/3} \left(7 - \sqrt{50} \right)^{1/3} \right)$$

$$\cdot \left(\left(7 + \sqrt{50} \right)^{1/3} + \left(7 - \sqrt{50} \right)^{1/3} \right)$$

$$= 14 + 3 \left(-1 \right)^{1/3} x$$

Evidently then, we find

$$x^3 = -3x + 14 ,$$

the equation of a depressed cubic. By standard means we find the solution to be $x = 2$, which finally means:

$$\left(7 + \sqrt{50} \right)^{1/3} + \left(7 - \sqrt{50} \right)^{1/3} = 2$$

6 Negative Radicals

It will turn out that the Ferro-Tartaglia formula isn't so straightforwardly applied in all cases, especially from the point of view of a 1500s mathematician. To creep up on the issue, let us factor the expression:

$$x^3 + 5x^2 - 2x - 24$$

Start with the transformation $x = z - B/3A$ and simplify to find

$$b = -\frac{31}{3}$$

$$c = \frac{308}{27} ,$$

and the corresponding depressed cubic equation reads

$$z^3 + z \left(\frac{-31}{3} \right) = \frac{308}{27} .$$

With b and c, a solution z_0 to the depressed cubic is generated from the Ferro-Tartaglia formula. Applying this, we inevitably hit

$$q = \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} = \sqrt{-\frac{25}{3}} .$$

Alarmingly, there is a minus sign inside the square root term.

6.1 Bombelli's Wild Thought

The courage to work with an equation that contains a negative term inside a square root came first to Rafael Bombelli, now known as his *wild thought*. To replicate Bombelli's wild thought, define a pair of numbers U, V where

$$\left(\frac{c}{2} + q \right)^{1/3} = U + \sqrt{-1}V$$

$$\left(\frac{c}{2} - q \right)^{1/3} = U - \sqrt{-1}V ,$$

such that the sum

$$z_0 = 2U$$

becomes an identity.

To proceed, raise the top equation to the third power to write

$$\frac{c}{2} + q = U \left(U^2 - 3V^2 \right) + \sqrt{-1}V \left(3U^2 - V^2 \right) ,$$

which was enough for Bombelli to see the solution. By separating the 'real' term without the factor of $\sqrt{-1}$ from its 'imaginary' counterpart, we end up with a pair of equations where *neither* makes reference to $\sqrt{-1}$. Moreover, Bombelli went on to seek integer solutions for U, V, which isn't always easy, or always possible.

Continuing the example on hand, we have

$$\frac{c}{2} + q = \frac{154}{27} + \sqrt{-\frac{25}{3}} \ ,$$

and comparing this to the expanded form in terms of U, V, gives

$$\frac{154}{27} = U\left(U^2 - 3V^2\right)$$

$$\sqrt{\frac{25}{3}} = V\left(3U^2 - V^2\right) \ .$$

To make headway on the problem, some fiddling leads one to the substitution

$$2\sqrt{3}V = \tilde{V}$$
$$6U = \tilde{U} \ ,$$

and the above becomes

$$11 \cdot 14 \cdot 8 = \tilde{U}\left(\tilde{U} + 3\tilde{V}\right)\left(\tilde{U} - 3\tilde{V}\right)$$
$$1 \cdot 12 \cdot 10 = \tilde{V}\left(\tilde{U} + \tilde{V}\right)\left(\tilde{U} - \tilde{V}\right) \ ,$$

solved by two integers

$$\tilde{U} = 11$$
$$\tilde{V} = 1 \ ,$$

meaning

$$U = \frac{11}{6}$$
$$V = \frac{1}{2\sqrt{3}} \ ,$$

from which we get

$$z_0 = 2U = \frac{11}{3} \ .$$

Restoring the x- variable, we finally have

$$x_0 = z_0 - \frac{B}{3A} = \frac{11}{3} - \frac{5}{3} = 2$$

as a solution to the original cubic expression.

With the hard part finished, we can find two more solutions to the example on hand by solving

$$0 = z^2 + \left(\frac{11}{3}\right)z + \frac{308 \cdot 3}{27 \cdot 11} \ ,$$

having solutions

$$z_1 = -\frac{7}{3}$$
$$z_2 = -\frac{4}{3} \ .$$

Restoring the x variable, these read

$$x_1 = -4$$
$$x_2 = -3 \ .$$

Finally then, we have

$$x^3 + 5x^2 - 2x - 24 = (x - 2)(x + 4)(x + 3) \ ,$$

finishing the example.