# Polynomial Division
# MANUSCRIPT

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# Chapter 1

# Polynomial Division

## 1 Introduction

A versatile and brutal method for manipulating a mathematical expression is the method of polynomial division. The setup and procedure for polynomial division is identical to elementary methods for arithmetic. To illustrate, consider a ratio such as

$$\frac{x^2-9x-10}{x+1},$$

and set up the corresponding division structure:

$$x+1 \overline{\big) \quad x^2 \quad -9x \quad -10}$$

For the process to yield useful results, the numerator should always contain a higher-degree polynomial than the denominator.

### 1.1 Long Division

Divide the first term in the dividend by the first term in the divisor to get $x^2/x = x$. Place the result ($x$) in the quotient field (above the line). Then, distribute $x$ into the divisor and subtract the result from the dividend:

$$\begin{array}{r|rrr} & & x & \\ \hline x+1 & x^2 & -9x & -10 \\ & x^2 & x & \\ \hline & & -10x & -10 \end{array}$$

The 'bottom line' of the above is $-10x-10$, which may now regard as the updated dividend, and the process is ready to repeat. Dividing the respective leading terms, we find $-10x/x = -10$, and update as follows:

$$\begin{array}{r|rrr} & & x & -10 \\ \hline x+1 & x^2 & -9x & -10 \\ & x^2 & x & \\ \hline & & -10x & -10 \\ & & -10x & -10 \\ \hline & & & 0 \end{array}$$

With a new dividend of zero, the process halts, and we can read off the answer:

$$\frac{x^2-9x-10}{x+1} = x-10$$

### 1.2 Remainder

Polynomial division doesn't always finish so cleanly as the example chosen above. Taking a more informative case, consider the ratio

$$\frac{\left(x^4+x+1\right)^2}{x^2-1}=\frac{x^8+2x^5+2x^4+x^2+2x+1}{x^2-1}\,.$$

Setting up and doing the hard work, we have:

$$\begin{array}{r|rrrrrrrr}
 & & x^6 & & +x^4 & +2x^3 & +3x^2 & +2x & +4 \\ \hline
x^2-1 & x^8 & & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & x^8 & -x^6 & & & & & & \\ \hline
 & & x^6 & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & & x^6 & & -x^4 & & & & \\ \hline
 & & & 2x^5 & +3x^4 & & +x^2 & +2x & +1 \\
 & & & 2x^5 & & -2x^3 & & & \\ \hline
 & & & & 3x^4 & +2x^3 & +x^2 & +2x & +1 \\
 & & & & 3x^4 & & -3x^2 & & \\ \hline
 & & & & & 2x^3 & +4x^2 & +2x & +1 \\
 & & & & & 2x^3 & & -2x & \\ \hline
 & & & & & & 4x^2 & +4x & +1 \\
 & & & & & & 4x^2 & & -4 \\ \hline
 & & & & & & & 4x & +5
\end{array}$$

The next step *would* be to try dividing $4x$ by $x^2$, however the result (and any following it) will contain factors of $x^{-1}$. This is a sign to halt the division process and tuck the leftovers into a remainder term, namely $(4x+5)\,/\left(x^2-1\right)$. In doing so, we write the final result:

$$\frac{x^8+2x^5+2x^4+x^2+2x+1}{x^2-1}=x^6+x^4+2x^3+3x^2+2x+4+\frac{4x+5}{x^2-1}$$

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### 1.3 Dividing Infinite Sums

Polynomial division also works well with infinite sums. A case worth exploring starts with the cosine and sine given by

$$\begin{aligned}
\cos(x) &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \\
\sin(x) &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\,.
\end{aligned}$$

The definition of $\tan(x)$ is the ratio of the sine to the cosine, which can be calculated by brute force using polynomial division. Setting up the problem carefully, one finds

$$\tan(x)\approx x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots\,.$$

This result comes with a warning, though. Unlike the sine and cosine, the above representation of the tangent is not periodic and *only* works near $x=0$.

## 2 Partial Fractions

When confronted with a ratio of polynomials where the denominator is of higher degree than the numerator, a technique called *partial fractions* can be used to break apart the ratio.

Starting with a simple case, consider the scenario where the denominator has a degree-two polynomial in factored form. With this, observe that such a ratio can be can be split into the sum of two terms, each containing a degree-one polynomial:

$$\frac{cx+d}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$$

The unknowns $A$, $B$ are easily determined in terms of $a$, $b$, $c$ $d$. By setting $x=0$, and then $x=1/c$, respectively, we gain two equations

$$\begin{aligned}
-d &= bA+aB \\
c &= A+B\,,
\end{aligned}$$

solved by

$$A = \frac{ac+d}{a-b}$$
$$B = \frac{bc+d}{b-a},$$

which could have also been inferred by choosing values $x = a$, $x = b$.

This method generalizes to higher-degree polynomial denominators, as shown for the degree-three case:

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

**Corollary**

In general, if a polynomial $p(x)$ occurs in the denominator and is already factored into linear and quadratic terms, then for each factor $x-a$, there exists a term

$$\frac{A}{x-a},$$

where $A$ must be determined in context.

Example 1

Find the equivalent ratio as a sum of partial fractions:

$$\frac{2x+1}{(x-3)(x-4)}$$

Step 1: Rewrite the ratio as a sum:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Step 2: Solve for $A$ and $B$ to get:

$$A = -7$$
$$B = 9$$

Step 3: Assemble the result:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{-7}{x-3} + \frac{9}{x-4}$$

Example 2

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{a^2-x^2}$$

Step 1: Factor the denominator:

$$\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$$

Step 3: Solve for $A$ and $B$ to get:

$$A = 1/2a$$
$$B = 1/2a$$

Step 4: Assemble the result:

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right)$$

## 2.1 Repeated Roots

Of course, the partial fraction expansion is prone to error if we run into division by zero, i.e. the case $a = b$. To handle a ratio having two repeated roots in the denominator, we use a partial fraction expansion

$$\frac{1}{(x-a)^2(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{x-b},$$

admitting a separate term for each instance of $(x-a)$. This pattern generalizes to three repeated roots, and so on:

$$\frac{1}{(x-a)^3(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \frac{B}{x-b}$$

## 2.2 Quadratic Factors

Factors of the form $x^2+ax+b$ occurring in the denominator can be balanced by an $Ax+B$-term according to

$$\frac{1}{(x^2+ax+b)(x-c)} = \frac{Ax+B}{x^2+bx+c} + \frac{C}{x-c}.$$

If a factor like $\left(x^2+ax+b\right)^2$ occurs, extra terms are needed:

$$\frac{1}{(x^2+ax+b)^2(x-c)} = \frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{C}{x-c}$$

Example 3

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{x^4-1}$$

Step 1: Factor the denominator:

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Step 3: Multiply through by the left-hand denominator:

$$\begin{aligned} 1 &= A(x+1)\left(x^2+1\right) + B(x-1)\left(x^2+1\right) \\ &\quad + (Cx+D)(x-1)(x+1) \end{aligned}$$

Step 4: Let $x=1$, $x=-1$, $x=0$, and $x=2$ to isolate each coefficient:

$$\begin{aligned} A &= 1/4 \\ B &= -1/4 \\ D &= -1/2 \\ C &= 0 \end{aligned}$$

Step 5: Assemble the result:

$$\frac{1}{x^4-1} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)$$

## 2.3 Mixed Division Cases

Certain situations call for polynomial division *and* partial fractions. For instance, in the ratio

$$\frac{x^3+4}{x^2+x},$$

the numerator contains a higher-degree polynomial than the denominator. Carrying out the division problem

$$x^2+x \overline{\big) \quad x^3 \quad +4 \quad},$$

we end up with a quotient and a remainder as follows:

$$\frac{x^3+4}{x^2+x} = (x-1) + \frac{x+4}{x^2+x}$$

Next, take the remainder term in isolation and use partial fraction analysis to write

$$\frac{x+4}{x^2+x} = \frac{x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1},$$

where we find $A=4$, $B=-3$. In summary then, we find:

$$\frac{x^3+4}{x^2+x} = x - 1 + \frac{4}{x} - \frac{3}{x+1}$$

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# 3 Factoring by Division

Consider the curious equation with a special $n$-degree polynomial

$$x^n - a^n = 0,$$

where $a$ is an arbitrary real constant. In the most general case, there are $n$ complex solutions to the above, which may or may not be difficult to come by. Regardless of $n$ though, we *can* be sure that $x_0 = a$ is a valid real solution.

With a solution on hand, it's instructive to factor $x_0$ from the left-hand expression to check if anything interesting happens, i.e.

$$\frac{x^n - a^n}{x-a} = ?$$

Setting this up, we have:

$$x-a \overline{\big) \quad x^n \quad -a^n \quad}$$

Without specifying $n$, it's not clear where the division process ought to terminate. Carrying out the division process *anyway*, we find, after four steps:

$$\frac{x^n - a^n}{x-a} = x^{n-1} + a^1 x^{n-2} + a^2 x^{n-3} + a^3 x^{n-4} + \frac{a^4 x^{n-4} - a^n}{x-a}$$

To be prudent, the maximum number of division steps should not exceed the degree number $n$, otherwise the exponent on $x$ becomes negative.

Tidy up the equation by multiplying $x - a$ into each side:

$$x^n - a^n = (x - a)\left(x^{n-1} + a^1 x^{n-2} + a^2 x^{n-3} + a^3 x^{n-4}\right) + \left(a^4 x^{n-4} - a^n\right)$$

Of course, there was no real reason to stop the division process at four steps, so the above must also be true for $j$ steps:

$$x^n - a^n = (x - a)\left(x^{n-1} + a^1 x^{n-2} + a^2 x^{n-3} + \cdots + a^{j-1} x^{n-n}\right) + \left(a^j x^{n-j} - a^n\right)$$

By choosing $j = n$, the remainder term vanishes entirely, leaving

$$x^n - a^n = (x - a)\left(x^{n-1} + a^1 x^{n-2} + a^2 x^{n-3} + \cdots + a^{n-1}\right) . \tag{1.1}$$

This we'll take as the answer to the curious factoring problem. Starting with the polynomial $x^n - a^n$, solutions other than $x = a$ are contained in another polynomial of order $n - 1$ given by the above.

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**Sigma Notation**

In order to avoid always writing Equation (1.1) as a long polynomial basted with exponents, we switch to sigma notation as follows:

$$x^n - a^n = (x - a)\left(\sum_{k=1}^{n} a^{k-1} x^{n-k}\right) \tag{1.2}$$

Sometimes it's convenient to work the modified version that replaces $a^n$ with $a$:

$$x^n - a = \left(x - a^{1/n}\right)\left(\sum_{k=1}^{n} a^{(k-1)/n} x^{n-k}\right) \tag{1.3}$$

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Example 4

Factor:

$$x^3 - 8$$

Step 1: Identify variables:

$$n = 3$$
$$a = 2$$

Step 2: Write the factored expression in summation notation:

$$x^3 - 8 = (x - 2)\left(\sum_{k=1}^{3} 2^{k-1} x^{3-k}\right)$$

Step 3: Simplify:

$$x^3 - 8 = (x - 2)\left(x^2 + 2x + 4\right)$$

Example 5

Factor:

$$x^4 - 9$$

Step 1: Identify variables:

$$n = 4$$
$$a = 3$$

Step 2: Write the factored expression in summation notation:

$$x^4 - 9 = \left(x - \sqrt{3}\right)\left(\sum_{k=1}^{4} 9^{(k-1)/4} x^{4-k}\right)$$

Step 3: Simplify:

$$\begin{aligned} x^4 - 9 & \\ &= \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}x^2 + 3x + 3\sqrt{3}\right) \\ &= \left(x - \sqrt{3}\right)\left(x^2\left(x + \sqrt{3}\right) + 3\left(x + \sqrt{3}\right)\right) \\ &= \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)\left(x^2 + 3\right) \end{aligned}$$

# 4 Recursive Sequences

Equation (1.1) representing the 'curious identity' lends to a variety of uses beyond factoring. Here we develop the notion of a *recursive sequence*, which in essence, is a sequence of numbers that uses itself to extend itself.

### 4.1 Applied Polynomial Division

By making the substitution $a = (-1/x)^n$, we use Equation (1.3) to write

$$\frac{x^n - (-1/x)^n}{x + 1/x} = \sum_{k=1}^{n} (-1)^{k-1} x^{n+1-2k} ,$$

where the right-hand sum is a sequence depending solely on $x$ and $n$.

Choosing $n = 1$, $n = 2$, $n = 3$, and so on, a simple-enough pattern emerges:

$$\frac{x^n - (-1/x)^n}{x + 1/x} =$$

$$\begin{cases} n = 1 : \; x^0 \\ n = 2 : \; x^1 - x^{-1} \\ n = 3 : \; x^2 - x^0 + x^{-2} \\ n = 4 : \; x^3 - x^1 + x^{-1} - x^{-3} \\ n = 5 : \; x^4 - x^2 + x^0 - x^{-2} + x^{-4} \\ n = 6 : \; x^5 - x^3 + x^1 - x^{-1} + x^{-3} - x^{-5} \end{cases}$$

Labeling the $n$th result as $C_n$, we equivalently write

$$C_n = \frac{x^n - (-1/x)^n}{x + 1/x} \tag{1.4}$$

$$= \begin{cases} C_1 = 1 \\ C_2 = x^1 - x^{-1} \\ C_3 = -C_1 + x^2 + x^{-2} \\ C_4 = -C_2 + x^3 - x^{-3} \\ C_5 = -C_3 + x^4 + x^{-4} \\ C_6 = -C_4 + x^5 - x^{-5} \end{cases}$$

#### Recursion Relations

By inspection of the above, the coefficients $C_n$ are subect to *recursion relations*:

$$C_n = -C_{n-2} + x^{n-1} - x^{-(n-1)} \tag{1.5}$$

$$C_{n+1} = -C_{n-1} + x^n + x^{-n} \tag{1.6}$$

### 4.2 Large-n Recursion

Supposing we choose any even-valued $n$, the coefficient $C_n$ and its next neighbor relate by the recursion relations (1.5), (1.6). The pair of these begs the ratio

$$R = \frac{C_{n+1}}{C_n} = \frac{-C_{n-1} + x^n + x^{-n}}{-C_{n-2} + x^{n-1} - x^{-(n-1)}} .$$

Within this ratio, let us examine the quantities $x^n$, $x^{-n}$ with $n$ growing very large. Regardless of whether $x$ is less than one or greater than one (but not equal to one), either $x^n$ or $x^{-n}$ will grow very large, whereas the other will grow very small.

Taking the case with $x > 1$, then $x^{-n}$ and $x^{-(n-1)}$ become negligible, and we find

$$R \approx x \left( \frac{-C_{n-1} + x^n}{-x \cdot C_{n-2} + x^n} \right) \approx x ,$$

suggesting that, for large $x^n$:

$$C_{n+1} \approx x \cdot C_n$$

Taking $x < 1$ instead, the same reasoning boils down to, for small $x^n$:

$$C_{n+1} \approx \frac{-C_n}{x}$$

## 5 Lucas Numbers

In deriving Equation (1.4), we manged to avoid specifying the variable $x$. While we're free to mess with $x$ directly, it's more interesting to direct this freedom into the $C_2$ coefficient.

Choosing the most nontrivial case of $C_2 = 1$, we have

$$1 = x - \frac{1}{x} ,$$

which forces $x$ to be found by the quadratic equation. There are two solutions $x_1 = \phi$, $x_2 = \psi$ to the above, obeying

$$\phi \cdot \psi = -1$$
$$\phi + \psi = 1 .$$

Then, making the association

$$x = \phi$$
$$1/x = -\psi ,$$

the coefficients $C_n$ represented in Equation (1.4) specialize to

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} . \tag{1.7}$$

The terms $F_n$ are labeled to foreshadow their formal name.

In terms of $\phi$, $\psi$, the above occurs in list form as

$$F_n = \begin{cases} F_1 = 1 \\ F_2 = \phi + \psi \\ F_3 = -F_1 + \phi^2 + \psi^2 \\ F_4 = -F_2 + \phi^3 + \psi^3 \\ F_5 = -F_3 + \phi^4 + \psi^4 \\ F_6 = -F_4 + \phi^5 + \psi^5 \end{cases} ,$$

or as a recursion relation,

$$F_n = -F_{n-2} + L_{n-1} . \tag{1.8}$$

### 5.1 Lucas Generating Formula

The terms
$$L_n = \phi^n + \psi^n \tag{1.9}$$
are called *Lucas numbers*. These are a bit tricky to evaluate, but nonetheless with some grit one can produce:

$$\begin{aligned}
L_1 &= \phi^1 + \psi^1 = 1 \\
L_2 &= \phi^2 + \psi^2 = (\phi + \psi)^2 - 2\phi\psi = L_1^2 + 2 \\
L_3 &= \phi^3 + \psi^3 = \phi^2 + 1 + \psi^2 = L_1 + L_2 \\
L_4 &= \phi^4 + \psi^4 = \left(\phi^2 + \psi^2\right)^2 - 2\phi^2\psi^2 = L_2^2 - 2 \\
L_5 &= \phi^5 + \psi^5 = \phi^4 + \phi^2 + 1 + \psi^2 + \psi^4 = L_3 + L_4 \\
L_6 &= \phi^6 + \psi^6 = \left(\phi^3 + \psi^3\right)^2 - 2\phi^3\psi^3 = L_3^2 + 2
\end{aligned}$$

Evidently, the pattern in $L_n$ splits between odd and even channels

$$\begin{aligned}
L_{n \text{ odd}} &= L_{n-1} + L_{n-2} \\
L_{n \text{ even}} &= L_{n/2}^2 - 2 \cdot (-1)^{n/2} \ ,
\end{aligned}$$

where explicitly:

$$\begin{aligned}
L_0 &= 2 \\
L_1 &= 1 \\
L_2 &= 1^2 + 2 = 3 \\
L_3 &= L_2 + L_1 = 4 \\
L_4 &= L_2^2 - 2 = 3^2 - 2 = 7 \\
L_5 &= L_4 + L_3 = 7 + 4 = 11 \\
L_6 &= L_3^2 + 2 = 4^2 + 2 = 18 \\
L_7 &= L_6 + L_5 = 18 + 11 = 29
\end{aligned}$$

#### Recursion Relations

Since the equation for $L_{n \text{ odd}}$ makes reference to the *two* previous terms, it suffices to write
$$L_n = L_{n-1} + L_{n-2} \tag{1.10}$$
as a single formula applying to both odd and even Lucas numbers.

### 5.2 Lucas Sequence

Listing the Lucas numbers in order gives the *Lucas sequence*:
$$\{L\} = \{2, 1, 3, 4, 7, 11, 18, 29, \dots\}$$

## 6 Fibonacci Numbers

In discovering the Lucas numbers, the intermediate relation
$$F_n = -F_{n-2} + L_{n-1}$$
emerged before focusing on $L_n$. In the above, $F_n$ is given by
$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \ ,$$
and $\phi$, $\psi$ are solutions to $1 = x - 1/x$. With the result for $L_n$ in hand, we can use the above to generate the *Fibonacci numbers*:

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= -1 + L_2 = 2 \\
F_4 &= -1 + L_3 = 3 \\
F_5 &= -2 + L_4 = 5 \\
F_6 &= -3 + L_5 = 8 \\
F_7 &= -5 + L_6 = 13 \\
F_8 &= -8 + L_7 = 21
\end{aligned}$$

Note that the Fibonacci numbers follow a recursion relation analogous to equation (1.10), namely
$$F_n = F_{n-1} + F_{n-2} \ . \tag{1.11}$$
Despite its obvious truth, the proof of the above is reserved for the more general treatment of the Lucas-Fibonacci system (see below).

### 6.1 Fibonacci Sequence

Listing the Fibonacci numbers in order gives the *Fibonacci sequence*:
$$\{F\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

### 6.2 Negative Fibonacci Numbers

The calculation that gives rise to the Fibonacci sequence can be repeated with a slight tweak that involves swapping $x$ for $1/x$. This gives rise to a modified generating formula
$$\tilde{F} = \frac{(1/x)^n - (-x)^n}{1/x + x} = \frac{(-\psi)^n - (-\phi)^n}{\phi - \psi} \ ,$$
inviting a similar algebraic puzzle to the one solved already.

Working this formula carefully, we find

$$
\begin{aligned}
\tilde{F}_1 &= 1 = F_1 \\
\tilde{F}_2 &= -\frac{(\phi+\psi)\cancel{(\phi-\psi)}}{\cancel{(\phi-\psi)}} = -1 = -F_2 \\
\tilde{F}_3 &= \frac{\cancel{(\phi-\psi)}\left(\phi^2+\phi\psi+\psi^2\right)}{\cancel{(\phi-\psi)}} = -1 + L_2 = F_3 \\
\tilde{F}_4 &= \frac{\left((-\psi)^2+(-\phi)^2\right)\cancel{(\phi-\psi)}\cancel{((-\psi)+(-\phi))}^{-1}}{\cancel{(\phi-\psi)}} \\
&= -\left(L_1^2+2\right) = -F_4 ,
\end{aligned}
$$

which is enough to see the pattern. Evidently, the even-indexed terms flip sign, where the odd-indexed terms remain the same. The extended Fibonacci sequence thus reads:

$$
\{F\} = \{\ldots,-8,5,-3,2,-1,1,0,1,1,2,3,5,8,\ldots\}
$$

## 6.3 Solving for x

Interestingly, we've made it this far without explicitly needing the numerical values of $x_1 = \phi$, $x_2 = \psi$. Recalling these are defined as solutions to

$$
1 = x - \frac{1}{x} ,
$$

it's straightforward to find

$$
\begin{aligned}
x_1 &= \phi = \frac{1+\sqrt{5}}{2} \approx 1.618034\ldots \\
x_2 &= \psi = -\frac{1}{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.618034\ldots .
\end{aligned}
$$

### Golden Ratio

The constant

$$
\phi = \frac{1+\sqrt{5}}{2} \approx 1.618034\ldots
$$

is the famed *golden ratio*. Little ado is made of this number in the fundamental sciences, but plenty of attention is given to this number in areas pertaining to graphics, architecture, and biology.

Without using symbols, the $n$th Lucas or Fibonacci number can be straightforwardly expressed:

$$
\begin{aligned}
L_n &= \frac{\left(1+\sqrt{5}\right)^n + \left(1-\sqrt{5}\right)^n}{2^n} \\
F_n &= \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}
\end{aligned}
$$

# 7 General L-F Numbers

The Lucas-Fibonnacci rabbithole was entered by setting $C_2 = 1$ as it occurs as Equation (1.4). Of course, this can all be be generalized by setting $C_2$ to an arbitrary constant $p$, leading to a generalized Lucas-Fibonacci regime characterized by

$$
C_2 = p = x - \frac{1}{x} ,
$$

having solutions $x_1$, $x_2$ obeying

$$
\begin{aligned}
x_1 \cdot x_2 &= -1 \\
x_1 + x_2 &= p .
\end{aligned}
$$

Pursuing this, we stumble upon a new recursion statement analogous to Equation (1.8), namely

$$
C_n = -C_{n-2} + \tilde{L}_{n-1} , \tag{1.12}
$$

where $\tilde{L}$ is a generalized Lucas number

$$
\tilde{L}_n = x_1^n + x_2^n ,
$$

obeying the recursion relation

$$
\tilde{L}_n = p\tilde{L}_{n-1} + \tilde{L}_{n-2} . \tag{1.13}
$$

### Recursion Relations

To derive a robust recursion relation, let us write three instances of Equation (1.8) based on respective indices $n$, $n+1$, and $n+2$:

$$
\begin{aligned}
C_n &= -C_{n-2} + \tilde{L}_{n-1} \\
C_{n+1} &= -C_{n-1} + \tilde{L}_n \\
C_{n+2} &= -C_n + \tilde{L}_{n+1}
\end{aligned}
$$

Multiply the first equation by a factor of $-1$, and the second by a factor of $-p$

$$
\begin{aligned}
-C_n &= C_{n-2} - \tilde{L}_{n-1} \\
-pC_{n+1} &= pC_{n-1} - p\tilde{L}_n \\
C_{n+2} &= -C_n + \tilde{L}_{n+1}
\end{aligned}
$$

Next, take the sum of all three equations and regroup terms:

$$
\begin{aligned}
&(C_{n+2} - pC_{n+1} - C_n) + (C_n - pC_{n-1} - C_{n-2}) \\
&\qquad = \left(\tilde{L}_{n+1} - p\tilde{L}_n - \tilde{L}_{n-1}\right)
\end{aligned}
$$

The right side is identically zero by Equation (1.13). The rest can only be true if

$$
C_n = pC_{n-1} + C_{n-2} , \tag{1.14}
$$

which we take as the generalized recursion relation. In the special case $p = 1$, the above reduces to the Fibonacci case represented by Equation (1.11).

### Modified Seed

The *seed* value $C_2 = p$ has direct bearing on the set of Lucas-Fibonacci-like numbers that emerge from the analysis. In general, the solutions $x$ depend on $p$ such that

$$x = \frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 + 4}\,,$$

where the special case $p = 1$ was studied in detail above. From this perspective, we see there is a continuous family of Lucas-Fibonacci numbers.

The first number in any Fibonacci-like sequence is always $C_1 = 1$, and the second number is always $C_2 = p$. Using the generalized recursion relation (1.14), we easily find the rest:

$$\{F_{p=1}\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$
$$\{F_{p=2}\} = \{1, 2, 5, 12, 29, 70, 169, 408, \dots\}$$
$$\{F_{p=3}\} = \{1, 3, 10, 33, 109, 360, 1189, 3927, \dots\}$$
$$\{F_{p=4}\} = \{1, 4, 17, 72, 305, 1292, 5473, 23184, \dots\}$$

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