Numbers and Arithmetic
MANUSCRIPT

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Chapter 1

Numbers and Arithmetic

1 Numbers

Numbers are symbols used for counting, measuring, or specifying a certain quantity. Each number is unique, which means no two numbers represent the same value.

1.1 Number Line

When two numbers are compared, one number must have a greater value, and the other will have the lesser value. It follows that all numbers can be arranged on a number line, from lesser (left) to greater (right) as shown in Figure 1.1.

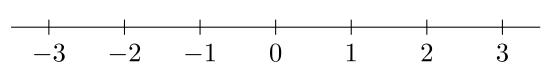


Figure 1.1: Number line (partial).

The number line is divided into two regions, positive and negative, with the special number 0 being neither of these. It extends infinitely in both directions, to $+\infty$ on the right and $-\infty$ on the left, and the whole number line is *continuous*.

Continuity

On the number line,'continuous' means that every conceivable decimal and fraction occurs somewhere on the line. This implies if we choose some number A and add or subtract from A to 'move over' to number B, we do so by strolling over all possible numbers between the two.

1.2 Real Numbers

The continuous set of all numbers on the number line is called the *real numbers*, and has the symbol $\mathbb{R}$. The values ∞ and $-\infty$ are not considered real numbers, nonetheless we can loosely capture all this by writing

$$-\infty < \mathbb{R} < \infty \,.$$

Infinity

Briefly, one reason that infinity is not considered a real number is for the lack of compatibility with ordinary arithmetic. Questions like 'what is infinity plus one?' are answered by 'infinity again':

$$\begin{aligned} \infty + 1 &= \infty \\ \infty + \infty &= \infty \\ \infty \times \infty &= \infty \end{aligned}$$

While things like the above may be true in a certain intuitive way, these are not really algebraic statements. Carrying away one example on hand, one might want to conclude

$$1 = \infty - \infty = 0$$

and claim to have broken mathematics. This is ultimately nothing but abuse of notation.

Rational and Irrational Numbers

Real numbers can be divided into two categories, *rational numbers* and *irrational numbers*. A rational number, coming from the word 'ratio', is a number that can be expressed as a non-repeating decimal. Rational numbers are things like 7, 0.25, or 33/3.

On the other hand, numbers like $1/3 = 0.333\ldots$ that require an infinite trail of 3's after the decimal are irrational. Perhaps the most frequently used irrational numbers are π, e, and $\sqrt{2}$, but it's straightforward to reason that there are *many* more irrational numbers than there are rational numbers.

Set Notation

The symbol assigned to rational numbers is $\mathbb{Q}$, and the symbol assigned to irrational numbers is $\mathbb{Q}'$. Each of these qualifies as a *subset* of the real numbers, and to denote this we write:

$$\{\text{rational numbers}\} = \mathbb{Q} \subset \mathbb{R}$$
$$\{\text{irrational numbers}\} = \mathbb{Q}' \subset \mathbb{R}$$

The *union* of $\mathbb{Q}$ and $\mathbb{Q}'$ reconstitute the real numbers:

$$\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$$

1.3 Integers

Rational numbers that have no decimal component belong to the *integers*, denoted $\mathbb{Z}$. The set of integers is still infinite in size but is certainly 'smaller' than $\mathbb{Q}$ and $\mathbb{R}$:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Whole and Natural Numbers

Next there are *whole numbers*, which is the subset of $\mathbb{Z}$ that excludes negatives but includes zero:

$$\text{Whole Numbers} = \{0, 1, 2, 3, \ldots\}$$

Finally, we have *natural numbers*, denoted $\mathbb{N}$, which simply shaves the zero from the set of whole numbers:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

2 Expressions

The *mathematical operators* are symbols that are situated among numbers. A valid *mathematical expression* usually has at least two numbers and one operation, such as:

$$1 + 2 \,.$$

In the above, the 'plus' operator ($+$) combines the pair of 'input numbers' 1 and 2.

Said another way, a mathematical expression is an encoding of a position on the number line.

Equations

The expression $1 + 2$ is equivalent to the *sum* of the two input numbers, namely 3. The expression and its result can be written together as a *mathematical equation* with an 'equality' ($=$) symbol balancing each side:

$$1 + 2 = 3$$

Simplifying Expressions

Using operators to reduce the complexity of an expression (toward a number) without introducing error is called *simplifying*, or *evaluating* the expression. The 'simplified' expression must be essentially equal to the original, and is given the term 'equivalent'. By convention, the equivalent expression is placed on the right side of the equality symbol ($=$).

Nested Expressions

Bracketing symbols called *parentheses* ( ) are used to embed an expression within an expression. There must always be parenthetical balance in an equation, meaning the number of opening- and closing-parentheses must be equal.

When simplifying an expression, *the most-embedded parenthesized contents must be evaluated first*. To keep your work organized, it's good practice to keep all equality (=) symbols in a column. For instance:

$$\begin{aligned} 4+(3-(1\times 2)) &= 4+(3-(1\times 2)) \\ &= 4+(3-2) \\ &= 4+1 \\ &= 5 \end{aligned}$$

Exponents

In the same way that multiplication is like generalized addition, i.e.

$$N+N+N = N\times 3\,,$$

the exponent operation is like generalized multiplication:

$$N\times N\times N = N^3$$

2.1 Arithmetic Operators

There are six mathematical operators for basic arithmetic. Listing in a particular order, these are:

Operator	Symbol	Result
Parentheses	(N)	N
Exponent	ˆ	product
Multiplication	$\times$ or $\cdot$	product
Division	$\div$ or /	ratio or quotient
Addition	$+$	sum
Subtraction	$-$	difference

At the top of the list is the set of *parentheses* ( ), which tell us to ignore everything else and solve whatever is inside the parentheses first. Next on the list is the exponent or 'power' (ˆ) operator, followed by multiplication ($\times$ or $\cdot$), and so on down to subtraction ($-$).

Order of Operations = PEMDAS

The so-called *order of operations* is summarized by the letters **P E M D A S**, and can be recovered from the phrase **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally. This means to look for parentheses first, then exponents, multiplication, division, addition, and subtraction follow in order.

The '(P)EMDAS' operators take two or more numbers as input and return one number as output.

Operator Precedence and Binding

For some more terminology, it is said that operators have an order of *precedence*, where, for instance, exponents have higher precedence than sums. The subtraction operator has the lowest precedence. Another term one encounters is *binding*. In practice, one could read that the multiplication operator binds more tightly (to a number) than an addition operator.

<u>Problem 1</u>

Use the order of operations to simplify

$$4+5\times 9$$

and make sure the following gives a different answer:

$$(4+5)\times 9$$

<u>Problem 2</u>

Verify each of the following:

$$\begin{aligned} 3-(-4)+5 &= 12 \\ 3^2+4^2 &= 5^2 \end{aligned}$$

<u>Problem 3</u>

Insert any combination of operators to make the following statement true:

$$1\;2\;3\;4\;5\;6\;7\;8\;9 = 100$$

Suggestions:

$$\begin{aligned} 1\times 2-3+4-5+6+7+89 &= 100 \\ 1+2+3+4+5+6+7+8\times 9 &= 100 \end{aligned}$$

2.2 Special Operators

There is a class of special operators that take one number instead of two, and these bind *more* tightly than the two-input operators in the 'PEMDAS' hierarchy.

Absolute Value

The *absolute value* of a number is an operation that converts any negative number to positive number while leaving positive numbers alone. A number enclosed by tall slashes (||) tells us to take the absolute value. For example:

$$\begin{aligned}|-3| &= 3\\ |2-7| &= 5\\ |4/3| &= 4/3\\ |0| &= 0\end{aligned}$$

Factorial

The *factorial* operator occurs as an exclamation symbol (!) after a number, and (for our purposes) is only defined for whole numbers. The factorial operator tells us to take the base number and multiply it by every whole number less than the base number.

For example, the quantity 5! is pronounced 'five factorial', and is given by

$$5! = 5\times 4\times 3\times 2\times 1 = 120\,.$$

For shorthand, the same number can also be written:

$$5! = 5\times 4! = 120$$

Since the factorial operator binds more tightly than multiplication, the above is quite different than:

$$20! = (5\times 4)! = 2432902008176640000$$

Something you may not guess is what mathematicians have agreed to do with the factorial of zero. By definition, the factorial of zero is one:

$$0! = 1$$

To establish this, sketch out the pattern

$$\begin{aligned}n! &= n\,(n-1)!\\ &= n\,(n-1)\,(n-2)!\,,\end{aligned}$$

and then

$$\frac{n!}{(n-3)!} = n\,(n-1)\,(n-2)\ .$$

Generalizing this, we eventually find

$$\frac{n!}{(n-n)!} = n\,(n-1)\,(n-2)\cdots(1) = n!\,,$$

which can only mean

$$(n-n)! = 0! = 1\,.$$

Problem 4

Insert any combination of parentheses and operators to make the following statement true:

$$1\;1\;1\;1 = 5$$

Suggestion:

$$(1+1+1)! - 1 = 5$$

Problem 5

Insert any combination of parentheses and operators to make the following statement true:

$$2\;0\;1\;6 = 1$$

Suggestion:

$$\frac{(2+0+1)!}{6} = 1$$

Floor and Ceiling

Two operations useful in computer science are *floor* and *ceiling*. The 'floor' operation ($\lfloor\rfloor$) encloses any real base number and returns the greatest integer that is less than the base number. The 'ceil' operation ($\lceil\rceil$) encloses any real base number and returns the smallest integer that is greater than the base number.

The following table demonstrates a few use cases of the floor and ceiling operations:

N	floor $\lfloor N\rfloor$	ceil $\lceil N\rceil$
3	3	3
3.1	3	4
3.4	3	4
3.9	3	4
−3.1	−4	−3
−3.9	−4	−3
−4	−4	−4

Implied Multiplication

An abuse of notation that students of mathematics quickly accept is the omission of any symbol for multiplication. When a number 'butts up' against a set of parentheses, it is assumed we multiply the number into the parenthesized content:

$$2\,(3) = 2\times 3 = 6$$

A classic example showing the trouble with this is the expression

$$6 \div 2\,(2+1)\ .$$

Some calculators simplify the above to 1, while others arrive at 9. Explicitly, one could turn the above into either correct statement:

$$\begin{aligned}6\div(2\times(2+1)) &= 1\\ (6\div 2)\times(2+1) &= 9\end{aligned}$$

The precedence of implied multiplication can vary per computation regime, which leads to different answers.

One could perhaps argue that choosing the arithmetic division operator ($\div$) versus the forward slash (/) could break the tie on which way the expression is interpreted. With no solid rule, one could play games like this forever, indicating why numerical systems have more serious (or less serious) evaluation rules than implied by PEMDAS.

2.3 Inequality

We've seen that the equality (=) sign is the 'balance' between two equivalent expressions. When two expressions are not equivalent, or conditionally equivalent, there are special symbols to denote the nature of imbalance. These are summarized as follows:

Symbol	Meaning	Example
$<$	Less than	$2 < 3$
$>$	Greater than	$4 > 3$
$\leq$	Less or Equal	$2 \leq 2 + 2$
$\geq$	Greater or Equal	$4 \geq \lvert -4 \rvert$

2.4 Zero and One

Zero and one are two numbers that behave unlike the rest in many ways. In the following suppose N is any nonzero real number.

Properties of Zero

- Adding or subtracting zero to any number N leaves the number unchanged:

$$N \pm 0 = N$$

- Multiplying any number by zero results in zero:

$$N \times 0 = 0$$

- Division by zero produces no useful information[1]:

$$\frac{N}{0} = \text{Undefined}$$

- The only number equal to the negative of itself is zero:

$$0 = -0$$

Properties of One

- Multiplying or dividing a number by one leaves the number unchanged:

$$N \times 1 = \frac{N}{1} = N$$

- Fractions with the same numerator and denominator are equivalent to one:

$$\frac{N}{N} = 1$$

- Raising one to any real power results in one:

$$1 \times 1 \times 1 \times 1 \times \cdots = 1^0 = 1^{-1.75} = 1$$

- The infinite repeating decimal $0.999\overline{9}$ is equivalent to one:

$$0.999\overline{9} = 1$$

  (Proven later with geometric series.)

3 Using Operators

Let the symbols A, B, C, D, N represent any real numbers. To be 'safe', which means to avoid subtle errors like division by zero, you can imagine each number being nonzero.

3.1 Properties of Addition

Commutative Property

Consider the sum

$$N = A + B\,.$$

The *commutative property of addition* tells us that the order in which the terms A, B occur in the operation does not change the resulting number N. That is:

$$\begin{aligned} N &= A + B \\ N &= B + A \end{aligned}$$

Note that the subtraction operator doesn't yield an analogously true statement. The difference $A - B$ is not the same as $B - A$.

[1] Unfortunately, the number of instructors who have been caught conveying the grave error $N/0 = 0$ is itself nonzero.

Associative Property

Consider the sum

$$N = A + B + C\,.$$

The *associative property of addition* tells us that the order in which the terms A, B, C are added does not change the resulting number N. That is:

$$\begin{aligned} N &= (A + B) + C \\ N &= A + (B + C) \end{aligned}$$

3.2 Properties of Multiplication

Commutative Property

Consider the product

$$N = A \times B\,.$$

The *commutative property of multiplication* tells us that the order in which the terms A, B occur in the operation does not change the resulting number N. That is

$$\begin{aligned} N &= A \times B \\ N &= B \times A\,, \end{aligned}$$

or equivalently:

$$A \times B = B \times A \tag{1.1}$$

Note that the division operator doesn't yield an analogously true statement. The ratio A/B is not the same as B/A.

Associative Property

Consider the product

$$N = A \times B \times C\,.$$

The *associative property of multiplication* tells us that the order in which the terms A, B, C are multiplied does not change the resulting number N. This means the two products

$$\begin{aligned} N &= (A \times B) \times C \\ N &= A \times (B \times C) \end{aligned}$$

are equivalent, or more concisely:

$$(A \times B) \times C = A \times (B \times C) \tag{1.2}$$

Distributive Property

The *distributive property of multiplication* tells what to do with expressions of the form

$$(A + B) \times (C + D)\ ,$$

where $(A + B)$, $(C + D)$ are called *binomials*. Note that the multiplication operator ($\times$) is optional, and the above is the same as

$$(A + B)\,(C + D)\ .$$

To multiply two binomials, multiply each term in the first binomial into each term in the second:

$$\begin{aligned} (A + B)\,(C + D) &= A\,(C + D) + B\,(C + D) \\ &= AC + AD + BC + BD \end{aligned} \tag{1.3}$$

By the commutative property, the same result is attained when swapping the binomials:

$$\begin{aligned} (C + D)\,(A + B) &= C\,(A + B) + D\,(A + B) \\ &= AC + BC + AD + BD \end{aligned}$$

FOIL Method for Binomials

A trick for remembering the steps to multiply two binomials is contained in the word 'FOIL', where each respective letter stands for **F**irst, **O**uter, **I**nner, **L**ast. For instance, consider the product:

$$(4 + 2)\,(5 - 3)$$

1. Multiply the **F**irst respective terms to get
$$4 \times 5 = 20\,.$$
2. Multiply the **O**uter terms, resulting in
$$4 \times -3 = -12\,.$$
3. Multiply the **I**nner terms, namely
$$2 \times 5 = 10\,.$$
4. Finally, the **L**ast terms multiply to give
$$2 \times -3 = -6\,.$$

The final answer is the sum of the four respective products. In our example, we have

$$(4 + 2)\,(5 - 3) = 20 - 12 + 10 - 6 = 12\,.$$

Of course, we this was visible from the outset, as

$$(4 + 2)\,(5 - 3) = (6) \times (2) = 12\,.$$

Multiplication with Trinomial

The FOIL method works on *all* binomials, but *only* works on binomials. This can cause confusion when confronted with a problem like

$$(A+B)(C+D+E)\ ,$$

where the combination $C+D+E$ is a *trinomial*.

Simplifying the above requires a direct application of the distributive property. Particularly, one must have:

$$\begin{aligned}&(A+B)(C+D+E)\\&\qquad= A(C+D+E)+B(C+D+E)\\&\qquad= AC+AD+AE+BC+BD+BE\end{aligned}$$

<u>Problem 6</u>

Simplify

$$(y-4)\left(y^2-3x-2\right)$$

in two different ways. First distribute each term in the first polynomial into each term of the second polynomial. Then do it again in the other order. Either way, find the result

$$y^3-4y^2-3xy+12x-2y+8\ .$$

3.3 Applied Division

Factoring a Number

Any real number N can be considered as the product of *factors* A, B, C, etc., so long as the product

$$N = A\times B\times C\times\cdots$$

holds.

Prime Numbers

Prime numbers are natural numbers that cannot be divided into smaller natural numbers (besides 1). Apart from 2, no even numbers are prime. There is otherwise no simple pattern to the prime numbers.

The following table shows the first fifteen prime numbers in boldface situated in the lattice of integers.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50

Prime Factorization

For a whole number N, the *prime factorization* of N is defined as a list of prime numbers whose product is N.

To find the prime factorization of a whole number, try dividing by the first prime number M, namely $M=2$. The result

$$Q=\frac{N}{M}$$

is handled as follows:

1. If Q is a prime number, stop. The prime factorization includes M, Q.

2. If Q is a non-prime whole number, then M is a factor. Replace N with Q and go to Step 1.

3. If Q is fraction or decimal, replace M with the next prime number and go to Step 1.

The above steps can be repeated until the prime being tested exceeds $N/2$. The original number N is the product of factors.

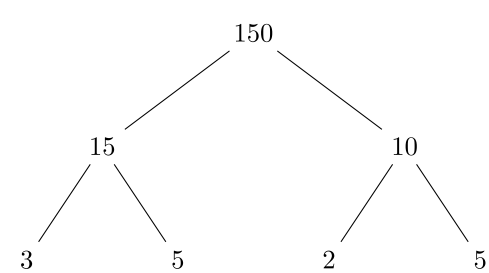
<u>Problem 7</u>

Use the above as a guide to verify the prime factorization for each:

$$\begin{aligned}12&=2\times2\times3\\231&=3\times7\times11\\150&=2\times3\times5\times5\\225&=3\times3\times5\times5\end{aligned}$$

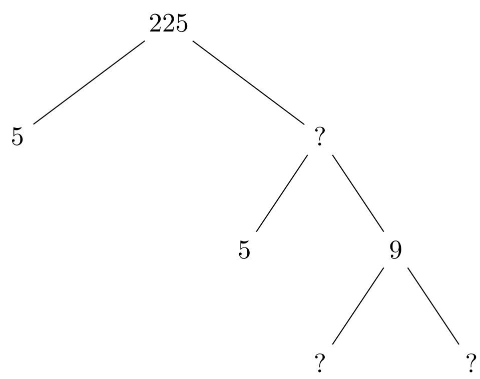
Tree Method

A pictorial technique for prime decomposition is the so-called *tree method*, which represents the factors of a number as ‘branches’. The final extremity of each branch is the set of prime factors as demonstrated for the number 150.



<u>Problem 8</u>

Fill in the missing numbers in the follwing tree method factorization of 225: Answer: 45, 3, 3



Greatest Common Factor

Given a list of real numbers, the *common factors* are those that occur in the factorization (prime or otherwise) of each member on the list. The greatest of these is called the *greatest common factor*, or GCF.

<u>Problem 9</u>

List all factors of 12, 30 and identify the GCF. Answer:

$$\begin{aligned} 12 &\to \mathbf{1}, \mathbf{2}, \mathbf{3}, 4, \mathbf{6}, 12 \\ 30 &\to \mathbf{1}, \mathbf{2}, \mathbf{3}, 5, \mathbf{6}, 10, 15, 30 \\ \text{GCF} &= 6 \end{aligned}$$

<u>Problem 10</u>

List all factors of 15, 30, 105 and identify the GCF. Answer:

$$\begin{aligned} 15 &\to \mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{15} \\ 30 &\to \mathbf{1}, 2, \mathbf{3}, \mathbf{5}, 6, 10, \mathbf{15}, 30 \\ 105 &\to \mathbf{1}, \mathbf{3}, \mathbf{5}, 7, \mathbf{15}, 21, 35, 105 \\ \text{GCF} &= 15 \end{aligned}$$

<u>Problem 11</u>

Use prime factorization to find the GCF of 24 and 108. Answer:

$$\begin{aligned} 24 &= 2 \times (2 \times 2 \times 3) \\ 108 &= (2 \times 2 \times 3) \times 3 \times 3 \\ \text{GCF} &= (2 \times 2 \times 3) = 12 \end{aligned}$$

<u>Problem 12</u>

Bobby has 160 red marbles and 144 blue marbles. He wants to split the marbles into identical groups (not necessarily the same number of each color in a group). What is the greatest number of groups he can make?

Answer: Solve this by finding the GCF of 160 and 144, or:

$$\begin{aligned} 160 &\to 10, \mathbf{16} \\ 144 &\to 9, \mathbf{16} \\ \text{GCF} &= 16 \end{aligned}$$

Thus 16 groups can be made. Each group has $160/16 = 10$ red marbles and $144/16 = 9$ blue marbles.

<u>Problem 13</u>

Alice has 9 forks and 6 spoons. She intends to lay the utensils out in groups around the dinner table with none left over. How many groups can she make, and how many of each utensil per group?

<u>Problem 14</u>

Betty is preparing Halloween treats for her classmates. She has 72 orange candies and 24 red candies in total, and wants to divide them into the greatest number of bags. How many bags can she prepare, and how much of each type of candy should be in each bag?

Least common Multiple

Given a list of real numbers, the *least common multiple* (LCM) is the product of common prime factors raised to the highest power in which they occur. In other words, the LCM is the smallest number that can be evenly divided by two or more numbers.

<u>Problem 15</u>

Find the LCM of 84, 147. Answer:

$$\begin{aligned} 84 &= 2^{\mathbf{3}} \times 3^{\mathbf{1}} \times 7 \\ 147 &= 3 \times 7^{\mathbf{2}} \\ \text{LCM} &= 2^3 \times 3 \times 7^2 = 588 \end{aligned}$$

<u>Problem 16</u>

What is the smallest number that can be divided by 3, 9, 21? Answer:

$$\begin{aligned} 3 &= 3 \\ 9 &= 3^{\mathbf{2}} \\ 21 &= 3 \times 7^{\mathbf{1}} \\ \text{LCM} &= 3^2 \times 7 = 63 \end{aligned}$$

<u>Problem 17</u>

Cindy works in the orchard picking peaches, and fits 8 peaches per bag. Bob has the same job, but fits 9 peaches per bag. At the end of the day, they have picked the same number of peaches. What is the smallest number of peaches they each could have picked? (Hint: Find the LCM of 8, 9.)

Problem 18

For the numbers 2940, 3150, find the GCF and LCM. Answer: GCF = 210, LCM = 210^2

3.4 Properties of Fractions

Any real number that is not an integer is some kind of fraction or decimal. These are numbers such as 1/3, 9.5, −0.6666666, and so on. The conventional symbol for division is the forward-slash (/), equivalent to the numerator-over-denominator notation (top number over the bottom number). That is, the following are equivalent:

$$2/3 \leftrightarrow \frac{2}{3}$$

Multiplying Fractions

When fractions are multiplied, first multiply all terms in the numerator, and then separately multiply all terms in the denominator. The resulting fraction is the ratio of the respective products. For example:

$$\frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{5}{6}$$

$$\frac{2}{3} \times \frac{5}{4} \times \frac{7}{9} = \frac{2 \times 5 \times 7}{3 \times 4 \times 9} = \frac{35}{54}$$

Notion of Reciprocal

The *reciprocal* or *inverse* of a fraction arises when exchanging the numerator and the denominator. For instance, the reciprocal of 2/7 is 7/2. In the most general sense, the reciprocal of any value equals one divided by that value:

$$\text{Reciprocal} = \frac{1}{\text{Fraction}}$$

For instance:

$$\frac{34}{256} = \frac{1}{256/34}$$

Dividing Fractions

With a word like 'reciprocal' under our belt, we think about fractions in a more refined fashion. Consider the equivalent fractions

$$2/3 = \frac{2}{3}$$

that plainly read as 'two over three' or 'two divided by three'. An equivalent interpretation of this fraction would read 'two times one third' because, division by a number is equivalent to multiplying by that number's reciprocal. For our example, this means

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

or

$$2/3 = 2 \times 1/3 \,.$$

Problem 19

Convert to a single fraction and simplify

$$(3/4) \,/\, (2/5) \;.$$

Answer: Identify the reciprocal of the fraction being divided, namely 5/2 Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$(3/4) \,/\, (2/5) = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$$

Problem 20

Convert to a single fraction and simplify

$$\frac{3/5}{7/8} \,.$$

Answer: Identify the reciprocal of the fraction being divided, namely 8/7 Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$\frac{(3/5)}{(7/8)} = \frac{3}{5} \times \frac{8}{7} = \frac{24}{35}$$

Change of Denominator

Fractions can be modified to have a different numerator and denominator, but only if the overall value of the fraction remains the same. For example, we know that

$$\frac{1}{2} \leftrightarrow \frac{3}{6}$$

are *equivalent*, and evaluate to 0.5, yet the actual numbers used in each fraction are different.

The conversion between equivalent fractions is done by multiplying by a carefully chosen factor of one. For example, we convert 1/2 to 3/6 by:

$$\frac{1}{2} = \frac{1}{2} \times \left(\frac{3}{3}\right) = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}$$

Choosing a different factor of one, we can convert to any fraction that is ultimately equivalent to the original ratio. For instance:

$$\frac{1}{2} = \frac{1}{2} \times \left(\frac{13}{13}\right) = \frac{1 \times 13}{2 \times 13} = \frac{13}{26}$$

Problem 21

Consider the ratio 9/3. Change the denominator to 17 without changing the numerical value of the fraction. Answer:

$$\frac{9}{3} = \frac{9/3}{1} \times \left(\frac{17}{17}\right) = \frac{3 \times 17}{17} = \frac{51}{17}$$

Problem 22

Consider the pair of fractions 2/3 and 3/4. Rewrite each fraction such that the denominators are the same.

Answer: The LCM of each denominator 3 and 4 is 12, thus the target denominator is 12. Thus each fraction becomes:

$$\frac{2}{3} = \frac{2}{3} \times \left(\frac{4}{4}\right) = \frac{8}{12}$$
$$\frac{3}{4} = \frac{3}{4} \times \left(\frac{3}{3}\right) = \frac{9}{12}$$

Adding and Subtracting Fractions

There is one single rule for adding and subtracting fractions, and that is: only fractions having the *same* denominator can be added or subtracted.

To demonstrate, consider the true statement:

$$\frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

The reason the above statement executes without error is the denominator of each fraction being added (subtracted in this case) is the same. On the left side, the two terms 3/4 and 1/4 share a denominator (namely 4), thus the two terms combine easily.

Common Denominator

Two fractions can be added or subtracted if they share a *common denominator*. In general, the common denominator is the least common multiple (LCM) of the fractions being combined.

Problem 23

Evaluate the sum of 2/3 and 3/4. Answer:

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12} \, .$$

Problem 24

Evaluate the expression $1/2 + 1/3 - 1/7$. Answer: The LCM of each denominator, namely, 2, 3, and 7 is 42, hence

$$\frac{1}{2} + \frac{1}{3} - \frac{1}{7} = \frac{21}{42} + \frac{14}{42} - \frac{6}{42} = \frac{29}{42} \, .$$

Terminating Decimal Conversion

Consider any real decimal number N where the digits are non-repeating, a case called *terminating*. Given N, there must be some power of ten that, when multiplied by N, results in an integer:

$$M = N \times 10^B = \text{Integer}$$

Then, the original number N can be written in fraction form

$$N = \frac{M}{10^B} \, .$$

For example, if $N = 1.2345$, then we must have $B = 4$ to have integer M, and thus

$$1.2345 = \frac{12345}{10000} \, .$$

3.5 Properties of Exponents

Exponent Notation

When a number is multiplied by itself more than twice, or many times, we don't want to be burdened by writing things like $6 \times 6 \times 6 \times 6$. This is remedied by exponent notation, demonstrated as follows:

$$4 = 4^1$$
$$3 \times 3 = 3^2$$
$$5 \times 5 \times 5 = 5^3$$
$$2 \times 2 \times 9 \times 9 = 2^2 \times 9^2$$

Exponents obey certain conventions that we shall establish here.

- The *base* number is the 'lower' number to which the exponent applies.
- The *exponent* is written smaller and to the upper-right of the base number.

Special Exponents

- When the exponent is zero, the result equals one:
  $$N^0 = 1$$
- When the exponent is one, the result equals the base number:
  $$N^1 = N$$
- When the exponent is 2, the result is the *square* of the base number.
- When the exponent is 3, the result is the *cube* of the base number.

Negative Base Number

Negative numbers with exponents must be treated carefully. For instance, if asked to square the number -3, one may write

$$(-3)^2 = -3 \times -3 = (-1)^2 \times 3^2 = 9\,,$$

which has the same result as (positive) $3^2 = 9$. Indeed, raising to any base number *even* power results in a positive number.

On the other hand, if a negative base number is raised to an *odd* number, factors of -1 occur an odd number of times, as in

$$(-2)^3 = (-1)^3 \times 2^3 = -8\,.$$

Beware the placement of parentheses when handling exponents. For instance the number -2^4 translates to

$$-2 \times 2 \times 2 \times 2 = -16\,.$$

However, if we were handed $(-2)^4$, the full statement would be

$$(-2) \times (-2) \times (-2) \times (-2) = 16\,.$$

Multiplying Numbers with Exponents

For multiplying two numbers having equal base and unequal exponents, the product is the base number raised to the sum of the exponents:

$$A^B \times A^C = A^{B+C} \tag{1.4}$$

For example, the product $3^2 \times 3^4$ can be immediately translated to 3^6:

$$(3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^{2+4} = 3^6$$

Negative Exponent

A number raised to a negative exponent is equivalent to the reciprocal of that number raised to the positive exponent:

$$A^{-B} = \left(\frac{1}{A}\right)^B = \frac{1}{A^B} \tag{1.5}$$

For an example, one may have

$$6^2 \times 3^{-2} = 6^2 \times \left(\frac{1}{3}\right)^2 = \left(\frac{6}{3}\right)^2 = 4\,,$$

or similarly

$$6^2 \times 3^{-2} = \frac{6^2}{3^2} = \left(\frac{6}{3}\right)^2 = 4\,.$$

Compound Exponent

A number that is raised to an exponent, and then raised to an exponent *again*, as in

$$\left(2^3\right)^2 = 8^2 = 64$$

is equivalent to *multiplying* the two exponents:

$$\left(2^3\right)^2 = 2^{3\times 2} = 2^6 = 64$$

In general the rule for compound exponents reads:

$$\left(A^B\right)^C = A^{B\times C} \tag{1.6}$$

Zero Exponent

By the established properties of exponents, it can only be that any number raised to the zero power equals one. In detail, consider the trivial ratio A^B/A^B, which simplifies according to:

$$1 = \frac{A^B}{A^B} = A^B \times \frac{1}{A^B} = A^{B-B} = A^0$$

Radicals

Numbers raised to non-integer exponents are sometimes called *radicals*. Valid radical numbers may look like:

$$3^{.5} = \sqrt{3}$$
$$4^{3/2} = \left(\sqrt{4}\right)^3 = \sqrt[3]{4}$$
$$7^{-1.5} = 7^{-3/2} = \frac{1}{7^{3/2}} = \frac{1}{\sqrt[3]{7}}$$

The base number is also denoted the *radicand*, and the exponent is called the *degree* of the radical. In the general case, we write:

$$A^{B/2} = \sqrt[B]{A} \tag{1.7}$$

Roots

When the exponent is precisely $1/2$, the result of the operation is the *square root* of the base number. Similarly, if the exponent is precisely $1/3$, the result of the operation is the *cube root* of the base number, and so on.

Square Root Table

The exponent 1/2 is common enough to justify having the following table:

x	$\sqrt{x}$	x	$\sqrt{x}$
1	1	26	5.0990195135
2	1.4142135620	27	5.1961524227
3	1.7320508075	28	5.2915026221
4	2	29	5.3851648071
5	2.2360679774	30	5.4772255750
6	2.4494897427	31	5.5677643628
7	2.6457513110	32	5.6568542494
8	2.8284271247	33	5.7445626465
9	3	34	5.8309518948
10	3.1622776601	35	5.9160797830
11	3.3166247903	36	6
12	3.4641016151	37	6.0827625302
13	3.6055512754	38	6.1644140029
14	3.7416573867	39	6.2449979983
15	3.8729833462	40	6.3245553203
16	4	41	6.4031242374
17	4.1231056256	42	6.4807406984
18	4.2426406871	43	6.5574385243
19	4.3588989435	44	6.6332495807
20	4.4721359549	45	6.7082039324
21	4.5825756949	46	6.7823299831
22	4.6904157598	47	6.8556546004
23	4.7958315233	48	6.9282032302
24	4.8989794855	49	7
25	5	50	7.0710678118

3.6 Logarithm Operator

Consider the equation

$$N = A^B ,$$

where A is the base number and B is the exponent. The *logarithm operator* can be used to solve for B in terms of A and N:

$$\log_A(N) = B$$

In words, the above reads 'log base A of N equals B'. To have it all in one statement, one may write:

$$\log_A\left(A^B\right) = B \tag{1.8}$$

Inverse Logarithm

The inverse to the logarithm is the exponent operator. To invert the above, start with the base number A and use each side as the exponent in the equation:

$$A^{\log_A(N)} = N = A^B$$

Isolating the Base

To solve for A, raise each side of the above to the $1/B$ power:

$$N^{1/B} = A^{B/B} = A$$

Log of One

Right away, we can establish an important identity for when the exponent is zero:

$$\log_A(1) = 0$$

That is, for any base number, the log of one equals zero.

Log Equals One

For another identity we start with $B = 1$ to have $N = A^1$, and thus

$$\log_A(A) = 1 ,$$

true for any base number.

Log of Zero

One should note that the logarithm of zero is undefined. In particular, suppose we have

$$0 = A^B .$$

Apart from the trivial solution of $A = 0$, $B \neq 0$, the only way to satisfy the above is to have B equal to an infinitely large negative number. For this reason, we establish

$$\log_A(0) = -\infty ,$$

which is technically undefined.

Base-Ten Logarithm

The base-ten logarithm operator quickly isolates the magnitude of any real number. For instance:

$$\log_{10}(1000) = 3$$
$$\log_{10}(10) = 1$$
$$\log_{10}\left(\frac{1}{1000}\right) = -3$$
$$\log_{10}\left(10^{-8}\right) = -8$$

Change of Base

From the equation $N = A^B$, we can attain an equivalent form $N = C^D$ by a *change of base*, which implies the exponent changes as well. Start with

$$\log_A(N) = B$$

to eliminate B from the original equation:

$$N = A^{\log_A(N)}$$

In this form, we see A can be replaced with any base number C without changing anything, i.e.

$$N = C^{\log_C(N)} .$$

From here we see

$$D = \log_C(N) ,$$

and solving for the base C yields

$$C = N^{1/D} .$$

Addition Identity

Consider the sum

$$N = \log_A(B) + \log_A(C) .$$

To simplify, use each side as an exponent with A as the base number:

$$A^N = A^{\log_A(B)+\log_A(C)} ,$$

simplifying to

$$A^N = A^{\log_A(B)} \times A^{\log_A(C)} = BC .$$

Apply the $\log_A$ operator to each side to find

$$N = \log_A(BC) ,$$

and in conclusion:

$$\log_A(B) + \log_A(C) = \log_A(BC) \tag{1.9}$$

Not surprisingly, multiplying the product BC by a factor $D = 1$ leaves the result unchanged:

$$\log_A(B) + \log_A(C) + \cancel{\log_A(1)} = \log_A(BC \times 1)$$

Subtraction Identity

Consider the difference

$$N = \log_A(B) - \log_A(C) .$$

To simplify, use each side as an exponent with A as the base number:

$$A^N = A^{\log_A(B)-\log_A(C)} ,$$

simplifying to

$$A^N = \frac{A^{\log_A(B)}}{A^{\log_A(C)}} = \frac{B}{C} .$$

Apply the $\log_A$ operator to each side to find

$$N = \log_A\left(\frac{B}{C}\right) ,$$

and in conclusion:

$$\log_A(B) - \log_A(C) = \log_A\left(\frac{B}{C}\right) \tag{1.10}$$

Product Identity

Start with the equation

$$N = C \times \log_A(B) ,$$

and use each side as an exponent with A as the base number:

$$A^N = A^{C\times\log_A(B)}$$

Recognizing the compound exponent, the above is also

$$A^N = \left(A^{\log_A(B)}\right)^C = B^C .$$

Apply the $\log_A$ operator to each side and simplify:

$$N = \log A\left(B^C\right)$$

Eliminating N gives the product identity:

$$C \times \log_A(B) = \log_A\left(B^C\right) \tag{1.11}$$

Natural Logarithm

There is a special base number called *Euler's constant*, namely

$$e \approx 2.7182818284590\ldots .$$

Precisely *why* this number is important is beyond the scope of this discussion. Nonetheless, when the base number is equal to e, the corresponding operation is called the *natural logarithm*:

$$\log_e(B) = \ln(B)$$

Correspondingly, we also have

$$e^{\ln(B)} = B$$
$$\ln\left(e^B\right) = B .$$

It's sometimes useful to convert a logarithm to base e. For a number N, take any base number A and use the $\log_A$ operator to establish

$$\log_A(N) = \log_A\left(e^{\ln(N)}\right) = \ln(N) \times \log_A(e) \ ,$$

or

$$\ln(N) = \frac{\log_A(N)}{\log_A(e)} \ .$$

Similarly, one may write

$$\ln(N) = \ln\left(A^{\log_A(N)}\right)$$

to find

$$\ln(N) = \log_A(N) \times \ln(A) \ ,$$

or equivalently

$$\log_A(N) = \frac{\ln(N)}{\ln(A)}$$

Problem 25

Derive the identity:

$$\ln(A) \times \log_A(e) = 1$$

Problem 26

For two base numbers A, B, derive

$$\log_A(N) = \log_B(N) \times \log_A(B)$$
$$\log_B(N) = \log_A(N) \times \log_B(A) \ ,$$

and also:

$$\log_A(B) \times \log_B(A) = 1 \tag{1.12}$$

Problem 27

Given

$$5^x = 17 \ ,$$

solve for x. Answer:

$$x = \frac{\log_A(17)}{\log_A(5)}$$

Problem 28

Spot the error in the following 'proof' that: $1 = 2 = 3 = \cdots$

$$1 = 1^2 = 1^3 = 1^4 = \cdots$$
$$\ln(1) = \ln\left(1^2\right) = \ln\left(1^3\right) = \ln\left(1^4\right) = \cdots$$
$$\ln(1) = 2\ln(1) = 3\ln(1) = 4\ln(1) = \cdots$$
$$\cancel{\ln(1)} = 2\cancel{\ln(1)} = 3\cancel{\ln(1)} = 4\cancel{\ln(1)} = \cdots$$
$$1 = 2 = 3 = 4 = \cdots$$

4 Variables and Equations

A *variable* is any symbol used to represent a number (or anything). Common symbols used are familiar letters x, y, z, a, b, c - and some unfamiliar (Greek) letters α, β, γ, etc.

Variables behave exactly as numbers do: any place where a number is needed, a variable will work just as well.

Variable Notation

When a variable is multiplied by a number or another variable, the multiplication symbol is usually ignored. This means $4 \times x$ or $4 \cdot x$ should be written $4x$. When a variable appears alone, there is always an invisible 1 next to it. That is, x is equivalent to $1x$ or $x/1$.

Coefficients

When a variable is multiplied by a number, i.e. $3x$, $-5y$, etc., the leading number is called a *coefficient*. When a variable is 'by itself', there is an implied coefficient of one.

Problem 29

Show that the GCF and LCM of $4a^2b$, $6ab$, $8ab^2$ are:

$$\text{GCF} = 2 \times a \times b = 2ab$$
$$\text{LCM} = 2^3 \times 3 \times a^2 \times b^2 = 24a^2b^2$$

Variable Assignment

To *set a variable* is to associate a number with a symbol. For example, the statement

$$x = 3$$

will 'tie' to the value of 3 to the symbol x. Any time x occurs, it really means 3. The value of a variable is the number represented by the symbol. That is, we say 3 is the 'value' of x.

Variable names such as x, y, z, etc. are heavily recycled. That is, the same letters are prone to show up in different problems. Needless to mention, variables in one problem are not the same in a different problem. (They wouldn't be variables otherwise.)

Combining Like Terms

When an expression contains two or more instances of the same variable raised to the same exponent, the coefficients may be added (or subtracted) in a process called *combining like terms*.

To demonstrate, consider the expression:

$$2x - 5x + 7x^2$$

Notice that $2x$ and $-5x$ each contain a single instance of x. It follows that these terms may be added together to give $-3x$. The $7x^2$ term can't be combined with anything, because there are no other x^2 terms in play. The simplified expression is then:

$$-3x + 7x^2$$

Problem 30

Add the two trinomials:

$$\left(2x^2 + 2y^2 - 3\right) + \left(y^2 - 9x^2 - 2\right)$$

to get

$$-7x^2 + 4y^2 - 5 \,.$$

Problem 31

Add the two polynomials

$$\left(2x^2 + 2y^2 - 3\right) - \left(2y^2 - 4x^2\right)$$

to get

$$6x^2 - 3 \,.$$

Problem 32

Multiply

$$(y - 4)\left(y^2 - 3x - 2\right)$$

to get

$$y^3 - 4y^2 - 3xy + 12x - 2y + 8 \,.$$

4.1 Manipulating an Equation

Formally, an *equation* is made of two expressions joined by an equal sign. However 'ugly' an equation may appear, the left side and right side are equal. Examples of equations are:

$$\begin{aligned} 5 + 5 &= 10 \\ 3 - x &= 2 \\ x^2 &= 24 + 25 \end{aligned}$$

Whatever manipulations are performed on an equation, the information it contains must not change: the left side must always equal the right side. The number of allowed manipulations boil down to just *two*: you may add zero to an equation, or multiply one into an equation, that's it. In the following we unpack this notion.

Adding or Subtracting a Number

Adding or subtracting the same number on both sides of an equation is the same as adding zero. For example, consider the equation

$$x - 5 = 3 \,.$$

If we simply add 5 to each side, the equation becomes

$$x \cancel{- 5 + 5} = 3 + 5 \,,$$

which simplifies very nicely:

$$x = 8$$

By isolating x, we have successfully 'solved for' its value.

Multiplying or Dividing a Number

Multiplying or dividing the same number on both sides of an equation is the same as multiplying by one. For example, let us isolate x in the equation

$$2x = 24 \,.$$

To do so, multiply both sides by a factor of $1/2$ to get

$$\cancel{\left(\frac{1}{2} \times 2\right)} x = \frac{1}{2} \times 24 \,,$$

and the coefficients cancel out on the left:

$$x = 12$$

Of course, one exception to the above occurs when attempting to divide by zero. If you ever do this, turn back, it's an error.

Raising to a Power

Raising each side of an equation to a power is another allowed operation. For example, consider the equation:

$$\sqrt{x} = 3$$

If we raise each side to the power 2, the equation becomes

$$\left(\sqrt{x}\right)^2 = 3^2 \,,$$

or

$$\left(x^{1/2}\right)^2 = x = 3^2 \,,$$

and thus $x = 9$.

Utilizing Logarithm

The logarithm operators $\log_A$, ln, can also be applied to each side of an equation. Recycling the previous example, we could solve $\sqrt{x} = 3$ by writing

$$\log_x \left(\sqrt{x}\right) = \frac{1}{2} \log_x (x) = \log_x (3) \ ,$$

or

$$\log_x (x) = \log_x \left(3^2\right) \ ,$$

reducing (again) to $x = 9$.

Taking a Root

Taking the square root (or cube root, etc.) of each side of an equation is an allowed operation for positive numbers. For example, consider the equation:

$$x^2 = 49$$

Putting the square root symbol around the entire left side and the entire right side, the equation becomes

$$\sqrt{x^2} = \sqrt{49} \, ,$$

where the square root and the 2-power cancel, giving

$$\sqrt{x^2} = x = 7 \, .$$

Paying closer attention, note that $x = -7$ also satisfies the original equation $x^2 = 49$. This in fact true for all square roots: whenever we take the square root of a number, there are *two* results, one positive and one negative. In the most general case, we will always have

$$\sqrt{a^2} = \begin{cases} +a \\ -a \end{cases} \, , \tag{1.13}$$

usually condensed using the 'plus-or-minus' symbol ($\pm$):

$$\sqrt{a^2} = \pm a$$

Strategy

To summarize the allowed manipulations on an equation, we say **you may multiply by one or add zero**. Anything else is is error. Do not be sloppy when applying operators across an equation. To illustrate, consider the equation

$$3x + 6 = 12y + 18 \, .$$

There are many ways to manipulate the equation above, but for the same of demonstration, let us try the square root operation. It would be WRONG to try:

$$\sqrt{3x} + \sqrt{6} = \sqrt{12y} + \sqrt{18} \, ,$$

as this approach forgets to treat the left side and right side equally. To proceed correctly, we must write

$$\sqrt{3x + 6} = \sqrt{12y + 18} \, .$$

Conversely, if we wanted to square each side of the equation, it would be WRONG to try

$$3x^2 + 6^2 = 12y^2 + 18^2 \, ,$$

which again forgets to treat each side equally. Instead, we must have

$$(3x + 6)^2 = (12y + 18)^2 \ .$$

Problem 33

Given $a = b$, spot the error in the following 'proof' that $2 = 1$:

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a - b)(a + b) &= b(a - b) \\ \cancel{(a - b)}(a + b) &= b\cancel{(a - b)} \\ a + b &= b \\ b + b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

4.2 Solving for a Variable

When a variable is embedded in an equation, we often need to determine its exact value to solve a problem. This is done by manipulating the equation so as to 'get the variable by itself' on the left side or the right side, a process called *solving for a variable.*

Problem 34

Solve for x:

$$\frac{4x}{5} + \frac{7x}{9} = 4$$

Step 1: Rewrite fractions with a common denominator:

$$\frac{36x}{45} + \frac{35x}{45} = 4$$

Step 2: Combine like terms:

$$\frac{71x}{45} = 4$$

Step 3: Multiply both sides by 45:

$$\frac{71x}{\cancel{45}} \times \cancel{45} = 4 \times 45$$

Step 4: Divide both sides by 71 and simplify:

$$x = \frac{4 \times 45}{71} = \frac{180}{71}$$

Problem 35

Solve for x:

$$6 = 2\sqrt{x+3}$$

Step 1: Divide by 2 on both sides:

$$3 = \sqrt{x+3}$$

Step 2: Raise each side to the power 2:

$$3^2 = \left(\sqrt{x+3}\right)^2$$

Step 3: Simplify:

$$9 = x + 3$$

Step 4: Subtract 3 from both sides:

$$6 = x$$

Problem 36

Find two solutions for x:

$$x^2 + 3 = 12$$

Step 1: Subtract 3 from both sides:

$$x^2 = 9$$

Step 2: Take the square root of each side:

$$\sqrt{x^2} = \sqrt{9}$$

Step 3: Simplify to write the result:

$$x = \pm 3$$

Problem 37

If $y = 2$, determine all solutions for x:

$$x^2 - 10y - 5 = 0$$

Step 1: Add $10y + 5$ to both sides:

$$x^2 = 10y + 5$$

Step 2: Take the square root of both sides:

$$x = \pm\sqrt{10y + 5}$$

Step 3: Substitute $y = 2$ and simplify:

$$x = \pm\sqrt{25}$$

Step 4: Use the a table, calculator, or memory to write the final answer:

$$x = \pm 5$$

Problem 38

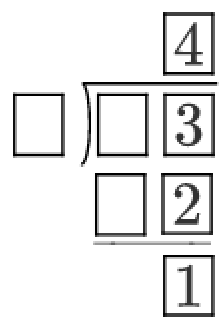
What three digits are represented by X, Y, and Z in the addition problem shown?

$$\begin{array}{r} \mathbf{XZY} \\ +\ \mathbf{XYZ} \\ \hline \mathbf{YZX} \end{array}$$

Answer: The first column indicates $Y \neq 0$. The second column suggests $Z + Y \geq 10$, and furthermore $10 + 10Z + 10Y = 10Z + 100$, giving $Y = 9$. Thus, the first column is only satisfied by $X = 4$. Finally, the third column tells us $Y + Z = 10 + X$, or $Z = 5$.

Problem 39

Shown is a long division with some digits hidden. Which of $(5, 6, 7, 8)$ could represent one of the empty boxes?



Answer: Label the divisor as x, where y and z are natural numbers. Observe $10x + 3 - (10z + 2) = 1$ to conclude $y = z$. Observe next that

$$\frac{10y + 3}{x} = 4 + \frac{1}{x},$$

which only delivers a natural number if $y = 3$ and $x = 8$.

Problem 40

At temperature T in Celsius, the speed of sound v in air (in meters per second) is given by

$$v \approx 331.1 \times \sqrt{1 + \frac{T}{273.15}}.$$

If the speed of sound in a certain place is measure to be exactly 350 m/s, calculate the required temperature.

Problem 41

The area of a circle is πR^2, where $\pi \approx 3.14$ and R is the radius (center-to-edge distance). If the area of a certain pizza is 100 units, calculate the length of a slice (same as R).

4.3 Systems of Equations

When handed multiple equations containing multiple variables, this is called a system of equations. The number of equations/variables is called the *order* of the system. For example, below we have an order-two system, having two equations and two variables:

$$4x + 2y = 20$$
$$x + 3y = 15$$

There is a reliable technique for solving a system of two equations and two variables. Using the above system as an example, the following procedure generally works:

1. Solve either equation for x or y, whichever is easier.

$$y = 10 - 2x$$
$$x + 3y = 15$$

2. Substitute the isolated variable into the unused equation.

$$x + 3(10 - 2x) = 15$$

3. The resulting equation should contain only one variable:

$$x = 3$$

4. Substitute the known variable value into either of the original equations.

$$3 + 3y = 15$$

5. Solve for the last unknown:

$$y = 4$$

If the number of equations/variables does not match, the system is either *underdetermined* (not enough information), or *overdetermined* (conflicting information). In either case, there is no clear solution to the system.

Problem 42

For the linear system of equations

$$2x + 3y = 19$$
$$4x - y = 3\,,$$

show that the solution to the system is $x = 2$, $y = 5$.

Celsius and Fahrenheit

In standard conditions, water freezes at $0°\,C = 32°\,F$, and it boils at $100°\,C = 212°\,F$. The two temperature scales (Fahrenheit and Centigrade) obey a linear relation

$$T_F = m \cdot T_C + b\,.$$

Use a system of equation to determine m and b (as in $y = mx + b$).

Generate two equations and two unknowns:

$$32°F = m \cdot 0 + b$$
$$212°F = m \cdot 100°C + b$$

First eliminate b to solve for m:

$$m = \frac{180°F}{100°C}$$

Next solve for b to get

$$b = 32°F\,,$$

and write the final answer:

$$T_F = \frac{9}{5} \cdot T_C + 32°F$$

Problem 43

Solve for T_C in terms of T_F.

4.4 Linearity of Equations

Linear System

A(n) (system of) equation(s) is considered *linear* if all variables x, y, z, etc. have an exponent of exactly *one*. A linear equation should never have terms such $\sqrt{x}$, or $x \cdot y$, or z^2, as these would be considered *nonlinear* terms. For example, the set of equations

$$4x + 2y = 20$$
$$x + 3y = 15$$

qualifies as an order-two linear system, and can always be solved by the procedure outlined above.

Nonlinear System

On the other hand, the system

$$\begin{aligned} x^2 + y &= 21 \\ xy &= 20 \end{aligned}$$

contains two nonlinear equations due to the x^2- and xy-terms. Nonetheless, the solution can be attained by reducing the system to one equation with one variable. In this case, substitute $y = 20/x$ into the first equation and simplify to get

$$x^3 - 21x + 20 = 0 ,$$

which is still nonlinear, but *can* be solved. (There are two solutions, one with $x = 1$ and the other with $x = 4$.)

Transcendental Equation

An equation is *transcendental* if the variable(s) being solved for cannot be isolated by the usual tools of algebra. For instance, the equation

$$2^x = x^2$$

qualifies as transcendental, as no ordinary process can be used to isolate x. Many transcendental equations can still be solved by analytic means, or when that fails, approximated by computer.

5 Rate, Interval, and Work

5.1 Units and Dimension

A number that is measured in a countable quantity such as dollars, seconds, or pounds are said to carry *units*, also called *dimension*. A plain number such as 3 (ant not 3 dollars or seconds) is called *dimensionless*.

Unit quantities can be manipulated as one would do with variables, and are subject to the same rules. For instance, we may add numbers carrying units by combining like terms as in

$$2\,\text{meter} + 3\,\text{meter} = 5\,\text{meter} ,$$

or multiply as in

$$2\,\text{second} \times 3\,\text{second} = 6\,\text{second}^2 .$$

As long as we obey the rules for combining fractions, rates can also be combined, as in

$$14\,\frac{\text{dollar}}{\text{hour}} - 3\,\frac{\text{dollar}}{\text{hour}} = 11\,\frac{\text{dollar}}{\text{hour}} .$$

Using fewer letters, the above calculations are identical to

$$\begin{aligned} 2x + 3x &= 5x \\ 2y \times 3y &= 6y^2 \\ 14z - 3z &= 11z , \end{aligned}$$

where x replaces meter, y replaces second, and z replaces dollar/hour.

5.2 Rate

A *rate* is defined as some quantity, such as a cost, weight, or area, measured against another quantity, such as time or distance. Rates are *always* ratios, containing two pieces of information, hence the similarity in name. Common examples of rates may occur as:

$$\begin{aligned} \text{Hourly Wage} &= 14\,\text{dollar/hour} \\ \text{Speed} &= 15\,\text{meter/second} \\ \text{Pressure} &= 55\,\text{pound/inch}^2 \\ \text{Density} &= 55\,\text{gram/cm}^3 \end{aligned}$$

In general, we shall denote a rate with the variable letter R. For instance, for a car traveling at 40 miles per hour (abbreviated mi/hr or mph), we may write

$$R = 40\,\frac{\text{mi}}{\text{hr}} = 40\,\text{mph} .$$

5.3 Interval

Looking again at the definition of a rate, there is a numerator of a certain dimension (dollars, meters, etc.), along with a denominator of a different dimension (usually *time*, but not always.)

For any given rate R, we can conceive a number called an *interval*, denoted I, having units matching those in the denominator of R. For instance, consider an example rate

$$R = 11\,\frac{\text{dollar}}{\text{hour}} .$$

To construct an interval, choose *some* quantity I measured specifically in hours, perhaps

$$I = 3\,\text{hour} .$$

By computing the product $R \times I$, we find

$$R \times I = 11\,\frac{\text{dollar}}{\cancel{\text{hour}}} \times 3\,\cancel{\text{hour}} = 33\,\text{dollar} ,$$

which cancels the ‘hours’ unit altogether, leaving the result in only dollars.

5.4 Work

The product of the rate and the interval is called *work*, denoted W:

$$W = R \times I \tag{1.14}$$

Reiterating the previous example, we say that a rate of $R = 11\,\text{dollar/hour}$ multiplied by an interval of $I = 3\,\text{hour}$ amounts to the work value of $W = 33\,\text{dollar}$.

The so-called 'work equation' can be solved for R, namely

$$R = \frac{W}{I}\,,$$

telling us that a rate can be interpreted as the work value divided by the interval. Similarly, the interval

$$I = \frac{W}{R}$$

is the ratio of the work value to the rate.

Problem 44

Joseph works as a journeyman electrician for a pay rate of 30 dollars per hour. After a 9-hour day, what are his daily earnings?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= 30\,\frac{\text{dollar}}{\text{hour}} \\ I &= 9\,\text{hour} \\ W &= ? \end{aligned}$$

Step 2: Apply the work equation with W as the unknown:

$$W = R \times I = 30\,\frac{\text{dollar}}{\text{hour}} \times 9\,\text{hour}$$

Step 3: Cancel the hour units, and simplify the fraction:

$$\begin{aligned} W = R \times I &= 30\,\frac{\text{dollar}}{\cancel{\text{hour}}} \times 9\,\cancel{\text{hour}} \\ &= 30 \times 9\,\text{dollar} = 270\,\text{dollar} \end{aligned}$$

Problem 45

Joseph keeps working as a journeyman electrician for a pay rate of 30 dollars per hour. After a long workday, his earnings were 330 dollars. How many hours did he work?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= 30\,\frac{\text{dollar}}{\text{hour}} \\ I &= ? \\ W &= 330\,\text{dollar} \end{aligned}$$

Step 2: Apply the work equation with I as the unknown:

$$I = \frac{W}{R} = \frac{330\,\text{dollar}}{30\,\text{dollar/hour}}$$

Step 3: Cancel the dollar units, and simplify the fraction:

$$I = \frac{W}{R} = \frac{330\,\cancel{\text{dollar}}}{30\,\cancel{\text{dollar}}/\text{hour}} = \frac{330}{30}\,\text{hour} = 11\,\text{hour}$$

Problem 46

Joseph agrees to work on a special project for 10 hours. His earnings for the project were 350 dollars. Calculate the pay rate for the special project.

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R &= ? \\ I &= 10\,\text{hour} \\ W &= 350\,\text{dollar} \end{aligned}$$

Step 2: Apply the work equation with R as the unknown:

$$R = \frac{W}{I} = \frac{350\,\text{dollar}}{10\,\text{hour}}$$

Step 3: Simplify to write the answer:

$$R = 35\,\frac{\text{dollar}}{\text{hour}}$$

Problem 47

A bee can visit 644 flowers in 7 hours. How many flowers can the bee visit in 9 hours?

Step 1: Set up a work equation containing known information

$$\begin{aligned} W &= R \times I \\ 644\,\text{flower} &= R \times 7\,\text{hour} \end{aligned}$$

Step 2: Solve for the rate R:

$$R = \frac{644}{7}\,\frac{\text{flower}}{\text{hour}} = 92\,\frac{\text{flower}}{\text{hour}}$$

Step 3: Step up a work equation with R known and $I = 9\,\text{hour}$ to get the result:

$$\begin{aligned} W &= R \times I \\ W &= 92\,\frac{\text{flower}}{\cancel{\text{hour}}} \times 9\,\cancel{\text{hour}} \\ W &= 92 \times 9\,\text{flower} = 828\,\text{flower} \end{aligned}$$

Mixed Units

Certain problems call for a rate-like quantity to have mixed units.

Suppose a certain brand of concentrated iced tea advises that 2 oz of product mixed with 12 oz of water makes the perfect serving. How much of each ingredient is needed to fill a 64 oz container at the proper ratio?

Proceed by writing an equation containing known information:

$$14\text{ oz tea} = 2\text{ oz product} + 12\text{ oz water} ,$$

and then divide each side side by the number 14 to write a 'unit work' equation:

$$1\text{ oz tea} = \frac{1}{7}\text{ oz product} + \frac{6}{7}\text{ oz water}$$

We can interpret the work value as 1 oz tea, but the quantity $I \times R$ is tangled up in the sum on the right. The natural choice for the I-variable is the number of ounces of each ingredient, so we write the above in equivalent form:

$$(1\text{ oz})\ \text{tea} = \left(\frac{1}{7}\text{ product} + \frac{6}{7}\text{ water}\right) \times (1\text{ oz})$$

Now, it's clear that 1 oz can be replaced with any 'work' quantity, not restricted to ounces. Choosing 64 oz, we find

$$\begin{aligned}(64\text{ oz})\ \text{tea} &= \left(\frac{1}{7}\text{ product} + \frac{6}{7}\text{ water}\right) \times (64\text{ oz}) \\ &= 9.14\text{ oz product} + 54.86\text{ oz water}\end{aligned}$$

5.5 Adding Rates

Certain situations call for individual rates to be combined into an *effective rate*. That is, if we have a handful of compatible rates R_1, R_2, etc., the effective rate is

$$R_{\text{eff}} = R_1 + R_2 + \cdots ,$$

obeying the same work equation

$$W = R_{\text{eff}} \times I .$$

The obvious assumption is that all rates being combined share the same units. It makes no sense (in any context) to add rates with dissimilar units.

Problem 48

An inlet pipe can fill a swimming pool in 5 hours, while an outlet pipe can drain the pool in 8 hours. By mistake, a maintenance worker left both pipes open. Will the pool overflow?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned}R &= R_{\text{eff}} = R_{fill} + R_{drain} \\ I &= \text{ (not needed)} \\ W &= 1\text{ pool}\end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 5, 8.)

$$\begin{aligned}R_{\text{eff}} &= \frac{1\text{ pool}}{5\text{ hour}} - \frac{1\text{ pool}}{8\text{ hour}} \\ R_{\text{eff}} &= \frac{8\text{ pool}}{40\text{ hour}} - \frac{5\text{ pool}}{40\text{ hour}} \\ R_{\text{eff}} &= \frac{3\text{ pool}}{40\text{ hour}} = 0.075\,\frac{\text{pool}}{\text{hour}}\end{aligned}$$

Step 3: Observe that R_{eff} came out positive, meaning the pool will overflow.

Problem 49

A cold water faucet can fill a bathtub in 12 minutes, and a hot water faucet can fill the bathtub in 18 minutes. The drain can empty the bathtub in 24 minutes. If both faucets are on and the drain is open, how long would it take to fill the bathtub?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned}R &= R_{\text{eff}} = R_{hot} + R_{cold} + R_{drain} \\ I &= ? \\ W &= 1\text{ tub}\end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 12, 18, and 24.)

$$\begin{aligned}R_{\text{eff}} &= \frac{1\text{ tub}}{12\text{ minute}} + \frac{1\text{ tub}}{18\text{ minute}} - \frac{1\text{ tub}}{24\text{ minute}} \\ R_{\text{eff}} &= \frac{6\text{ tub}}{72\text{ minute}} + \frac{4\text{ tub}}{72\text{ minute}} - \frac{3\text{ tub}}{24\text{ minute}} \\ R_{\text{eff}} &= \frac{7\text{ tub}}{72\text{ minute}}\end{aligned}$$

Step 3: Apply the work equation with I as the unknown:

$$I = \frac{W}{R_{\text{eff}}} = W \times \frac{1}{R_{\text{eff}}} = 1\text{ tub} \times \frac{72\text{ minute}}{7\text{ tub}}$$

Step 4: Cancel the tub units, and simplify the fraction:

$$\begin{aligned}I &= 1\,\cancel{\text{tub}} \times \frac{72\text{ minute}}{7\,\cancel{\text{tub}}} \\ &= \frac{72}{7}\text{ minute} = 10.29\text{ minute}\end{aligned}$$

Problem 50

It takes Tom 4 hours to build a fence. If he hires Jack to help him, together they can do the job in just 3 hours. If Jack built the same fence alone, how long would it take him?

Step 1: Discern the following equations from the problem statement:

$$\begin{aligned} 1\,\text{fence} &= R_{Tom} \times 4\,\text{hour} \\ 1\,\text{fence} &= (R_{Tom} + R_{Jack}) \times 3\,\text{hour} \\ 1\,\text{fence} &= R_{Jack} \times I_{Jack} \end{aligned}$$

Step 2: Solve the middle equation for R_{Jack}:

$$\begin{aligned} R_{Jack} &= \frac{1\,\text{fence}}{3\,\text{hour}} - R_{Tom} \\ &= \frac{1\,\text{fence}}{3\,\text{hour}} - \frac{1\,\text{fence}}{4\,\text{hour}} = \frac{1\,\text{fence}}{12\,\text{hour}} \end{aligned}$$

Step 3: Solve the third equation for I_{Jack}:

$$\begin{aligned} I_{Jack} &= \frac{1\,\text{fence}}{R_{Jack}} \\ &= 1\,\cancel{\text{fence}} \times \frac{12\,\text{hour}}{1\,\cancel{\text{fence}}} = 12\,\text{hour} \end{aligned}$$

Problem 51

A woodworking shop is open eight hours per day. Tony can build three birdhouses in the eight-hour day. Joe is new, and accidentally destroys one birdhouse every four hours. If the shop hires a third builder for six hours per day, how many birdhouses must he build during his shift in order for the shop to produce five birdhouses in an eight-hour day? Hint: Let x be the number of birdhouses the new worker builds per six hours.

Step 1: Let H denote 'house' and 'hr' stand for 'hour'. Identify the rate, the interval, and the work value:

$$\begin{aligned} R = R_{\text{eff}} &= R_{\text{Tony}} + R_{\text{Joe}} + \frac{x\,H}{6\,\text{hr}} \\ I &= 8\,\text{hr} \\ W &= 5\,H \end{aligned}$$

Step 2: Write the work equation using the information provided and simplify:

$$\begin{aligned} W &= R_{\text{eff}} \times I \\ 5\,H &= \left(\frac{3\,H}{8\,\text{hr}} - \frac{1\,H}{4\,\text{hr}} + \frac{x\,H}{6\,\text{hr}}\right) \times 8\,\text{hr} \\ 5\,\cancel{H} &= \left(\frac{3\,\cancel{H}}{8\,\cancel{\text{hr}}} - \frac{1\,\cancel{H}}{4\,\cancel{\text{hr}}} + \frac{x\,\cancel{H}}{6\,\cancel{\text{hr}}}\right) \times 8\,\cancel{\text{hr}} \\ 5 &= \left(\frac{3}{8} - \frac{1}{4} + \frac{x}{6}\right) \times 8 \end{aligned}$$

Step 3: Solve the above for x:

$$x = 6 \cdot \left(\frac{5}{8} - \frac{3}{8} + \frac{1}{4}\right) = 3$$

Problem 52

On a canoe trip, you paddle upstream for one mile, at which point your hat falls into the river. Ten minutes later, you turn around and catch up to the hat precisely where the trip began. Calculate the speed of the river.

Answer: Assume you can paddle the canoe at speed v on still water. If the speed of the river is $\tilde{v}$, then the canoe's net speed is $v - \tilde{v}$ upstream, and $v + \tilde{v}$ downstream.

Let D equal the distance traveled before the canoe turns around, thus we write

$$1\,\text{mile} + (v - \tilde{v})\,10\,\text{min} = D$$

for upstream. For downstream we have

$$(v + \tilde{v})\,T = D\,,$$

where T is the time required to paddle back to where the canoe began.

Note that the time required for the hat to float downstream to its starting point is T plus ten minutes, so we may also write

$$1\,\text{mile} = \tilde{v}\,(T + 10\,\text{min})\ .$$

Eliminate D between the first two equations and eliminate factors of $\tilde{v}$ using the third equation to find $T = 10\,\text{min}$. It quickly follows that

$$\tilde{v} = \frac{1\,\text{mile}}{20\,\text{min}}\ .$$

5.6 Unit Conversion

Multiplying Rates

Quantities carrying dimension can be converted from one set of units to another using *dimensional analysis*, also known as *unit conversion*. One may convert inches to centimeters, or seconds to hours, or liters to gallons because these conversions preserve the fundamental dimension of the quantity. On the other hand, trying to convert microseconds to pounds is a nonsensical attempt to equate a time to a weight.

Unit Conversion Table

Common unit equivalanes are given in the following table:

Time

$$1 \text{ hour} = 60 \text{ minute} = 3600 \text{ second}$$
$$1 \text{ day} = 24 \text{ hour}$$
$$1 \text{ year} = 365.25 \text{ day}$$

Length

$$1 \text{ inch} = 2.54 \text{ centimeter}$$
$$1 \text{ foot} = 12 \text{ inch} = (1/3) \text{ yard}$$
$$1 \text{ mile} = 5280 \text{ foot} = 1.609 \text{ kilometer}$$
$$1 \text{ meter} = 1.094 \text{ yard}$$
$$1 \text{ kilometer} = 1000 \text{ meter}$$

Volume

$$1 \text{ gallon} = 128 \text{ fluid ounce} = 3.785 \text{ liter}$$
$$1 \text{ millileter} = (1/1000) \text{ liter}$$
$$1 \text{ cm}^3 = 1 \text{ milliliter}$$

Mass

$$1 \text{ kg} = 1000 \text{ gram (anywhere)}$$

Weight

$$1 \text{ kg} = 2.20462 \text{ pound (at sea level)}$$

Dimensionless

$$360^\circ = 2\pi \text{ radian}$$
$$1 \text{ dozen} = 12$$
$$1 \text{ mole} = 6.023 \times 10^{23}$$

Dimensional Analysis

Starting with any entry from the table above, for instance, $1\,\text{in} = 2.54\,\text{cm}$, note this can be converted into a *unit rate* by dividing through by either side, giving

$$1 = \frac{2.54 \text{ cm}}{1 \text{ in}}$$

or

$$1 = \frac{1 \text{ in}}{2.54 \text{ cm}}\,.$$

In each case, we have a dimensionless number 1 on the left side of the equation. On the right, we have a rate with definite units.

From algebra, we know that multiplying by one leaves the quantity unchanged. It follows that multiplying, say, twelve inches by a carefully chosen factor of one should yield the equivalent in new units:

$$\begin{aligned} 12 \text{ in} &= 12 \text{ in} \times (1) = 12\,\cancel{\text{in}} \times \left(\frac{2.54 \text{ cm}}{1\,\cancel{\text{in}}}\right) \\ &= 12 \times 2.54 \text{ cm} = 30.48 \text{ cm} \end{aligned}$$

Dimensional analysis must also work in reverse. Driving the previous example backwards, start with with 30.48 cm, and multiply by another carefully-chosen factor of one so that centimeters cancel:

$$\begin{aligned} 30.48 \text{ cm} &= 30.48 \text{ cm} \times (1) \\ &= 30.48\,\cancel{\text{cm}} \times \left(\frac{1 \text{ in}}{2.54\,\cancel{\text{cm}}}\right) = 12 \text{ in} \end{aligned}$$

Problem 53

The speed limit on a Canadian highway is 120 kilometers per hour. Convert this speed to miles per hour. Answer: 74.6 mph

Problem 54

How long is 30 years in seconds? (Guess first: thousands, millions, or tens of millions?)

Problem 55

Joe found a wrench on the shop floor. Comparing to his own wrench set, he finds that the unknown wrench is slightly larger than 9/16 in, and slightly smaller than 5/8 in. Estimate the size of the wrench in millimeters. Answer ≈ 15 mm

Problem 56

Convert 70 meters per second to miles per hour. Answer: 156.6 mph

Area and Volume

Often, quantities of length in inches, centimeters, etc., occur as their square or cube, i.e. in^2, cm^3, etc. For these we take the corresponding conversion factor to the appropriate power so that units cancel properly.

For instance, converting from square inches to square centimeters looks like:

$$\begin{aligned} 25 \text{ in}^2 &= 25 \text{ in}^2 \times \left(1^2\right) = 25\,\cancel{\text{in}^2}\left(\frac{2.54^2 \text{ cm}^2}{1^2\,\cancel{\text{in}^2}}\right) \\ &= 25 \cdot 2.54^2 \text{ cm}^2 \approx 161 \text{ cm}^2 \end{aligned}$$

Problem 57

Convert 4500 gallons to cubic meters. Answer: $17.03\,\mathrm{m}^3$

Problem 58

Calculate the volume of one gallon of liquid in cubic feet. Answer:

$$\begin{aligned}1\text{ gal} &= 3.785\,\cancel{L}\frac{1000\,\cancel{\text{mL}}}{\cancel{L}}\frac{1\,\cancel{\text{cm}^3}}{\cancel{\text{mL}}}\frac{1\,\cancel{\text{in}^3}}{2.54^3\,\cancel{\text{cm}^3}}\frac{1\text{ ft}^3}{12^3\,\cancel{\text{in}^3}}\\ &= \frac{3.785\times 1000}{2.54^3\times 12^3}\text{ ft}^3 = 0.13367\text{ ft}^3\end{aligned}$$

Problem 59

One cubic foot of liquid is equal to how many gallons? Answer:

$$1\text{ ft}^3 = \frac{1\text{ gal}}{0.13367} = 7.4813\text{ gal}$$

Problem 60

Three gallons of paint are uniformly spread over a wall having area $600\,\mathrm{ft}^2$. Calculate the thickness of the paint in millimeters. Answer: 0.204 mm

Problem 61

Suppose one cubic foot of liquid passes through a hole of diameter 1.25 inches in one minute. Calculate the speed of the water through the hole.

Answer: Take a cylinder of water having volume

$$1\text{ ft}^3 = \pi\left(\frac{1.25}{2}\text{ in}\right)^2 L\,,$$

so

$$L = \frac{1\text{ ft}^3}{\pi\,(0.625\text{ in})^2} = 1408\text{ in} = 117.3\text{ ft}\,.$$

It takes one minute for the whole volume of water to move through length L, so the speed is:

$$|v| = \frac{1408\text{ in}}{60\text{ sec}} = 23.47\frac{\text{in}}{\text{sec}} = 117.3\frac{\text{ft}}{\text{min}}$$

Density

The *density* of a given substance, denoted ρ (Greek *rho*), is defined as the mass per unit volume of material present:

$$\rho = \frac{M}{V}$$

Density of Water

At atmospheric pressure, the density of liquid water varies per temperature as suggested:

$$\begin{aligned}\rho_{0^\circ\text{C}} &= 999.84\text{ kg/m}^3\\ \rho_{4^\circ\text{C}} &= 998.97\text{ kg/m}^3\\ \rho_{20^\circ\text{C}} &= 998.20\text{ kg/m}^3\\ \rho_{22^\circ\text{C}} &= 997.77\text{ kg/m}^3\\ \rho_{25^\circ\text{C}} &= 997.05\text{ kg/m}^3\\ \rho_{30^\circ\text{C}} &= 995.65\text{ kg/m}^3\\ \rho_{40^\circ\text{C}} &= 992.22\text{ kg/m}^3\\ \rho_{60^\circ\text{C}} &= 983.20\text{ kg/m}^3\\ \rho_{80^\circ\text{C}} &= 971.79\text{ kg/m}^3\\ \rho_{100^\circ\text{C}} &= 958.40\text{ kg/m}^3\end{aligned}$$

Problem 62

Using $\rho_{\text{water}} = 1000\text{ kg/m}^3$, calculate the weight in pounds of one gallon of water at sea level. Express the final answer as a weight density. Answer:

$$\begin{aligned}M &= \rho V\\ &= \frac{1000\text{ kg}}{\cancel{\text{m}^3}}(1\,\cancel{\text{gal}})\frac{3.785\,\cancel{L}}{1\,\cancel{\text{gal}}}\left(\frac{1000\,\cancel{\text{cm}^3}}{1\,\cancel{L}}\right)\frac{1\,\cancel{\text{m}^3}}{100^3\,\cancel{\text{cm}^3}}\\ &= \frac{1000\times 3785}{10^6}\text{ kg} = 3.785\text{ kg}\\ W &= 3.785\,\cancel{\text{kg}}\left(\frac{2.20462\text{ lb}}{1\,\cancel{\text{kg}}}\right) = 8.344\text{ lb}\\ \tilde{\rho} &= 8.344\text{ lb/gal}\end{aligned}$$

Problem 63

The volume of a cylindrical pipe is $V = \pi R^2 L$, where R is the radius and L is the length. Calculate the volume of water in a length $L = 1$ ft of pipe having diameters (i) 8 in, (ii) 6 in, (iii) 4 in, (iv) 3 in, (v) 2 in, (vi) 1.5 in, (vii) 1 in, (viii) 0.5 in. Also calculate the weight in pounds (at sea level) of each volume using $\tilde{\rho}$ calculated previously. Answer:

Diam. (in)	**Volume** (gal)	**Weight** (lb)
8	2.6115	21.791
6	1.4689	12.258
4	0.65287	5.4478
3	0.3672	3.0644
2	0.16322	1.3620
1.5	0.091809	0.76610
1	0.040804	0.34049
0.5	0.010201	0.085122

6 Scientific Notation

Consider the list of real numbers:

$$\begin{aligned} I &= 1 \\ J &= 10003 \\ K &= 0.0001357 \\ L &= 31.4159 \\ M &= 28318530718 \end{aligned}$$

While each is a valid real number, it would be neater to use a unified notation for each of these. For this we employ *scientific notation*, in where any number N is represented by:

$$N = A \times 10^B$$

Precision and Magnitude

Any number represented in scientific notation consists of a 'precision number' $|A| < 10$, and an order-of-ten 'magnitude' number B. Both A and B are themselves real numbers.

In this convention, the above list of numbers reads:

$$\begin{aligned} I &= 1 \times 10^0 \\ J &= 1.0003 \times 10^4 \\ K &= 1.357 \times 10^{-4} \\ L &= 3.14159 \times 10^1 \\ M &= 2.8318530718 \times 10^{10} \end{aligned}$$

In this form, one can quickly read off the magnitude of the number from the exponent attached to 10. The rest of the information carries the precision.

Large Numbers and Scientific Notation

Converting a standard fraction or decimal to scientific notation is a straightforward task. For exotically-large numbers (or small) numbers, some care must be taken.

To work an example, let us convert the number the number

$$N = 55^{99}$$

to scientific notation, which means to find A and B such that

$$55^{99} = A \times 10^B \ .$$

To proceed, recast the variable A via $A = 10^C$, where because $|A| < 10$, it follows that C obeys $0 < C < 1$:

$$55^{99} = 10^{B+C}$$

Apply the $\log_{10}$ operator to each side and simplify to write

$$99 \times \log_{10}(55) = B + C \ .$$

Supposing B and C are positive, it turns out that the 'floor' operator can be used to tease the values apart. In particular, we must have

$$B = \texttt{floor}\,(B + C) = \texttt{floor}\,(99 \times \log_{10}(55)) \ ,$$

and subsequently

$$C = 99 \times \log_{10}(55) - B \ .$$

For the example on hand, we use a machine or lookup table to find, approximately,

$$99 \times \log_{10}(55) \approx 172.296 \ ,$$

and thus:

$$\begin{aligned} B &= 172 \\ C &\approx 0.296 \end{aligned}$$

From C we calculate $A \approx 10^{0.296} \approx 1.977$, and the work is done:

$$N \approx 1.977 \times 10^{172}$$

Despite the approximate nature of the result, the order of magnitude B is exact and the precision C can be made finer by stressing the calculator that produced it.

<u>Problem 64</u>

Convert the number

$$X = 5^{256}$$

to scientific notation. Answer:

$$\log_{10}(X) = \log_{10}\left(5^{256}\right) = 256 \log_{10}(5) \approx 178.936$$

$$X \approx 10^{0.936} \times 10^{178} \approx 8.63 \times 10^{178}$$

Factorials and Scientific Notation

The factorial operator, which to remind, occurs as

$$N! = N\,(N-1)\,(N-2) \cdots (2)\,(1) \ ,$$

gives very large results for increasing N, and often such numbers can outsize the ability of a standard calculator.

To work with very large factorials, it helps to attain $N!$ in scientific notation. To proceed, apply the $\log_{10}$ operator to both sides of the above to write

$$\begin{aligned} \log_{10}(N!) &= \log_{10}(N) + \log_{10}(N-1) \\ &\quad + \log_{10}(N-2) + \cdots + \log_{10}(2) + 0 \end{aligned}$$

Sigma Notation

To contain the large sum on the right, we employ sigma notation as follows:

$$\log_{10}(N!) = \sum_{j=0}^{N-1} \log_{10}(N-j) = M$$

Rather than deal with $N!$ as a whole, the work turns into adding up the base-ten logarithm values for all natural numbers up to N. The sum of this effort is stored in the variable M. With a value for M on hand, use the exponent operator to solve for $N!$, particularly

$$N! = 10^M \,.$$

Converting to scientific notation, we write, as previously,

$$N! = 10^M = A \times 10^B = 10^{B+C} \,,$$

where $|A| < 10$ and thus $|C| < 1$. Moreover, we have

$$M = B + C \,,$$

which means we simply look at M and assign the whole number part to B and the decimal to C.

Problem 65

Convert the number $X = 1000!$ to scientific notation. Answer:

Step 1: Using a computer or calculator, compute:

$$M = \sum_{j=0}^{1000-1} \log_{10}(1000-k) = 2567.604644$$

Step 2: Identify

$$B = 2567$$
$$C = 0.604644$$

Step 3: Assemble the result:

$$1000! \approx 10^{0.604644} \times 10^{2567} \approx 4.02387 \times 10^{2567}$$

6.1 Order of Magnitude

The advantage to scientific notation is we don't have to waste time writing zeros. Many of the powers of ten have a special name, listed below from small to large. Common entries are emphasized in bold.

Power	**Prefix**	**Symbol**
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	**nano**	**n**
10^{-6}	**micro**	**u**
10^{-3}	**milli**	μ
10^{-2}	**centi**	**c**
10^{-1}	**deci**	**m**
10^{3}	**kilo**	**k**
10^{6}	**mega**	**M**
10^{9}	**giga**	**G**
10^{12}	**tera**	**T**
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z
10^{24}	yotta	Y

One can easily see the utility of scientific notation in practice. It would be a great burden to write, for instance, thirty extra zeros whenever we want to calculate with the solar mass. Following is a brief tour through the magnitude of typical quantities used in the sciences.

Order	**Example**
10^{-30}	Mass of an electron in kilograms.
10^{-27}	Mass of a hydrogen atom in kilograms.
10^{-15}	Diameter of a hydrogen atom in meters.
10^{-11}	Radius of a hydrogen atom in meters.
10^{-9}	Volume of a flea in cubic meters.
10^{-6}	Lifetime of a muon particle in seconds.
10^{-3}	Oscillation period of a guitar string in seconds.
10^{0}	Height of a human child in meters.
10^{3}	Mass of a car in kilograms.
10^{6}	Radius of Earth in meters.
10^{9}	Duration of a century in seconds.
10^{12}	Number of cells in human body.
10^{16}	Radius of the solar system in meters.
10^{24}	Number of carbon-12 atoms in twelve grams.
10^{27}	Mass of Jupiter in kilograms.
10^{30}	Mass of the Sun in kilograms.

Lightyears

The speed of light as defined (or measured) in meters per second is equal to

$$c = 2.99792498 \times 10^8 \text{ meter/sec} ,$$

which is close enough to

$$c \approx 3.00 \times 10^8 \text{ m/s} ,$$

much easier to remember. A lightyear is defined as the *distance* that light will travel in one year. With this, it follows that one lightyear is

$$1 \text{ lightyear} = c \times 1 \text{ year} .$$

We can calculate the length of a lightyear in meters by unit conversion. Start with

$$1 \text{ lightyear} = c \times 1 \cancel{\text{year}} \left(\frac{365.25 \cancel{\text{day}}}{1 \cancel{\text{year}}}\right) \left(\frac{24 \text{ hr}}{1 \cancel{\text{day}}}\right) ,$$

yielding the intermediate step

$$1 \text{ lightyear} = c \times 365.25 \times 24 \text{ hr} .$$

The units on the right need to be seconds to cleanly multiply by c, so keep going:

$$\begin{aligned} 1 \text{ lightyear} &= c \times 365.25 \times 24 \cancel{\text{hr}} \left(\frac{3600 \text{ sec}}{1 \cancel{\text{hr}}}\right) \\ &= c \times 365.25 \times 24 \ \times 3600 \text{ sec} \end{aligned}$$

Finally, we have

$$\begin{aligned} 1 \text{ lightyear} &= 2.9979 \times 10^8 \times 31557600 \frac{\cancel{\text{sec}}}{\cancel{\text{sec}}} \text{ m} \\ &= 9.4607 \times 10^{15} \text{ m} \end{aligned}$$

Problem 66

Calculate the lightyear distance in miles.

Problem 67

Express the speed of light in miles per second and miler per hour. Answer: 1.86×10^5 mi/sec, 6.71×10^8 mi/hr

Problem 68

Convert the Massachusetts highway speed limit of 65 mph into lightyears per second. Answer: 3.071×10^{-15} lightyear/sec

Problem 69

The thickness of paper is closest to:

10^{-4} m

10^{-1} m

10^{1} cm

10^{-7} m

Problem 70

The length of an American football field (100 yards) is closest to:

10^{4} m

10^{-1} km

10^{5} mm

10^{3} foot

Problem 71

The radius of Earth is about 10^6 meters. If the volume of a sphere is given by $V = (4/3)\pi R^3$, roughly estimate Earth's volume in cubic meters. Answer: $\approx 10^{18}$ m^3

Problem 72

The mass of Earth is approximately $M = 5.972 \times 10^{24}$ kg. Using radius $R = 6371$ km, calculate the average density (total mass over total volume) of Earth in grams per cubic centimeter. Answer: $5.513\,\mathrm{g/cm^3}$

Problem 73

At sea level, a 2.205 pound rock weighs the same as a 1 kilogram object. On Mt. Everest, a 2.199 pound rock weighs the same as a 1 kilogram object. What is the weight in pounds of an 80.0 kg person at each location? Answer: 176.4 lb, 175.9 lb

Folding Paper

Problem 74

Suppose a very wide sheet of paper has thickness 0.1mm. If the paper is folded once, the thickness doubles to 0.2 mm. If the resulting sheet is folded again, the total thickness doubles again to 0.4 mm. A third fold makes 0.8 mm, a fourth fold makes 1.6 mm, and so on. How thick is the paper after 20 folds? Answer:

$$T = 2^{20} \times 0.1\ \mathrm{mm} \approx 104.9\ \mathrm{m}$$

Problem 75

If a very wide sheet of paper with thickness 0.1mm is folded N times, determine N such that the resultant thickness is the radius of the solar system, approximately 10^{16} m. Answer:

$$N = \log_2\left(\frac{10^{16}\ \mathrm{m}}{0.1\ \mathrm{mm}}\right) \approx 63$$

6.2 Numerical Precision

In any laboratory, engineering, or other real-world setting, a number's precision is subject to the methods or apparatus used to attain that number. For instance, a length measured by a ruler might be precise to one millimeter, which means two similar things: (i) Any object smaller than one millimeter won't be 'visible' to the ruler. (ii)

Any length measured by the millimeter-precision ruler cannot be known to greater precision than 1mm. For instance, suppose your lab assistant measures his shoelace with the above-mentioned ruler and reports the result $L = 26.5912$ cm. You should immediately round this to $L = 26.6$ cm, because the 'junk' digits 912 occupy decimal places that are smaller than the precision of the ruler.

A way to justify throwing away the 'junk' digits is to consider what happens when the measurement is repeated. While it's likely that the length L will be close to 26.6 cm again, it's unlikely that the digits 912 would follow a second time. In fact, we aren't confident that the shoelace really is 26.6 cm until the measurement is repeated.

Significant Figures

The precision in a number is represented by how many decimal places are *confidently* known, where each decimal place is called a *significant figure* or *significant digit*. Supposing a quantity is handed to us *without* any junk digits, there are several rules for counting the number of significant figures. These are:

1. Zeros to the left of the number are not significant.
2. Zeros to the right of the number are significant.
3. All nonzero digits in the number are significant.
4. Any zeros between significant digits are significant.

The following table lists a few numbers with varying significant figures as expressed in scientific notation:

Number	**Sig. Figs.**	**Sci. Not.**
7	1	7×10^0
10^{-3}	1	1×10^{-3}
3.0×10^8	2	3.0×10^8
000.0000010	2	1.0×10^{-6}
2.998×10^8	4	2.998×10^8
0623.53	5	6.2353×10^2
12.340	5	1.2340×10^1
90000	5	9.0000×10^4
90000.00	7	9.000000×10^4

Strategy

Sometimes the precision of a number is not obvious as given. If someone says they earned \$4500 last month, does this precisely mean \$4500.00 dollars and zero cents, or was \$4500 rounded from some nearby number? Should we trust only the 4 and the 5? This issue is resolved by the following convention: *When the precision is ambiguous, do intermediate calculations to four significant figures, and report answers to three significant figures.* For the example on hand, we can agree that an honest representation of \$4500 is $\$4.50 \times 10^3$.

6.3 Propagation of Precision

When two numbers of finite precision are combined, the natural question arises: what's the precision of the result? The answer depends on the operation being performed, but there are two main ways to proceed.

Multiplication, Division, Roots, Powers

Be pessimistic: In your calculation, seek the quantity with the lowest precision, and adopt *that* precision for the final answer, as illustrated in the following examples:

$$2.345 \times 5.4 \approx 13$$
$$55555 \times 0.0100 \approx 556$$
$$\sqrt{\pi \times 2.56/5.567} \approx 1.20$$
$$70 \times 23.846251 \approx 1.7 \times 10^3$$
$$\left((2/3) \times \left(6.984 \times 10^{-8}\right)\right)^2 \approx 2.170 \times 10^{-15}$$

Notice that purely mathematical factors like 2 or $\pi = 3.14159265358979323846264\ldots$ or $2/3 = 0.666\bar{6}$ are more-or-less infinitely precise numbers, causing no penalty in the overall precision of a calculation.

Addition and Subtraction

If the quantities being added or subtracted do not involve any decimals, you're off the hook, and may proceed as usual. When there *are* decimals though, be pessimistic again: the number of decimal places carried by the answer is determined by the contributing factor with the *least* number of decimal places. See the examples that follow.

$$12551 + 3114 = 15665$$
$$1000 - 3 = 997$$
$$2.345 + 5.4 \approx 7.7$$
$$100.2 + 15.438 \approx 115.6$$
$$1000 - 0.3 \approx 1000$$

7 Fundamental Units

7.1 Physical Units

Any physical quantity carries some kind of dimensionality, i.e. physical units. Whether the quantity represents duration in years, force in pounds, distance in lightyears, etc., the associated physical units are always reducible to some combination of (i) length, (ii) mass, and (iii) time. These are called the *fundamental dimensions*.

Following are several combinations of the fundamental dimensions linked with their colloquial names:

Combination	**Label**
L^2	Area
L^3	Volume
L/T	Speed
L/T^2	Acceleration
$M \cdot L/T$	Momentum
$M \cdot L/T^2$	Force
M/L^3	Mass Density
$M/\left(LT^2\right)$	Pressure
$M \cdot L^2/T^2$	Energy

SI System

The abbreviation *SI* stands for Système International, more commonly known as the *metric system*. There are two factions within the metric system, one called *MKS*, and the other called *CGS*. The letters *M-K-S* tell us to use meters, kilograms, seconds as fundamental units. Similarly, *C-G-S* uses centimeters, grams, seconds.

Imperial System

As of 2024, three countries have resisted the metric system's campaign for domination: Liberia, Myanmar, and the United States. Due to this, we still accomodate the the *English system*, whose fundamental dimensions are (i) the foot, (ii) the slug, and (iii) seconds.

Expensive Mistake by NASA

Forgetting to properly handle units can be an expensive mistake, namely $328 million when NASA lost the Mars Climate Orbiter. In September 1999, NASA attempted to put a satellite close enough to Mars to study its atmosphere and surface features. However, a ground-based computer in communication with the Orbiter was sending information using the *English* system of units! Specifically, the Orbiter expected a packet of information expressed in Newtons, but it was interpreted as Pounds. The Orbiter didn't steer correctly and slammed into Mars.

7.2 Standard Units

Standard of Length

Length standards trace back to ancient history, with one popular unit of measurement being the the elbow-to-fingertip distance on a human arm in an L-shape, known as the 'cubit'. Obviously, the cubit varies form person to person, and more drastically between separate populations.

From Europe we gained another measure of length, the familiar 'foot', which was once defined by the anatomy of the prevailing royalty. It wasn't until the days of Louis XIV that the foot was standardized in France. The name was upgraded to the 'Royal Foot'.

Around 1795, the standard length unit was decidedly the *meter*, and was defined as 1/20000000 the length of Earth's prime meridian (this was before orthodox geology was developed).

By the late 1800's, the standard length unit was determined by the length of one carefully protected platinum-iridium bar, whose length was simply declared to equal precisely one meter. Precise copies of this bar were produced and shipped off to spread the length standard around the world.

Eventually, experimental physics would demand a more precise definition for the length standard that was robust with respect to special relativity, which inevitably involves the speed of light. *The meter is precisely defined as the distance light travels in* 1/299792458 *sec in vacuum.*

Standard of Mass

The modern definition of the Kilogram was settled in 2019. Before the decision to uproot tradition, the *kilogramme* was a another carefully-protected chunk of metal housed in France made of 90% platinum, 10% iridium, whose mass was decidedly one kilogram by popular vote. It's likely that the calibration of all science-grade mass scales and standards of the time could be traced back to the *kilogramme*.

There was curious problem with the *kilogramme* though: it's mass has slowly *increased* over history, and much effort has been devoted to finding out why this appears so. With or without a perfect explanation for this, the conclusion we must draw is that a chunk of metal is ultimately *not* a good mass standard.

The General Conference on Weights and Measures (CGPM) tried repeatedly to settle a definition of the kilogram that arises from quantum mechanics, specifically involving the Planck mass. They couldn't decide in 2011, and procrastinated again in 2014. Eventually, laboratory measurements of Planck's constant became sufficiently refined (about 13 ppb uncertainty) such that the definition of one kilogram is simply back-read out of the result.

Standard of Time

One reasonable definition for the *second* is to slice a 24-hour interval into $60 \cdot 60 \cdot 24$ parts. This was considered reliable until 1967, until it was decided that the fundamental unit of time had better take on a less 'Earthly' definition. It was then decided that one second would be defined as precisely 9192631770 periods of the light waves emitted from cesium decay.

There was nothing special about cesium except it was an available atom for study. There's nothing terribly special about the number 9192631770, except it seemed good enough for reproducing the 'old' value of one second. Going through the trouble to define the second this way allows physicists to deal more easily with many practical issues, such as adding leap-seconds to the calendar, and calibrating clocks far from Earth. (The latter was on their minds in the 1960's more so than it is today.)

7.3 Elemental Mass Units

Avogadro's Number

Certain problems in thermodynamics, chemistry, and quantu mechanics require calculating with a large number of particles, atoms, or molecules. To this end, we emplot a quantity like a 'chemists dozen', called a *mole*. One mole of any material is equal to 6.022×10^{23} particles, known as *Avogadro's Number*, denoted N_A:

$$N_A = 6.022 \times 10^{23}$$

Mass Units

By definition, one mole of carbon-12 contains N_A atoms and weighs precisely 12 g. From this we define the *unified atomic mass unit*, a single number used to approximate both the proton mass and the neutron mass having value 1.660539×10^{-24} g. In *physical units*, the unified amu is roughly 0.9928 proton masses or 0.9914 neutron masses. The *molecular mass* is defined as the number of *atomic mass units* (masses of protons and neutrons) in an element or molecule, ignoring electron masses.

Elemental masses are not reported as whole numbers in the periodic table. This arises from averaging over various *isotopes* of a given element occurring in nature, whose number of neutrons vary from the number in the pure element.

For example, 99.2% of hydrogen atoms observed in nature contain one proton and zero neutrons, denoted ${}^1_1H$. Meanwhile, 0.8% of hydrogen occurs with one proton and one neutron, denoted ${}^2_1H$. Statistically 0% of hydrogen contains two neutrons per atom, meaning ${}^3_1H$ is exceedingly rare. The weighted average of the respective masses per isotope is therefore

$$(0.992)\,1\,\text{amu} + (0.008)\,2\,\text{amu} + 0 = 1.008\,\text{amu}\,,$$

matching the reported molecular mass of hydrogen, and the same goes for all other elements in the periodic table.

Problem 76

An H_2O molecule consists of 10 protons, 8 neutrons, and 10 electrons. Use the periodic table values $m_H = 1.008$ and $m_O = 15.999$ to compute the mass of a single water molecule in grams. Answer: 2.99×10^{-23} g

7.4 Units in Quantum Mechanics

Fundamental constants in quantum theory can be expressed in terms of Planck's length (L), Placnk mass (M), and Planck time (T). Particularly, one can state:

- The speed of light c in units LT^{-1}.
- The electron mass m in units of M.
- Planck's reduced constant $\hbar = h/2\pi$ in units ML^2T^{-1}.
- The squared electron charge $e^2/4\pi\epsilon_0$ in units ML^3T^{-2}.

Reduced Compton Wavelength

Consider three unknown integers x, y, z such that the quantity

$$\lambda_c = \hbar^x c^y m^z$$

has units of length. From dimensional analysis, we generate three equations

$$\begin{aligned} x + z &= 0 \\ 2x + y &= 1 \\ x + y &= 0 \,, \end{aligned}$$

solved by $x = 1$ with $y = z = -1$. Such a length is called the *reduced Compton wavelength*:

$$\lambda_C = \frac{\hbar}{mc}$$

Fine Structure Constant

For the proper choice of x, y, z, the combination

$$\alpha = \hbar^x c^y \left(e^2/4\pi\epsilon_0\right)^z$$

can be made dimensionless. From dimensional analysis, we generate three equations

$$\begin{aligned} x + z &= 0 \\ -x - y - 2z &= 0 \\ 2x + y + 3z &= 0 \,, \end{aligned}$$

solved by $x = y = -1$, $z = 1$. The result is called the *fine structure constant*:

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c} \approx \frac{1}{137}$$

Bohr Radius

The scale of a hydrogen atom is approximated by the so-called *Bohr radius*, denoted a_0, which is the ratio of the reduced Compton wavelength to the fine structure constant:

$$a_0 = \frac{\lambda_c}{\alpha} = \frac{\hbar^2}{m\left(e^2/4\pi\epsilon_0\right)} \approx 137\lambda_C \approx 5.29 \times 10^{-11}\ \text{m}$$

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