Linear Systems
MANUSCRIPT

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Chapter 1

Linear Systems

1 Matrix Formalism

Formally, a *matrix* is a collection of numbers or variables arranged in a block with fixed rows M and columns N. Each element, i.e. *component* in the matrix requires two subscripts.

Matrix-Operator Equivalence

A primary use for a matrix is to 'operate' on a vector of dimension N, yielding a new vector of dimension M. (The term 'matrix' is often interchanged with the term 'operator'.) Symbolically, this is written

$$A\vec{x} = \vec{y} \, ,$$

and in full *block notation*, the same statement looks like

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2N} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{M1} & A_{M2} & A_{M3} & \cdots & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_M \end{bmatrix}$$

More compactly, we use *index notation* to express the same calculation:

$$\sum_{k=1}^{N} A_{jk}\, x_k = y_j \tag{1.1}$$

$$j = 1, 2, 3, \ldots, M$$

Matrix Components

Consider two vectors $\vec{x}$, $\vec{y}$, each a linear combination in some N-dimensional basis such that

$$\vec{x} = \sum_{j=1}^{N} x_j\, \hat{e}_j$$

$$\vec{y} = \sum_{j=1}^{M} y_j\, \hat{e}_j \, .$$

While $\vec{y}$ is perfectly happy being expressed as a linear combination in the basis $\{\hat{e}_j\}$, it's instructive to re-express $\vec{y}$ in terms of its brother, $\vec{x}$. To do so, we propose an operator A such that

$$\vec{y} = A\vec{x} \, .$$

To proceed, write the above as

$$\sum_{k=1}^{M} y_k \, \hat{e}_k = \sum_{k=1}^{N} x_k \, A\hat{e}_k \; ,$$

and multiply the basis vector $\hat{e}_j$ (via dot product) into both sides:

$$\sum_{k=1}^{M} y_k \, \hat{e}_j \cdot \hat{e}_k = \sum_{k=1}^{N} x_k \, \hat{e}_j \cdot A\hat{e}_k$$

On the left, every term in the sum vanishes except that with $j = k$, and the above becomes

$$y_j = \sum_{k=1}^{N} \left(\hat{e}_j \cdot A\hat{e}_k \right) x_k$$
$$j = 1, 2, 3, \ldots, M \; .$$

The parenthesized quantity is what we're after:

$$A_{jk} = \hat{e}_j \cdot A\hat{e}_k \tag{1.2}$$

The term A_{jk} is the component of the matrix A corresponding to the jth row, kth column.

1.1 Projector

Consider the curious quantity

$$P_x = \vec{x} \, \vec{x} \; , \tag{1.3}$$

called the the *projector* of $\vec{x}$. By itself, P_x does nothing - there is no operation between the two copies of $\vec{x}$. What the projector *does* is 'wait' to be multiplied into another vector, resulting in a scaled version of $\vec{x}$. For example, applying the projector to a different vector $\vec{y}$ (of the same dimension as $\vec{x}$) goes like

$$P_x \vec{y} = \vec{x} \left(\vec{x} \cdot \vec{y} \right) \; .$$

1.2 Identity Operator

Consider a vector $\vec{x}$ as a linear combination in some N-dimensional basis:

$$\vec{x} = \sum_{j=1}^{N} x_j \, \hat{e}_j$$

For any one of the basis vectors $\hat{e}_k$, write the projector

$$P_{e_k} = \hat{e}_k \, \hat{e}_k \; ,$$

and then multiply $\vec{x}$ onto the right side to get

$$P_{e_k} \vec{x} = \hat{e}_k \, \hat{e}_k \cdot \vec{x} = x_k \, \hat{e}_k$$

By summing over the index k, the right side is identically $\vec{x}$:

$$\left(\sum_{k=1}^{N} P_{e_k} \right) \vec{x} = \sum_{k=1}^{N} x_k \, \hat{e}_k = \vec{x}$$

For the left side to also equal $\vec{x}$, the parenthesized quantity must be equivalent to 'multiplying by one', which we call the *identity* operator:

$$I = \sum_{k=1}^{N} P_{e_k} \tag{1.4}$$

The identity operator leaves a vector unchanged:

$$I\vec{x} = \vec{x}$$

The matrix-equivalence of I is square, has no mixed components, and has ones along the diagonal:

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1.5}$$

1.3 Unified Matrix Notation

Recall that a matrix A relates to its components A_{jk} in a way given by (1.2), namely

$$A_{jk} = \hat{e}_j \cdot A\hat{e}_k \; .$$

To establish this directly using projectors, start with $A = IAI$ and watch what happens:

$$\begin{aligned} A = IAI &= \sum_{j=1}^{M} \sum_{k=1}^{N} P_{e_j} A P_{e_k} \\ &= \sum_{j=1}^{M} \sum_{k=1}^{N} \hat{e}_j \left(\hat{e}_j \cdot A\hat{e}_k \right) \hat{e}_k \end{aligned}$$

The parenthesized quantity is precisely A_{jk}. Evidently, the symbolic notation unifies with the index notation in the equation

$$A = \sum_{j=1}^{M} \sum_{k=1}^{N} \hat{e}_j \left(A_{jk} \right) \hat{e}_k \tag{1.6}$$

The presence of $\hat{e}_j \, \hat{e}_k$ is like a projector - it couples the component to the operator.

2 Matrix Operations

2.1 Matrix Addition

Two matrices A and B of identical dimensions, meaning M rows, N columns, can be combined to form a new matrix C such that

$$A + B = C \,,$$

or, to elaborate:

$$A_{jk} + B_{jk} = C_{jk} \qquad (1.7)$$
$$\begin{cases} j = 1, 2, 3, \ldots, M \\ k = 1, 2, 3, \ldots, N \end{cases}$$

2.2 Scalar Multiplication

A scalar α can be multiplied into each component of a matrix A to form a new matrix B such that

$$\alpha A = B \,,$$

or:

$$\alpha A_{jk} = B_{jk} \qquad (1.8)$$
$$\begin{cases} j = 1, 2, 3, \ldots, M \\ k = 1, 2, 3, \ldots, N \end{cases}$$

2.3 Matrix Multiplication

Two matrices A, B, of equal or different dimensions can be multiplied to form a new matrix C:

$$AB = C$$

The main 'rule' is that the number of *columns* in A must equal the number of *rows* in B:

$$A_{(M,K)} \times B_{(K,N)} = C_{(M,N)}$$

Matrix Non-Commutativity

If you're paying attention, the commutated product BA may violate the above, and no product is defined. In any case, we should assume that the multiplication of two matrices is not commutative:

$$AB \neq BA \qquad (1.9)$$

Multiplication Formula

To derive the formula for matrix multiplication, begin with the following 'unified' representation (1.6) of the respective matrices:

$$A = \sum_{m=1}^{M} \sum_{k=1}^{K} \hat{e}_m \left(A_{mk}\right) \hat{e}_k$$
$$B = \sum_{k'=1}^{K} \sum_{n=1}^{N} \hat{e}_{k'} \left(B_{k'n}\right) \hat{e}_n$$

Then, the product AB reads

$$\begin{aligned} AB &= \sum_{m=1}^{M} \sum_{k=1}^{K} \hat{e}_m \left(A_{mk}\right) \hat{e}_k \sum_{k'=1}^{K} \sum_{n=1}^{N} \hat{e}_{k'} \left(B_{k'n}\right) \hat{e}_n \\ &= \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{k'=1}^{K} \sum_{n=1}^{N} \hat{e}_m \left(A_{mk}\right) \hat{e}_k \, \hat{e}_{k'} \left(B_{k'n}\right) \hat{e}_n \end{aligned}$$

Note that the quantity $\hat{e}_m \; \hat{e}_k \; \hat{e}_{k'} \; \hat{e}_n$ is the juxtaposition of two projectors, readily translating to $\hat{e}_m \left(\hat{e}_k \cdot \hat{e}_{k'}\right) \hat{e}_n$. Note further that the parenthesized product obeys

$$\hat{e}_k \cdot \hat{e}_{k'} = \begin{cases} 1 & k = k' \\ 0 & k \neq k' \end{cases} \,,$$

which has the effect of equating $k = k'$ in the above, eliminating one of the sums. So far then, we have

$$AB = C = \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{n=1}^{N} \left(A_{mk} B_{kn}\right) \hat{e}_m \; \hat{e}_n$$
$$C = \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\sum_{k=1}^{K} A_{mk} B_{kn} \right) \hat{e}_m \; \hat{e}_n \,.$$

The symbol C has replaced the quantity AB on the left. Comparing the right side to (1.6), we conclude that the component C_{mn} of matrix C is given by the famed *matrix multiplication* formula:

$$C_{mn} = \sum_{k=1}^{K} A_{mk} B_{kn} \qquad (1.10)$$
$$\begin{cases} m = 1, 2, 3, \ldots, M \\ n = 1, 2, 3, \ldots, N \end{cases}$$

Equation (1.10) reminds that it's only required that the number of columns in A match the number of rows in B. For instance, the operation $A_{(2,4)} \times B_{(4,3)} = C_{(2,3)}$, explicitly written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

is perfectly valid, whereas the commuted product $B_{(4,3)} \times A_{(2,4)}$ is undefined.

Matrix Associativity

A direct consequence of matrix multiplication is the associativity rule:

$$(AB)\,C = A\,(BC) \tag{1.11}$$

2.4 Change of Basis

A square matrix A with components A_{jk} in the basis $\{\hat{e}_j\}$ can be represented by (1.6):

$$A = \sum_{j=1}^{N}\sum_{k=1}^{N} \hat{e}_j\,(A_{jk})\,\hat{e}_k\ .$$

Under a change of basis $\{\hat{e}_j\} \to \{\hat{u}_j\}$, we can use

$$\hat{e}_j = \sum_{k=1}^{N} \tilde{U}_{jk}\,\hat{u}_k$$

to replace the unit vectors, leading to

$$A' = \sum_{m=1}^{N}\sum_{n=1}^{N} \hat{u}_m \left(\sum_{j=1}^{M}\sum_{k=1}^{N} U_{mj} A_{jk} \tilde{U}_{kn} \right) \hat{u}_n\ ,$$

where the (first) term $\tilde{U}_{jm}$ has been replaced by U_{mj}. The parenthesized quantity is precisely the formula for the component A'_{mn} of the transformed matrix

$$A'_{mn} = \sum_{j=1}^{N}\sum_{k=1}^{N} U_{mj} A_{jk} \tilde{U}_{kn}\ , \tag{1.12}$$

or in symbolic form,

$$A' = U A \tilde{U}\ .$$

Note that the above verifies the associativity rule (1.11) for matrix multiplication. The order in which the sums are taken directly corresponds to which matrices are multiplied first. As a bonus, (1.12) tells us exactly how to take the product of three square matrices.

3 Linear Systems

3.1 Order-Two Formalism

Motivation

Consider the linear system of two equations containing two unknowns x and y,

$$\begin{aligned} ax + by &= e \\ cx + dy &= f\ , \end{aligned}$$

where all coefficients are assumed nonzero. One way to solve the system begins by eliminating y, which means to multiply the top and bottom equations by d, b, respectively, and add the results:

$$x\,(ad - bc) = de - bf$$

Similarly, we can eliminate x to end up with

$$y\,(ad - bc) = af - ec\ ,$$

and it is now trivial to solve for x and y.

If it just so happens that $ad - bc = 0$, the equations for x and y become indeterminate, meanwhile implying $de = bf$ and $af = ec$. To visualize this, treat each equation as a separate line

$$\begin{aligned} y_1 &= -\frac{a}{b}x + \frac{e}{b} \\ y_2 &= -\frac{c}{d}x + \frac{f}{d}\ , \end{aligned}$$

having respective slopes $m_1 = -a/b$, $m_2 = -c/d$. Take the difference of slopes to find

$$m_1 - m_2 = -\frac{a}{b} + \frac{c}{d} = \frac{1}{bd}\,(-ad + bc) = 0\ ,$$

implying the lines are parallel. Moreover, $de = bf$ causes the lines to have the same y-intercept, thus the two lines y_1, y_2 are *identical*. This reduces the number of equations in the system back down to one equation that has an infinite number of solutions on the line $y_{1,2}$. In the special case that $e = 0$ or $f = 0$, the lines $y_{1,2}$ are parallel but non-overlapping, in which case no solutions exist at all.

Only when $ad - bc$ is *nonzero* does the line y_1 cross y_2 somewhere in the Cartesian plane at one point (x_0, y_0). The intersection point is the 'solution' to the system of equations.

Matrix Formulation

Start with the same two-dimensional system and relabel all coefficients such that

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= b_1 \\ A_{21}x_1 + A_{22}x_2 &= b_2\ , \end{aligned} \tag{1.13}$$

admitting a clean matrix representation

$$A\vec{x} = \vec{b}\,,$$

or

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\ .$$

Determinant

The classification of solutions $\vec{x}$ depends on the quantity $A_{11}A_{22} - A_{12}A_{21}$, called the *determinant* of the matrix A:

$$\det A = \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A_{11}A_{22} - A_{12}A_{21} \quad (1.14)$$

If $\det A$ is nonzero, the vector $\vec{x}$ solves the system. On the other hand, any matrix with $\det A = 0$ is called *singular*, having a non-obvious number of solutions (zero or infinite, depending on $\vec{b}$). For the non-singular case $\det A \neq 0$, the solution to the system is given by:

$$\begin{aligned} x_1 &= \frac{1}{\det A}\left(A_{22}b_1 - A_{12}b_2\right) \\ x_2 &= \frac{1}{\det A}\left(A_{11}b_2 - A_{21}b_1\right) \end{aligned}$$

Cramer's Rule

In the above solutions for x_1, x_2, observe that the quantities

$$\begin{aligned} &A_{22}b_1 - A_{12}b_2 \\ &A_{11}b_2 - A_{21}b_1 \end{aligned}$$

are themselves the determinants of new matrices C_1, C_2 such that

$$\begin{aligned} x_1 &= \frac{1}{\det A} \det \begin{bmatrix} b_1 & A_{12} \\ b_2 & A_{22} \end{bmatrix} = \frac{\det C_1}{\det A} \\ x_2 &= \frac{1}{\det A} \det \begin{bmatrix} A_{11} & b_1 \\ A_{21} & b_2 \end{bmatrix} = \frac{\det C_2}{\det A} \end{aligned}$$

That is, the solution to the two-dimensional linear system (1.13) with nonzero determinant is

$$\begin{aligned} x_j &= \frac{\det C_j}{\det A} \\ j &= 1, 2\,, \end{aligned} \quad (1.15)$$

with

$$\begin{aligned} C_1 &= \begin{bmatrix} b_1 & A_{12} \\ b_2 & A_{22} \end{bmatrix} \\ C_2 &= \begin{bmatrix} A_{11} & b_1 \\ A_{21} & b_2 \end{bmatrix}\,. \end{aligned} \quad (1.16)$$

The 'recipe' that got us to this point is called *Cramer's Rule*: if $\det A \neq 0$, there's a solution to the system.

3.2 Order-N Formalism

Generalizing the 2×2 linear system to have M equations and N unknowns, we begin with

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots + A_{3N}x_N &= b_3 \\ &\cdots \\ A_{M1}x_1 + A_{M2}x_2 + A_{M3}x_3 + \cdots + A_{MN}x_N &= b_M\,, \end{aligned} \quad (1.17)$$

admitting the matrix representation $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2N} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{M1} & A_{M2} & A_{M3} & \cdots & A_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_M \end{bmatrix} \quad (1.18)$$

At this stage, the relationship between M and N indicates the quality of solutions (if any) to the system. If $N > M$, the system is said to be *under-determined*, and there is not enough information to choose a solution. On the other hand, if $M > N$, the system is *over-determined*, and there may be zero, or perhaps infinite solutions.

In order to proceed, the matrix A is taken to be *square* with $M = N$. Then, the recipe for solving a two-dimensional linear system extrapolates to N dimensions. Although its not (yet) straightforward how to calculate the determinant of A, we can still use Cramer's rule to write down the components of the solution vector $\vec{x}$, namely

$$\begin{aligned} x_j &= \frac{\det C_j}{\det A} \\ j &= 1, 2, 3, \ldots, N \end{aligned} \quad (1.19)$$

The matrix C_j is constructed by starting with A and replacing the jth column with $\vec{b}$. That is:

$$C_j = \begin{bmatrix} A_{11} & A_{12} & \cdots & b_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & b_{2j} & \cdots & A_{2N} \\ A_{31} & A_{32} & \cdots & b_{3j} & \cdots & A_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & b_{Nj} & \cdots & A_{NN} \end{bmatrix} \quad (1.20)$$

3.3 Row Operations

An important tool set called *row operations* can be applied to matrices. To illustrate these, consider again the N-dimensional linear system (1.17). The 'game' we play is to find ways to manipulate the system of equations without losing any information.

Without much trouble, one finds the allowed operations to be:

- Exchange two rows.
- Multiply a row by a (nonzero) scalar.
- Replace a row by the sum of itself and another row.

For the sake of assigning symbols to the above row operations, let us denote row exchanges as E, scalar multiplication as M, and a row replacement as R. Using a four-dimensional square matrix as an example, row operations explicitly look like:

$$E^{23}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

$$M^2_\alpha A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ \alpha A_{31} & \alpha A_{32} & \alpha A_{33} & \alpha A_{34} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

$$R^2_3 A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{31}+A_{21} & A_{32}+A_{22} & A_{33}+A_{23} & A_{34}+A_{24} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

Note that the subscripts and superscripts on the symbols E, M, R are mere convenience of notation, often-omitted.

4 Determinants

The *determinant* is a scalar calculated from the components of a square ($N \times N$) matrix A. Of the many things the determinant can tell us, we've already seen that $\det A$ indicates the 'quality' of solutions to a linear system. Namely, if the determinant is nonzero, the linear system has a solution given by Cramer's rule. The determinant of a two-dimensional square matrix is given by (1.14).

Three Dimensions

Consider the three-dimensional linear system

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3 \ , \end{aligned} \tag{1.21}$$

and we're interested in the determinant of the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} .$$

Labeling each row R_1, R_2, R_3, respectively, we deploy row operations to (i) multiply R_2 by a factor of A_{11}/A_{21}, and then (ii) replace R_2 with $R_2 - R_1$:

$$A \to \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & \frac{A_{11}A_{22}}{A_{21}} - A_{12} & \frac{A_{11}A_{23}}{A_{21}} - A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Next, (iii) multiply R_3 by a factor of A_{11}/A_{31}, and (iv) replace R_3 with $R_3 - R_1$:

$$A \to \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & \frac{A_{11}A_{22}}{A_{21}} - A_{12} & \frac{A_{11}A_{23}}{A_{21}} - A_{13} \\ 0 & \frac{A_{11}A_{32}}{A_{31}} - A_{12} & \frac{A_{11}A_{33}}{A_{31}} - A_{13} \end{bmatrix}$$

With the matrix configured as such, observe that the 'important' information is crammed into the lower 2×2 portion of the transformed matrix. Accordingly, we deploy the determinant formula (1.14) to write

$$\begin{aligned} \det A \propto & \left(\frac{A_{11}A_{22}}{A_{21}} - A_{12} \right) \left(\frac{A_{11}A_{33}}{A_{31}} - A_{13} \right) \\ & - \left(\frac{A_{11}A_{23}}{A_{21}} - A_{13} \right) \left(\frac{A_{11}A_{32}}{A_{31}} - A_{12} \right) , \end{aligned}$$

which after simplifying, becomes

$$\begin{aligned} \det A \left(\frac{A_{21}A_{31}}{A_{11}} \right) \propto & \ A_{11} \left(A_{22}A_{33} - A_{32}A_{23} \right) \\ & - A_{12} \left(A_{21}A_{33} - A_{31}A_{23} \right) \\ & + A_{13} \left(A_{21}A_{32} - A_{31}A_{22} \right) . \end{aligned}$$

Keepping in mind that the determinant of A is a single number that characterizes the solutions to the system, it follows that the right-side quantity in the above contains all of the required information. It's also easy (enough) to see that exchanging two rows in the original A will lead to the same final form of $\det A$, up to numerical factors and/or negative signs. In conclusion, we take the the order-three determinant to be

$$\begin{aligned} \det A = & \ A_{11} \left(A_{22}A_{33} - A_{32}A_{23} \right) \\ & - A_{12} \left(A_{21}A_{23} - A_{31}A_{33} \right) \\ & + A_{13} \left(A_{21}A_{32} - A_{31}A_{22} \right) . \end{aligned} \tag{1.22}$$

Four Dimensions

Having witnessed the trick of performing row operations on an order-three square matrix A to condense all relevant information into a 2×2 square sub-matrix, this should also work for higher-order matrices. Indeed, the order-four matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} ,$$

permits a similar process to reduce the order of the problem. Doing so, the order-four determinant becomes the sum of four terms:

$$\begin{aligned} \det A = A_{11} \det \begin{bmatrix} A_{22} & A_{23} & A_{24} \\ A_{32} & A_{33} & A_{34} \\ A_{42} & A_{43} & A_{44} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{bmatrix} \\ + A_{13} \det \begin{bmatrix} A_{21} & A_{22} & A_{24} \\ A_{31} & A_{32} & A_{34} \\ A_{41} & A_{42} & A_{44} \end{bmatrix} - A_{14} \det \begin{bmatrix} A_{12} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \end{aligned} \tag{1.23}$$

4.1 Sub-Matrix and Matrix Minor

Taking another look at the three- and four-dimensional determinant formulas, we see the right side contains the sum of several 'cross sections' of the original matrix, each having dimension $N - 1$.

Sub-Matrix

The *sub-matrix* S_{jk} removes the jth row and the kth column from the original matrix A.

Matrix Minor

The *matrix minor*, denoted M_{jk} is the determinant of the sub-matrix S_{jk}. With matrix minor notation, equations (1.22), (1.23) can be written:

$$\begin{aligned} \det A_{(3)} &= A_{11}M_{11} - A_{12}M_{12} + A_{13}M_{13} \\ \det A_{(4)} &= A_{11}M_{11} - A_{12}M_{12} + A_{13}M_{13} - A_{14}M_{14} \end{aligned}$$

N Dimensions

Using matrix minor notation, the three and four-dimensional determinant formulas suggest of how to handle the N-dimensional case. Formally, the procedure is to use row operations to condense down-andd-right all of the information on the matrix. After the dust settles, the general formula for the determinant of an order-N matrix is remarkably simple:

$$\text{if } N = 1 : \quad \det A = \det \begin{bmatrix} A_{11} \end{bmatrix}$$

$$\text{if } N > 1 : \quad \det A = \sum_{\substack{j \leq N \\ k=1}}^{N} (-1)^{j+k} A_{jk} M_{jk} \tag{1.24}$$

Note too that the summation can take place over rows *or* columns, meaning

$$\text{if } N > 1 : \quad \det A = \sum_{\substack{j \leq N \\ k=1}}^{N} (-1)^{j+k} A_{kj} M_{kj}$$

also holds. Note that the variable j is fixed at some integer less than N. Only the k-variable is summed over.

Multiplication Rules

Readily shown from the properties of determinants are various multiplication rules. A scalar α multiplied by a matrix A of dimension N has the result

$$\det (\alpha A) = \alpha^N \det A \,.$$

Meanwhile, for the product of two matrices A, B:

$$\det (AB) = \det (A) \, \det (B)$$

Row Operations

The row operations E (row exchange), M (multiply by scalar), R (combine rows) have the following effect on the determinant:

$$\begin{aligned} \det (EA) &= - \det A \\ \det (MA) &= \alpha \det A \\ \det (RA) &= \det A \end{aligned}$$

Transpose Rule

Interestingly but not surprisingly, the matrix and its transpose have the same determinant:

$$\det \left(A^T \right) = \det A$$

5 Inverse Matrix

Definition

Given a square matrix A of dimension N, there *may* exist a special matrix A^{-1} that obeys the property

$$A^{-1}A = AA^{-1} = I\,, \tag{1.25}$$

where A^{-1} is called the *inverse matrix* to A, and I is the identity matrix of dimension N. The product of A and its inverse, or vise versa, results in the identity matrix.

Existence

For the notion of the inverse to make sense, the matrix A must perform a one-to-one mapping of a vector $\vec{x}$ to a vector $\vec{b}$ as in $A\vec{x} = \vec{b}$. By multiplying A^{-1} into both sides of $A\vec{x} = \vec{b}$, we end up with

$$A^{-1}A\vec{x} = A^{-1}\vec{b}\,,$$

effectively 'solving' for the vector $\vec{x}$:

$$\vec{x} = A^{-1}\vec{b} \tag{1.26}$$

Formula

To come up with a formula for the inverse of A, consider another matrix B defined from the components A_{jk} such that

$$B_{jk} = (-1)^{j+k} M_{kj}\,, \tag{1.27}$$

where M_{kj} are the matrix minors of A. As innocent as it looks, (1.27) is quite 'computationally expensive', which means as N grows, it requires preposterous efforts to calculate the B-matrix by hand.

To proceed, calculate the product AB by matrix multiplication. Starting with the formula for matrix multiplication, and replacing the components of B using (1.27), we find

$$(AB)_{mn} = \sum_{k=1}^{N} A_{mk}B_{kn} = \sum_{k=1}^{N} A_{mk}\,(-1)^{k+n} M_{nk}\,.$$

In the case $m = n$, the above reduces to the formula for the determinant of A, namely (1.24). Any other case $m \neq n$ causes the right side to resolve to zero:

$$(AB)_{mn} = \begin{cases} \det A & m = n \\ 0 & m \neq n \end{cases}$$

In symbolic terms, the above reads

$$AB = (\det A)\, I\,,$$

where by comparison to (1.25), suggests the combination $B/\det A$ is equal to the inverse of A:

$$A^{-1} = \frac{1}{\det A} B \tag{1.28}$$

Products

The inverse of the product of two matrices is equal to the product of the individual inverses in reversed order:

$$(AB)^{-1} = B^{-1}A^{-1} \tag{1.29}$$

5.1 Cramer's Rule Derived

For a linear system of N dimensions, we start with the dichotomy

$$\begin{aligned} A\vec{x} &= \vec{b} \\ \vec{x} &= A^{-1}\vec{b}\,, \end{aligned}$$

where accounting for (1.28), the 'solution' vector $\vec{x}$ is written

$$\vec{x} = \frac{1}{\det A} B\vec{b}\,,$$

or using index notation,

$$x_j = \frac{1}{\det A}\sum_{k=1}^{N} B_{jk}b_k = \frac{1}{\det A}\sum_{k=1}^{N} (-1)^{j+k} M_{kj}b_k\,.$$

The product $M_{kj}b_k$ has the k, j indices in 'reverse' order, in the sense that the calculation $M\vec{b}$ does *not* represent this situation. Instead, the sum constitutes the determinant of a matrix modified from A such that the kth column is replaced by $\vec{b}$. If this situation sounds familiar, it precisely describes the matrix introduced as equation (1.20)

$$C_j = \begin{bmatrix} A_{11} & A_{12} & \cdots & b_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & b_{2j} & \cdots & A_{2N} \\ A_{31} & A_{32} & \cdots & b_{3j} & \cdots & A_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & b_{Nj} & \cdots & A_{NN} \end{bmatrix}\,,$$

and the above reduces to Cramer's rule (1.19) for the soltuion of the system:

$$\begin{aligned} x_j &= \frac{\det C_j}{\det A} \\ j &= 1, 2, 3, \ldots, N \end{aligned}$$

Two Dimensions

Consider the two-dimensional matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

whose determinant is given by (1.14). To calculate the inverse, begin with the B matrix given by (1.27), coming out to

$$B = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} .$$

Then, by the inverse formula (1.28), the inverse of the 2×2 matrix reads:

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{1.30}$$

Three Dimensions

The three-dimensional matrix with

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

has a total of nine minors M_{jk}, readily readable from A. Constructing the matrix B using

$$B_{jk} = (-1)^{j+k} M_{kj} ,$$

we find

$$\begin{aligned}
B_{11} &= (-1)^2 (A_{22}A_{33} - A_{32}A_{23}) \\
B_{12} &= (-1)^3 (A_{12}A_{33} - A_{32}A_{13}) \\
B_{13} &= (-1)^4 (A_{12}A_{23} - A_{22}A_{13}) \\
B_{21} &= (-1)^3 (A_{21}A_{33} - A_{31}A_{23}) \\
B_{22} &= (-1)^4 (A_{11}A_{33} - A_{31}A_{13}) \\
B_{23} &= (-1)^5 (A_{11}A_{23} - A_{21}A_{13}) \\
B_{31} &= (-1)^4 (A_{21}A_{32} - A_{31}A_{22}) \\
B_{32} &= (-1)^5 (A_{11}A_{32} - A_{31}A_{12}) \\
B_{33} &= (-1)^6 (A_{11}A_{22} - A_{21}A_{12}) ,
\end{aligned}$$

In matrix form, the above reads:

$$B = \begin{bmatrix} (A_{22}A_{33} - A_{32}A_{23}) & -(A_{12}A_{33} - A_{32}A_{13}) & (A_{12}A_{23} - A_{22}A_{13}) \\ -(A_{21}A_{33} - A_{31}A_{23}) & (A_{11}A_{33} - A_{31}A_{13}) & -(A_{11}A_{23} - A_{21}A_{13}) \\ (A_{21}A_{32} - A_{31}A_{22}) & -(A_{11}A_{32} - A_{31}A_{12}) & (A_{11}A_{22} - A_{21}A_{12}) \end{bmatrix}$$

With the matrix B fully specified in terms of A, the inverse A^{-1} is given by (1.28), namely

$$A^{-1} = \frac{1}{\det A} B .$$

Note that $\det A$ was already calculated as equation (1.22).

6 Special Matrices

Transpose Matrix

Given a matrix A, there always exists the *transpose* of A, which swaps all rows for columns and vice versa. The transpose of a matrix A is denoted A^T, particularly

$$A^T_{jk} = A_{kj} . \tag{1.31}$$

Symmetric Matrix

A square matrix is said to be *symmetric* if the original matrix A is equal to the transposed matrix A^T:

$$\begin{aligned} A &= A^T \\ A_{jk} &= A_{kj} \end{aligned} \tag{1.32}$$

Anti-symmetric Matrix

A square matrix is said to be *anti-symmetric* if the original matrix A is equal to the negative transposed matrix A^T:

$$\begin{aligned} A &= -A^T \\ A_{jk} &= -A_{kj} \end{aligned} \tag{1.33}$$

This is sometimes known as *skew-symmetric*.

Orthogonal Matrix

A square matrix A whose transpose A^T is equal to the inverse A^{-1} is called an *orthogonal matrix*:

$$A^T = A^{-1} \tag{1.34}$$

Role of Row Operations

A general $M \times N$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix}$$

can be *reduced* by any operations E, M, R to produce a different matrix A' that contains the same information or similar information to A. This process can be applied sequentially to achieve various matrix forms cataloged below.

6.1 Triangular Forms

Square matrices with $M = N$ admit two special reduced forms called *triangular forms.*

Upper Triangular Form

If (by row operations or otherwise) a square matrix has $A_{jk} = 0$ when $j > k$, the form is called *upper triangular*:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & A_{nn} \end{bmatrix} \tag{1.35}$$

Lower Triangular Form

If a square matrix has $A_{jk} = 0$ when $j < k$, the form is called *lower triangular*:

$$A = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ \cdots & \cdots & \cdots & 0 \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \tag{1.36}$$

6.2 Diagonal Form

If a square matrix has $A_{jk} = 0$ when $j \neq k$, the form is called *diagonal*:

$$A = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{nn} \end{bmatrix} \tag{1.37}$$

For any triangular or diagonal matrix A, the determinant is equal to the product of its diagonal entries:

$$\det A = A_{11} A_{22} \cdots A_{NN} = \prod_{j=1}^{N} A_{jj}$$

6.3 Augmented Matrix

For linear systems characterized by $A\vec{x} = \vec{b}$, where A is an $M \times N$ matrix, we can construct the *augmented* matrix by appending the components of $\vec{b}$ as an extra column:

$$A|b = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2N} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} & b_M \end{bmatrix} \tag{1.38}$$

In the general case, $\vec{b}$ can be replaced with any matrix with M rows.

6.4 Row-Reduced Echelon Form

If (by any means) the a square matrix and a vector $\vec{x}$ can be written as

$$I|x = \begin{bmatrix} 1 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & \cdots & 0 & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_N \end{bmatrix} , \tag{1.39}$$

this is called the *row-reduced echelon form.*

7 Elimination

7.1 Linear Systems

Consider an N-dimensional linear system $A\vec{x} = \vec{b}$, represented by the $M = N$-case of (1.17):

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1N}x_N &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2N}x_N &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots + A_{3N}x_N &= b_3 \\ &\cdots \\ A_{N1}x_1 + A_{N2}x_2 + A_{N3}x_3 + \cdots + A_{NN}x_N &= b_N \end{aligned}$$

Equivalently, the above is represented by an augmented matrix $A|b$ of the form (1.38). Next, imagine having done all of the hard work to solve the system

$$\begin{aligned} x_1 + 0 + 0 + \cdots + 0 &= x_1 \\ 0 + x_2 + 0 + \cdots + 0 &= x_2 \\ 0 + 0 + x_3 + \cdots + 0 &= x_3 \\ &\cdots \\ 0 + 0 + 0 + \cdots + x_N &= x_N , \end{aligned}$$

which appears like a tautological thing to write, but is in fact the row-reduced echelon form, $I|x$ cataloged as equation (1.39). Written this way, the solutions x_j to the system are readily exportable as the right side of each equation.

The natural question is, how can we start with $A|b$ and somehow end up with with $I|x$ using matrix trickery? The answer is called *elimination*, which is a sequence of row operations E, M, R that we carry out on the augmented matrix $A|b$ to bring it the form $I|x$. Representing the exact sequence of row operations as one 'operator' $\tilde{O}\,(E, M, R)$ or simply $\tilde{O}$, one writes

$$\tilde{O}\,(A|b) = I|x\,. \tag{1.40}$$

One may think of $\tilde{O}$ as a sequential list of procedures to carry out on $A|b$, much as a program receives input and returns output.

Example

Consider a linear system represented by the augmented matrix

$$A|b = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{bmatrix}\,.$$

Denoting the rows of $A|b$ as R_j, the first three operations may go as follows: (i) Subtract $2R_1$ from R_2. (ii) Subtract $4R_1$ from R_3. (iii) Add $4R_2$ to R_3.

$$A|b \xrightarrow{\text{(i)}} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{bmatrix} \xrightarrow{\text{(ii)}} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{bmatrix} \xrightarrow{\text{(iii)}} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{bmatrix}$$

Note the new matrix has zeros down and left of the diagonal, i.e. upper triangular form. Don't stop here though: (iv) Divide R_3 by 13 and subtract $3R_3$ from R_2. (v) Subtract R_3 from R_1. (vi) Subtract R_2 from R_1.

$$\xrightarrow{\text{(iv)}} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{(v)}} \begin{bmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{(vi)}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} = I|x$$

Elimination halts when the 'matrix part' of the above reduces to the identity. Reading off the right-hand column, we see the solution to the system of equations is

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 4 \\ x_3 &= -2\,. \end{aligned}$$

Reconciling what just happened with equation (1.40), we see the operator $\tilde{O}$ is comprised of steps (i)-(vi), each being one particular E, M, R operation.

7.2 Matrix Inverse

Looking again at Equation (1.40), note that the sequence of row operations $\tilde{O}$ applies to A and $\vec{b}$ separately:

$$\begin{aligned} \tilde{O}A &= I \\ \tilde{O}\vec{b} &= \vec{x} \end{aligned}$$

While $\tilde{O}$ is not established as a matrix, it does precisely same job as A^{-1}, and must contain the same information as A^{-1}. As a point of comparison, note the similarity between the above versus familiar relations

$$\begin{aligned} A^{-1}A &= I \\ A^{-1}\vec{b} &= \vec{x}\,. \end{aligned}$$

Going with the hunch that $\tilde{O}$ can be treated as an operator that obeys the associativity rule of matrix multiplication, we would be able to do the following:

$$\begin{aligned} \tilde{O}A &= I \\ \left(\tilde{O}A\right)A^{-1} &= IA^{-1} \\ \tilde{O}\left(AA^{-1}\right) &= A^{-1} \\ \tilde{O}I &= A^{-1} \end{aligned}$$

Once again, we see the sequence $\tilde{O}$ is doing the same job as A^{-1}. Rounding up the circumstancial evidence, we see the set of steps $\tilde{O}$ that carries $A \to I$ is the *same* set of steps that carries $I \to A^{-1}$. In the language of augmented matrices, this is summarized by

$$\tilde{O}\,(A|I) = I|A^{-1} \,. \tag{1.41}$$

This conspiracy of mathematics is otherwise known as *Gauss-Jordan elimination.*

Two Dimensions

Demonstrating on a 2×2 matrix, begin with $A|I$ as

$$A|I = \begin{bmatrix} A_{11} & A_{12} & 1 & 0 \\ A_{21} & A_{22} & 0 & 1 \end{bmatrix} \,,$$

and perform row operations until form $I|A^{-1}$ is attained. In brief detail, the augmented matrix develops as:

$$A|I \to \begin{bmatrix} A_{12}A_{21} - A_{22}A_{11} & 0 & -A_{22} & A_{21} \\ A_{21} & A_{22} & 0 & 1 \end{bmatrix}$$

$$A|I \to \frac{1}{\det A} \begin{bmatrix} 1 & 0 & -A_{22} & A_{21} \\ 0 & 1 & -A_{21} & A_{11} \end{bmatrix}$$

The final result is none other than (1.30), the formula for the inverse of a 2×2 square matrix:

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

8 Eigenvectors and Eigenvalues

An important situation that arises in mathematics and physics is the so-called *eigenvalue problem*

$$A\vec{u} = \lambda\vec{u} \,. \tag{1.42}$$

The matrix A is taken to be square and N-dimensional. The vectors $\vec{u}^{(j)}$ that satisfy (1.42) are called *eigenvectors*, and the corresponding scalar $\lambda^{(j)}$ is called an *eigenvalue.*

8.1 Calculating Eigenvalues

The eivenvalue problem (1.42) can be equivalently framed as

$$(A - \lambda I)\,\vec{u} = 0 \,, \tag{1.43}$$

where I is the identity matrix to match the dimension of A.

Two Dimensions

Taking a two-dimensional case as an example, we have

$$A - \lambda I = \begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix} \,,$$

which, as a set of equations, looks like

$$\begin{aligned} (A_{11} - \lambda)\,x_1 &= -A_{12}x_2 \\ (A_{22} - \lambda)\,x_2 &= -A_{12}x_1 \end{aligned}$$

Multiply the pair of equations and cancel the product x_1x_2 to get

$$(A_{11} - \lambda)\,(A_{22} - \lambda) - A_{12}A_{21} = 0 \,. \tag{1.44}$$

The only unknown in the equation is λ, which can be isolated using the quadratic formula:

$$\lambda_\pm = \frac{A_{11} + A_{22}}{2} \pm \frac{1}{2}\sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}} \tag{1.45}$$

Note there are two solutions for λ, which we label λ_+, and λ_-, respectively.

8.2 Characteristic Equation

When confronted with the eigenvalue problem (1.43), the first order of business, usually, is to calculate the eigenvalues λ. As we've seen for the two-dimensional case, this process boiled down to equation (1.44). Pausing on this result for a moment, note that a quicker way to get there is to write

$$\det\,(A - \lambda I) = 0 \,, \tag{1.46}$$

which is in fact true in any number of dimensions. Equation (1.46) is called the *characteristic equation* of the system.

Characteristic Polynomial

The characteristic equation always 'simplifies' to the *characteristic polynomial*, a single equation embedding λ:

$$P_N\,(\lambda) = C_0 + C_1\lambda + C_2\lambda^2 + \cdots + C_N\lambda^N = 0 \tag{1.47}$$

The characteristic polynomial is suggestive of the *fundamental theorem of algebra*, stating that there are exactly N (complex) roots of a polynomial of degree N.

8.3 Calculating Eigenvectors

Once the eigenvalues λ are known, the components of each eigenvector $\vec{u}$ are readily calculated directly from

$$\begin{aligned} A\vec{u}^{(j)} &= \lambda_j\vec{u}^{(j)} \\ j &= 1, 2, 3, \ldots, N \,. \end{aligned}$$

Two Dimensions

Developing the eigenvalue problem in two dimensions, there are two eigenvalues $\lambda_\pm$ given by (1.45), and let us label the two corresponding eigenvectors $\vec{u}$, $\vec{v}$ such that

$$A\vec{u} = \lambda_+\vec{u}$$
$$A\vec{v} = \lambda_-\vec{v} \,.$$

Working with the left equation first, it expands into two equations

$$A_{11}u_1 + A_{12}u_2 = \lambda_+ u_1$$
$$A_{12}u_1 + A_{22}u_2 = \lambda_+ u_2 \,,$$

which gives us two ways to solve for the ratio u_1/u_2:

$$\frac{u_1}{u_2} = \frac{-A_{12}}{A_{11} - \lambda_+} \tag{1.48}$$
$$\frac{u_1}{u_2} = \frac{-(A_{22} - \lambda_+)}{A_{21}}$$

As a sanity check, we may eliminate the ratio u_1/u_2 and recover the characteristic equation (1.44). A similar set of steps isolates the ratio v_1/v_2 for the second eigenvalue/eigenvector

$$\frac{v_1}{v_2} = \frac{-A_{12}}{A_{11} - \lambda_-} \tag{1.49}$$
$$\frac{v_1}{v_2} = \frac{-(A_{22} - \lambda_-)}{A_{21}} \,, \tag{1.50}$$

which also combine to reproduce the characteristic equation, so we're on the right track.

Symmetric Matrix

Suppose the matrix A is given as

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \,.$$

The eigenvalues of A are given by (1.45), and simplify very nicely:

$$\lambda_\pm = a \pm b$$

Denoting the respective eigenvectors $\vec{u}$, $\vec{v}$, we apply (1.48) directly to find

$$\frac{u_1}{u_2} = \frac{-b}{-b} = 1 \,.$$

Meanwhile, (1.49) similarly tells us

$$\frac{v_1}{v_2} = \frac{-b}{b} = -1 \,,$$

and we're done. Evidently, the two eigenvectors are

$$\vec{u} = \langle 1, 1\rangle$$
$$\vec{v} = \langle 1, -1\rangle \,,$$

or in normalized form,

$$\hat{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$$
$$\hat{v} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle \,.$$

Hermitian Matrix

For the Hermitian matrix

$$A = \begin{bmatrix} a & -ib \\ ib & a \end{bmatrix} \,,$$

the characteristic equation is

$$(a - \lambda) + i^2 b^2 = 0 \,,$$

or

$$\lambda_\pm = a \mp b \,.$$

Despite having complex components, the eigenvalues are real-valued.

Complex Eigenvalues

Modifying the above example, consider

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \,.$$

Following the same steps, we find the eigenvalues to be complex:

$$\lambda_\pm = a \pm ib$$

This is no hindrance, however. The complexity passes to the eigenvectors, which turn out to be:

$$\hat{u} = \frac{1}{\sqrt{2}}\langle 1, i\rangle$$
$$\hat{v} = \frac{1}{\sqrt{2}}\langle 1, -i\rangle$$

8.4 Diagonalization

For the eigenvalue problem (1.42)

$$A\vec{u} = \lambda\vec{u}$$

of dimension N, suppose we already have the list of N eigenvalues λ and N eigenvectors $\vec{u}$.

Modal Matrix

It's instructive to condense all of the eigenvector information into a new object called the *modal matrix*, denoted C, whose jth column is comprised of the components of the jth eigenvector:

$$\begin{aligned} C &= \begin{bmatrix} u_1^{(1)} & u_1^{(2)} & \cdots & u_1^{(N)} \\ u_2^{(1)} & u_2^{(2)} & \cdots & u_2^{(N)} \\ \cdots & \cdots & \cdots & \cdots \\ u_N^{(1)} & u_N^{(2)} & \cdots & u_N^{(N)} \end{bmatrix} \\ &= \begin{bmatrix} \vec{u}^{(1)} & \vec{u}^{(2)} & \cdots & \vec{u}^{(N)} \end{bmatrix} \end{aligned} \tag{1.51}$$

Then, the matrix product AC can be written

$$\begin{aligned} AC &= \begin{bmatrix} \lambda_1 \vec{u}^{(1)} & \lambda_2 \vec{u}^{(2)} & \cdots & \lambda_N \vec{u}^{(N)} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 u_1^{(1)} & \lambda_2 u_1^{(2)} & \cdots & \lambda_N u_1^{(N)} \\ \lambda_1 u_2^{(1)} & \lambda_2 u_2^{(2)} & \cdots & \lambda_N u_2^{(N)} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 u_N^{(1)} & \lambda_2 u_N^{(2)} & \cdots & \lambda_N u_N^{(N)} \end{bmatrix} . \end{aligned}$$

Diagonal Matrix

The product AC, especially in matrix form, looks like the product of C with another, much simpler matrix. Consider a *diagonal* matrix Λ (Greek uppercase lambda) whose off-diagonal entries are all zero, and the eigenvalues occupy the diagonal:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \tag{1.52}$$

Indeed, the right result of AC is reproduced by the product $C\Lambda$, meaning the matrix products are equal:

$$AC = C\Lambda$$

Supposing the inverse of C can be attained, the diagonal matrix Λ can be isolated:

$$\Lambda = C^{-1} AC \tag{1.53}$$

The process of attaining Λ is called the *diagonalization* of the matrix A. If the columns of C happen to form an orthonormal basis, the inverse matrix C^{-1} may be replaced with its transpose C^T.

8.5 Hermitian Matrix

Eigenvectors as a Basis

It is no coincidence that a system of N dimensions has N eigenvectors. It makes sense to wonder if an arbitrary linear combination can be expressed via the change-of-basis formula for vectors.

$$\vec{V} = \sum_{j=1}^{N} V_j \, \hat{e}_j \overset{?}{\to} (\vec{V})' = \sum_{j=1}^{N} V_j' \, \hat{u}^{(j)}$$

In the above, the eigenvectors are assumed to be normalized (unit magnitude), which is always possible for nonzero vectors. However, we are *not* to assume that the eigenvectors $\left\{\hat{u}^{(j)}\right\}$ form an orthogonal basis. That is, it's not always the case that any two eigenvectors are orthogonal.

Consider two solutions to the eigenvalue problem (1.42),

$$\begin{aligned} A\vec{u}^{(j)} &= \lambda_j \vec{u}^{(j)} \\ A\vec{u}^{(k)} &= \lambda_k \vec{u}^{(k)} , \end{aligned}$$

and multiply $\vec{u}^{(k)}$, $\vec{u}^{(j)}$, onto the left and right sides respectively into each:

$$\begin{aligned} \vec{u}^{(k)} \cdot A\vec{u}^{(j)} &= \lambda_j \vec{u}^{(k)} \cdot \vec{u}^{(j)} \\ A\vec{u}^{(k)} \cdot \vec{u}^{(j)} &= \lambda_k \vec{u}^{(k)} \cdot \vec{u}^{(j)} \end{aligned}$$

Looking at the left side of each equation, it appears as if

$$\vec{u}^{(k)} \cdot A\vec{u}^{(j)} = A\vec{u}^{(k)} \cdot \vec{u}^{(j)} \tag{1.54}$$

wants to be true, but simply isn't in the general case. The special case that satisfies (1.54) is called a *Hermitian* matrix.

Non-equal Eigenvalues

Pursuing the case where A is Hermitian, the above condenses to:

$$\lambda_j \vec{u}^{(k)} \cdot \vec{u}^{(j)} = \lambda_k \vec{u}^{(k)} \cdot \vec{u}^{(j)}$$

Now, if we assume that $\lambda_j \neq \lambda_k$, the *only* way to reconcile this result is that *non-equal eigenvectors of a Hermitian matrix are orthogonal*:

$$\vec{u}^{(k)} \cdot \vec{u}^{(j)} = 0$$

Just as importantly, this reinforces that the eigenvectors of a non-Hermitian matrix may not be orthogonal.

Equal Eigenvalues

If m of the N eigenvalues are equal, one speaks of *m-fold degeneracy*. In this case, the corresponding eigenvectors form a *vector subspace* of the original vector space that might admit its own orthonormal basis.

9 Degenerate Systems

Concerning the eigenvalue problem (1.42)

$$A\vec{u} = \lambda\vec{u} \, ,$$

it could turn out that two eigenvalues λ_j, λ_k are equal, in which case we *may* be able to construct N unique eigenvectors $\vec{u}^{(j)}$, but not always. Specifically, for each repeated eigenvalue λ_j of multiplicity m_j, there must be m_j linearly independent eigenvectors. The ability to successfully do this depends on the system on hand.

Dead-end Case

Consider the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \, ,$$

having a characteristic polynomial

$$(-2-\lambda)(1-\lambda)(-2-\lambda) = 0 \, .$$

Evidently we find three eigenvalues, with two identical:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -2 \\ \lambda_3 &= -2 \end{aligned}$$

Handling the easy case first, the eigenvector corresponding to λ_1 is calculated from $A\vec{u} = 1\vec{u}$, resulting in

$$\vec{u}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \, .$$

Proceeding to the repeated eigenvalue case, we solve $A\vec{u} = -2\vec{u}$ to get a single eigenvector

$$\vec{u}^{(2,3)} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \, .$$

Note that $\vec{u}^{(1)}$ is linearly independent from $\vec{u}^{(2,3)}$, but not orthogonal. Since there is no obvious way to 'peel apart' the eigenvectors $\vec{u}^{(2,3)}$, the show stops here. The matrix A cannot be diagonalized.

Salvageable Case

Consider the matrix

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \, ,$$

having a characteristic polynomial

$$0 = \lambda^3 + \lambda^2 - 5\lambda + 3 \, ,$$

which factors into

$$0 = (\lambda - 1)(\lambda - 1)(\lambda + 3) \, .$$

We have three eigenvalues, with two identical:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 \\ \lambda_3 &= -3 \end{aligned}$$

Handling the easy case first, the eigenvector corresponding to λ_3 is calculated from $A\vec{u} = -3\vec{u}$, leading to the relations

$$\begin{aligned} 2u_1 - u_2 + u_3 &= 0 \\ u_1 - u_2 + 2u_3 &= 0 \\ 3u_1 - 2u_2 + 3u_3 &= 0 \, , \end{aligned}$$

telling us the corresponding eigenvector is

$$\vec{u}^{(3)} = \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \, ,$$

or any multiple.

Proceeding to the repeated eigenvalue case, we solve $A\vec{u} = \vec{u}$ to generate three copies of

$$u_1 - u_2 + u_3 = 0 \, .$$

With one equation and three unknowns, we may choose any *two* values to be arbitrary. For instance, we may choose $u_1 = 1$ with $u_2 = 0$, causing $u_3 = -1$. We then construct an eigenvector from these numbers:

$$\vec{u}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

On the other hand, we may choose $u_1 = 0$, $u_2 = 1$, causing $u_3 = 1$, to create another eigenvector, linearly independent from the others:

$$\vec{u}^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

With three eigenvectors in hand, a modal matrix can be defined such that

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \, ,$$

allowing the matrix A to be diagonalized using $\Lambda = C^{-1}AC$.

Normalizable Case

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix} \,,$$

having a characteristic polynomial

$$0 = (2-\lambda)\left(-\lambda^2 + \lambda + 2\right) \,,$$

indicating three eigenvalues, with two identical:

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \\ \lambda_3 &= -1 \end{aligned}$$

Handling the easy case first, the eigenvector corresponding to λ_3 is calculated from $A\vec{x} = -\vec{x}$, leading to

$$\vec{u}^{(3)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix} \,.$$

Proceeding to the repeated eigenvalue case, we solve $A\vec{u} = 2\vec{u}$ to get a single eigenvector

$$\vec{u}^{(1,2)} = \frac{1}{\sqrt{3\alpha^2/2 + \beta^2}} \begin{bmatrix} \alpha \\ \beta \\ \alpha/\sqrt{2} \end{bmatrix} \,,$$

for two arbitrary constants α, β. The aim here is to tease two mutually orthogonal eigenvectors from the above, which means to require

$$\vec{u}^{(1)} \cdot \vec{u}^{(2)} = 0 \,.$$

This amounts to finding pairs of α_j, β_j that satisfy

$$\frac{3}{2}\alpha_1\alpha_2 + \beta_1\beta_2 = 0 \,.$$

Choosing $\alpha_1 = 0$ begins a fast avalanche that requires $\beta_1 = 1$, and also $\beta_2 = 0$, with α_2 remaining arbitrary. The remaining eigenvectors therefore read

$$\begin{aligned} \vec{u}^{(1)} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \vec{u}^{(2)} &= \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix} \,. \end{aligned}$$

With three eigenvectors in hand, a modal matrix can be defined such that

$$C = \begin{bmatrix} 1 & 0 & \sqrt{2/3} \\ 0 & 1 & 0 \\ -\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix} \,,$$

allowing the matrix A to be diagonalized via $\Lambda = C^{-1}AC$. However, since the set of eigenvectors $\left\{\vec{u}^{(j)}\right\}$ form an orthonormal basis, we may further simplify the above using $C^{-1} = C^T$.

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