Numbers and Arithmetic
MANUSCRIPT

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[1]

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Chapter 1

Numbers and Arithmetic

1 Numbers

Numbers are symbols used for counting, measuring, or labeling a certain quantity. Each number is unique, meaning no two numbers represent the same value.

1.1 Number Line

When two numbers are compared, one number must have a greater value, and the other will have the lesser value. It follows that all numbers can be arranged on a number line, from lesser (left) to greater (right) as shown in Figure 1.1.

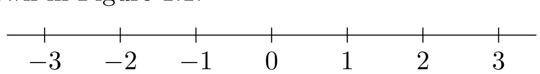

Figure 1.1: Number line (partial).

The number line is divided into two regions, positive and negative, with the special number 0 being neither of these. It extends infinitely in both directions, to $+\infty$ on the right and $-\infty$ on the left, and the whole number line is *continuous*.

Continuity

On the number line,'continuous' means that every conceivable decimal and fraction occurs somewhere on the line. This implies if we choose some number A and add or subtract from A to 'move over' to number B, we do so by strolling over all possible numbers between the two.

1.2 Real Numbers

The continuous set of all numbers on the number line is called the *real numbers*, and has the symbol $\mathbb{R}$. The values ∞ and $-\infty$ are not considered real numbers, nonetheless we can loosely capture all this by writing

$$-\infty < \mathbb{R} < \infty \,.$$

Infinity

Briefly, one reason that infinity is not considered a real number is for the lack of compatibility with ordinary arithmetic. Questions like 'what is infinity plus one?' are answered by 'infinity again':

$$\begin{aligned} \infty + 1 &= \infty \\ \infty + \infty &= \infty \\ \infty \times \infty &= \infty \end{aligned}$$

While things like the above may be true in a certain intuitive way, these are not really algebraic statements. Carrying away one example on hand, one might want to conclude

$$1 = \infty - \infty = 0$$

and claim to have broken mathematics. This is ultimately nothing but abuse of notation.

Rational vs. Irrational

Real numbers can be divided into two categories, *rational numbers* and *irrational numbers*. A rational number, coming from the word 'ratio', is a number that can be expressed as a non-repeating decimal. Rational numbers are things like 7, 0.25, or 33/3.

On the other hand, numbers like $1/3 = 0.333\ldots$ that require an infinite trail of 3's after the decimal are irrational. Perhaps the most frequently used irrational numbers are π, e, and $\sqrt{2}$, but it's straightforward to reason that there are *many* more irrational numbers than there are rational numbers.

Set Notation

The symbol assigned to rational numbers is $\mathbb{Q}$, and the symbol assigned to irrational numbers is $\mathbb{Q}'$. Each of these qualifies as a *subset* of the real numbers, and to denote this we write:

$$\{\text{rational numbers}\} = \mathbb{Q} \subset \mathbb{R}$$
$$\{\text{irrational numbers}\} = \mathbb{Q}' \subset \mathbb{R}$$

The *union* of $\mathbb{Q}$ and $\mathbb{Q}'$ reconstitute the real numbers:

$$\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$$

1.3 Integers

Rational numbers that have no decimal component belong to the *integers*, denoted $\mathbb{Z}$. The set of integers is still infinite in size but is certainly 'smaller' than $\mathbb{Q}$ and $\mathbb{R}$:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Whole and Natural

Next there are *whole numbers*, which is the subset of $\mathbb{Z}$ that excludes negatives but includes zero:

$$\text{Whole Numbers} = \{0, 1, 2, 3, \ldots\}$$

Finally, we have *natural numbers*, denoted $\mathbb{N}$, which simply shaves the zero from the set of whole numbers:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

1.4 Prime Numbers

Prime numbers are whole numbers that cannot be divided into smaller whole numbers (besides 1). Apart from 2, no even numbers are prime. There is otherwise no simple pattern to the prime numbers. The following table shows the first fifteen prime numbers in boldface situated in the lattice of integers.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50

Prime Decomposition

For a whole number N *prime decomposition* of N is defined as a list of prime numbers whose product is N. To find the prime decomposition of an odd number, try dividing by 2. The result can be one of three things:

- If $N/2$ is a prime number, stop.
- If $N/2 = M$ is a whole number, then 2 is a factor. Repeat using M.
- If $N/2$ is fraction or decimal, replace 2 by the next prime number (3).
- If $N/3$ is fraction or decimal, replace 3 by the next prime number (5).
- If $N/5$ is fraction or decimal, replace 5 by the next prime number (7), and so on.
- When the prime being tested exceeds $N/2$, stop.

Problem 1

Use the above as a guide to verify the prime decomposition for each:

$$12 = 2 \times 2 \times 3 = 2^2\, 3$$
$$231 = 3 \times 7 \times 11$$
$$150 = 2 \times 3 \times 5 \times 5 = 2\,(3)\,5^2$$
$$225 = 3 \times 3 \times 5 \times 5 = 3^2\, 5^2$$

2 Expressions

The *mathematical operators* are symbols that are situated among numbers. A valid *mathematical expression* usually has at least two numbers and one operation, such as:

$$1 + 2\,.$$

In the above, the 'plus' operator (+) combines the pair of 'input numbers' 1 and 2.

Equations

The expression $1 + 2$ is equivalent to the *sum* of the two input numbers, namely 3. The expression and its result can be written together as a *mathematical* *equation* with an 'equality' (=) symbol balancing each side:

$$1 + 2 = 3$$

Simplifying Expressions

Using operators to reduce the complexity of an expression (toward a number) without introducing error is called *simplifying*, or *evaluating* the expression. The 'simplified' expression must be essentially equal to the original, and is given the term 'equivalent'. By convention, the equivalent expression is placed on the right side of the equality symbol (=).

Nested Expressions

Bracketing symbols called *parentheses* ( ) are used to embed an expression within an expression. There must always be parenthetical balance in an equation, meaning the number of opening- and closing-parentheses must be equal.

When simplifying an expression, *the most-embedded parenthesized contents must be evaluated first.* To keep your work organized, it's good practice to keep all equality (=) symbols in a column. For instance:

$$\begin{aligned} 4 + (3 - (1 \times 2)) &= 4 + (3 - (1 \times 2)) \\ &= 4 + (3 - 2) \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

2.1 Operators

There are six mathematical operators for basic arithmetic. Listing in a particular order, these are:

Operator	Symbol	Result
Parentheses	(N)	N
Exponent	ˆ	product
Multiplication	× or ·	product
Division	÷ or /	ratio or quotient
Addition	+	sum
Subtraction	−	difference

At the top of the list is the set of *parentheses* ( ), which tell us to ignore everything else and solve whatever is inside the parentheses first. Next on the list is the exponent (ˆ) operator, followed by multiplication (× or ·), and so on down to subtraction (−).

Order of Operations = PEMDAS

The so-called *order of operations* is summarized by the letters **P E M D A S**, and can be recovered from the phrase **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally. This means to look for parentheses first, then exponents, multiplication, division, addition, and subtraction follow in order.

The '(P)EMDAS' operators take two or more numbers as input and return one number as output.

Precedence and Binding

For some more terminology, it is said that operators have an order of *precedence*, where, for instance, exponents have higher precedence than sums. The subtraction operator has the lowest precedence. Another term one encounters is *binding.* In practice, one could read that the multiplication operator binds more tightly (to a number) than an addition operator.

Problem 1

Use the order of operations to simplify

$$4 + 5 \times 9$$

and make sure the following gives a different answer:

$$(4 + 5) \times 9$$

2.2 Special Operators

There is a class of special operators that take one number instead of two, and these bind *more* tightly than the two-input operators in the 'PEMDAS' hierarchy.

Absolute Value

The *absolute value* of a number is an operation that converts any negative number to positive number while leaving positive numbers alone. A number enclosed by tall slashes (||) tells us to take the absolute value. For example:

$$\begin{aligned} |-3| &= 3 \\ |2 - 7| &= 5 \\ |4/3| &= 4/3 \\ |0| &= 0 \end{aligned}$$

Factorial

The *factorial* operator shows up as an exclamation symbol (!) after a number, and (for our purposes) is only defined for whole numbers. The factorial operator tells us to take the base number and multiply it by every whole number less than the base number.

For example, the quantity 5! is pronounced 'five factorial', and is given by

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120\,.$$

For shorthand, the same number can also be written:

$$5! = 5 \times 4! = 120$$

Since the factorial operator binds more tightly than multiplication, the above is quite different than:

$$20! = (5+4)! = 2432902008176640000$$

Floor and Ceiling

Two operations useful in computer science are *floor* and *ceiling*. The 'floor' operation ($\lfloor\rfloor$) encloses any real base number and returns the greatest integer that is less than the base number. The 'ceil' operation ($\lceil\rceil$) encloses any real base number and returns the smallest integer that is greater than the base number.

The following table demonstrates a few use cases of the floor and ceiling operations:

N	floor $\lfloor N \rfloor$	ceil $\lceil N \rceil$
3	3	3
3.1	3	4
3.4	3	4
3.9	3	4
-3.1	-4	-3
-3.9	-4	-3
-4	-4	-4

Implied Multiplication

An abuse of notation that students of mathematics quickly accept is the omission of any symbol for multiplication. When a number 'butts up' against a set of parentheses, it is assumed we multiply the number into the parenthesized content:

$$2\,(3) = 2 \times 3 = 6$$

A classic example showing the trouble with this is the expression

$$6 \div 2\,(2+1)\ .$$

Some calculators simplify the above to 1, while others arrive at 9. Explicitly, one could turn the above into either correct statement:

$$6 \div (2 \times (2+1)) = 1$$
$$(6 \div 2) \times (2+1) = 9$$

The precedence of implied multiplication can vary per computation regime, which leads to different answers.

One could perhaps argue that choosing the arithmetic division operator ($\div$) versus the forward slash (/) could break the tie on which way the expression is interpreted. With no solid rule, we could play games like this forever.

When dealing with equations in a serious way, it's best to disambiguate as much possible, which means to make liberal use of parentheses if any expression risks misinterpretation by human or machine.

2.3 Inequality

We've seen that the equality (=) sign is the 'balance' between two equivalent expressions. When two expressions are not equivalent, or conditionally equivalent, there are special symbols to denote the nature of imbalance. These are summarized as follows:

Symbol	Meaning	Example
$<$	Less than	$2 < 3$
$>$	Greater than	$4 > 3$
$\leq$	Less or Equal	$2 \leq 2+2$
$\geq$	Greater or Equal	$4 \geq \lvert -4 \rvert$

2.4 Zero and One

Zero and one are two numbers that behave unlike the rest in many ways. In the following suppose N is any nonzero real number.

Properties of Zero

- Adding or subtracting zero to any number N leaves the number unchanged:

$$N \pm 0 = N$$

- Multiplying any number by zero results in zero:

$$N \times 0 = 0$$

- Division by zero produces no useful information[1]:

$$\frac{N}{0} = \text{Undefined}$$

- The only number equal to the negative of itself is zero:

$$0 = -0$$

[1] Unfortunately, the number of instructors who have been caught conveying the grave error $N/0 = 0$ is itself nonzero.

Properties of One

- Multiplying or dividing a number by one leaves the number unchanged:

$$N \times 1 = \frac{N}{1} = N$$

- Raising one to any real power results in one:

$$1 \times 1 \times 1 \times 1 \times \cdots = 1^0 = 1^{-1.75} = 1$$

- Fractions with the same numerator and denominator are equivalent to one:

$$\frac{N}{N} = 1$$

- The infinite repeating decimal $0.999\overline{9}$ is equivalent to one:

$$0.999\overline{9} = 1$$

(Proven later with geometric series.)

3 Using Operators

Let the symbols A, B, C, D, N represent any four real numbers. To be 'safe', which means to avoid subtle errors like division by zero, you can imagine each number being nonzero.

3.1 Properties of Addition

Commutative Property

Consider the sum

$$N = A + B\,.$$

The *commutative property* of addition tells us that the order in which the terms A, B occur in the operation does not change the resulting number N. That is:

$$N = A + B$$
$$N = B + A$$

Note that the subtraction operator doesn't yield an analogously true statement. The difference $A - B$ is not the same as $B - A$.

Associative Property

Consider the sum

$$N = A + B + C\,.$$

The *associative property* of addition tells us that the order in which the terms A, B, C are added does not change the resulting number N. That is:

$$N = (A + B) + C$$
$$N = A + (B + C)$$

3.2 Properties of Multiplication

Commutative Property

Consider the product

$$N = A \times B\,.$$

The commutative property of multiplication tells us that the order in which the terms A, B occur in the operation does not change the resulting number N. That is:

$$N = A \times B$$
$$N = B \times A$$

Note that the division operator doesn't yield an analogously true statement. The ratio A/B is not the same as B/A.

Associative Property

Consider the product

$$N = A \times B \times C\,.$$

The associative property of multiplication tells us that the order in which the terms A, B, C are multiplied does not change the resulting number N. That is:

$$N = (A \times B) \times C$$
$$N = A \times (B \times C)$$

...