Limits, Functions, Sequences
MANUSCRIPT

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Contents

Chapter 1

Limits, Functions, Sequences

1 Limits

In mathematics, one speaks of *limit* when a starting value, call it A, is 'becoming' another value B:

$$A \to B$$

The use of the right-arrow ($\to$) is to remind that using an equal sign, i.e. simply writing $A = B$ is a hasty, possibly illegal jump through the real numbers.

1.1 Notion of Limit

The 'notion of limit' probably seems pedantic and perhaps mundane at first sight, but the ancient Greeks, particularly Zeno, famously failed to reconcile limits as anything but mathematical barriers in the universe and deemed the whole thing a paradox. The understanding of limits eluded popular thinking until the days of Newton and Leibniz.

Precisely 'how' a limit is traversed has significance, and is usually facilitated by the assumption that there is a smooth continuum of values, usually real numbers represented by x, between the initial and final points of the limit, i.e.

$$A \leq x < B \; .$$

Notice the mismatch in comparison symbols in the above, as we have $\leq$ on the A-side and $<$ on the B-side. This is because we shouldn't outright assume that x ever reaches B precisely. The condition $x = B$ is considered a special case.

1.2 Differential Limit

Compressing the picture of 'limit', suppose the gap between two values were extremely small, or 'vanishingly' small, or 'arbitrarily' small. There are synonyms for a *differential limit*.

To continue, suppose we are interested in the distance from any fixed point x_0 to a neighboring point x. To capture this one may begin with

$$\Delta x = x - x_0 \; ,$$

where on the left is the usual 'delta x' symbol for representing net displacement.

If the distance Δx is to be very small, i.e. if x is close to x_0, we can write the above as a differential limit:

$$dx = \lim_{x \to x_0} x - x_0 \tag{1.1}$$

The term Δx is replaced by dx, which means 'differential x'. On the right, there is a new 'limit' term slipped in to remind that $x - x_0$ is close to zero.

Note that the differential limit does not apply to integers, whole numbers, or natural numbers. Only real (and complex, but never mind) numbers make sense in light of the differential limit.

Very Small Numbers

In the differential limit, the interval dx refers to a 'small' number. It should make sense that things like dx^2 and dx^3 are absurdly small numbers, and can often be omitted when they pop up in calculations.

To illustrate, suppose we have a number A that is the sum of a close-by number A_0 and a differential quantity dx

$$A = A_0 + dx \; ,$$

and let us become interested in the quantity A^2, attained by squaring both sides of the above:

$$A^2 = A_0^2 + 2A_0 dx + dx^2$$

Now, we know already that A^2 is approximately equal to A_0^2 plus some small correction having to do with dx, namely $2A_0 dx + dx^2$.

One can see that, especially in the infinite limit, that dx^2 will vanish toward zero much 'faster' than the middle term, thus dx^2 can be omitted outright in this case:

$$A^2 \approx A_0^2 + 2A dx$$

Problem 1

Let $A = A_0 + dx$ and $B = B_0 + dy$. Calculate the product AB and rank resulting terms by magnitude.

1.3 Infinite Limit

Another use for limits is pushing a variable toward infinity, which occurs in the formula for the natural exponential named after Euler:

$$e^x = \lim_{h \to \infty} \left(1 + \frac{x}{h}\right)^h \tag{1.2}$$

In the above, we see a curious struggle arise with increasing h. The $1/h$-term inside the parentheses tries to drag the result downward, but the exponent h tries to drag the result back upward. The end result is $e \approx 2.71828\ldots$.

Problem 2

Show that:

$$\lim_{x \to 1} x^{1/(1-x)} = e$$

Infinite Sum

An infinite limit can take the form of an *infinite sum*, as is the case with geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{1.3}$$

As long as $|x| < 1$, the above always holds, quite astonishingly.

On the left is the simple fraction $1/(1-x)$, where on the right is all whole-number powers of x, all the way to x^∞, summed together.

We can see there is some kind of limit at play in the geometric series even though x is fixed. Rather, the number of terms in the series, which can only be a whole number, is the variable being pushed.

Summation Notation

An equivalent statement of the above that uses the 'lim' nomenclature goes like

$$\frac{1}{1-x} = \lim_{n \to \infty} \sum_{j=0}^{n} x^j \; ,$$

where j is an index variable and n is its upper limit (a positive integer). Keep in kind $|x| < 1$ for the formula to work.

The uppercase E-like symbol is the clue for a summation. The variable j is called the *index.*, which in this case, assumed all integer values 0, 1, 2, ..., n.

One

A psychological trick used in marketing is to price an item that is, say, \$7.00 (U.S. dollars, but currency doesn't matter) as \$6.99 instead. The idea is that the shopper only sees the 6 and forgets that the price is closer to \$7. It's typical for fuel prices to be advertised with a third digit after the decimal, always a 9, so drivers are tricked out of an extra $\approx$ \$0.01 per gallon purchased.

Suppose the price of an item for sale is $p = \$0.9999\overline{9}$, with the 9s carrying on *forever*. You hand the clerk a \$1 note to pay for the item. Did you overpay?

The instinct that says 'yes, the clerk owes you some change for the transaction, even if it's a small amount' is overturned by geometric series. Observe that an infinite string of 9s following a decimal can be converted to an infinite sum of fractions:

$$p = 0.9999\cdots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$$

Factor 9/10 from the right side

$$p = \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots\right),$$

and then see that the parenthesized quantity is precisely the geometric series as the right side of Equation (1.3) with $x = 1/10$. This means the infinite sum can be replaced by $1/(1-x)$:

$$p = \frac{9}{10}\left(\frac{1}{1-1/10}\right) = \frac{9}{10}\left(\frac{10}{9}\right) = 1$$

See what just happened? We have two expressions for the variable p, which can only mean they are equal:

$$0.9999\overline{9} = p = 1$$

Thus the clerk owes you nothing back, and the item price is exactly \$1 despite the advertising.

1.4 Convergence and Divergence

When a limit 'settles on' a reasonable answer, i.e., a finite real number, one speaks of *convergence*. For instance, Equation (1.2) converges to Euler's constant as h goes to infinity. The larger we make h, the slower the sum changes. Convergence also applies to the geometric series as Equation (1.3). When $|x| < 1$, the terms on the right decrease in magnitude until making vanishingly small contribution to the overall sum.

When convergence does not occur, it is likely that the object in hand exhibits *divergence*, which means tending toward $\pm\infty$. This is easy to see with the geometric series when purposefully choosing $|x| > 1$. On the right, there are ever-increasing powers of x, and for every new term added the sum changes drastically, refusing to settle down.

1.5 Sidedness

An important subtlety regarding limits pertains to the 'direction' in which a value is approached. For a variable x that is to approach, or 'limit to' a fixed value x_0, it could very well be that the initial case $x < x_0$ produces one answer, and the other case $x > x_0$ produce an entirely different answer.

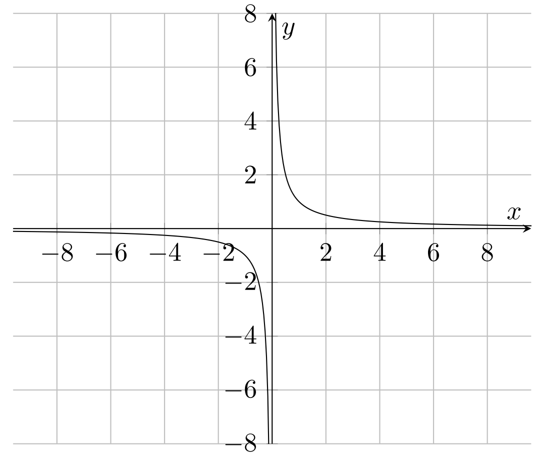

Figure 1.1: Reciprocal curve $y = 1/x$.

To demonstrate, consider the reciprocal curve $y = 1/x$ shown in Figure 1.1, which occupies the first and third quadrants and is symmetric about the origin. Notice that the point $x = 0$ is troublesome for this curve, as $y(0)$ itself is undefined. However, we can set up a limit to approach $x = 0$ from either the left- or the right-hand side. For the left-side limit, we have

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty\,,$$

whereas on the right, we get a similar statement with all $-$ signs as $+$ signs

$$\lim_{x\to 0^+} \frac{1}{x} = \infty\,,$$

which couldn't possibly be more different.

A new superscript has been slipped into each of the above equations, but the supporting term is already a subscript so it's easy to miss. Reiterating, we denote an approach from the negative or positive direction (left or right) using

$$(x < x_0) \text{ or Left-Sided: } \lim_{x\to x_0^-}$$
$$(x > x_0) \text{ or Right-Sided: } \lim_{x\to x_0^+}\,,$$

and the results need not be the same.

One-Sidedness

It is possible for only the left- or right-sided limit to exist, but not both. For instance, the square root curve $y = \sqrt{x}$ can only handle positive inputs, with the lowest allowable being $x = 0$, giving $y = 0$. Meanwhile, negative inputs lead to imaginary numbers or worse. Near the point $(0,0)$, the best we can say is

$$\lim_{x\to 0^-} \sqrt{x} = \text{Undefined}$$
$$\lim_{x\to 0^+} \sqrt{x} = 0 \, .$$

Zero to the Zero

For another example, amusing debate arises among students and hobbyists when discussing the quantity 0^0. Basic laws of exponents tell us that any number x raised to the zero-power equals one, however if we start with zero and raise it to any power x, the answer ought to be zero. So when with both numbers are zero, i.e. 0^0, what happens?

Letting x be a real number, we set up the problem by writing a one-sided limit

$$\lim_{x\to 0^+} x^x = A \, ,$$

where A stands for 'Answer'. Next, explicitly check descending x values (starting from a reasonable guess) and inspect for an emerging trend or pattern:

$$\begin{aligned} 0.1^{0.1} &= 0.794328\ldots \\ 0.01^{0.01} &= 0.9549926\ldots \\ 0.001^{0.001} &= 0.99311605\ldots \\ 0.0001^{0.0001} &= 0.999079390\ldots \end{aligned}$$

Evidently, the answer is getting closer to $A = 1$ as x is going to zero. This is motivation enough to write

$$\lim_{x\to 0^+} x^x = 1 \, ,$$

begging the conclusion

$$0^0 = 1 \, .$$

As a note of caution, not all numerical systems will treat 0^0 this way, particularly those that deal in only in integers or a subset of them.

Two-Sidedness

A two-sided limit is more 'well-behaved' than a one-sided limit. In the two-sided case, the left- and right-sided limits are in agreement at a given x_0:

$$\lim_{x\to x_0^-} \leftrightarrow \lim_{x\to x_0^+}$$

Two-sidedness apples to 'unbroken' curves (we'll refine this word later), such as $y = \sqrt{x}$ at any point except $x = 0$, or instead $y = \cos(x)$ at any x at all.

1.6 Limits in Geometry

Limits have their place in the Cartesian plane as well as on the number line.

In Figure 1.2, a unit circle is centered at $(1,0)$, and then a point (x,y) on the perimeter is chosen. If the distance from $(0,0)$ to (x,y) is r, consider a line from $(0,r)$ through (x,y) that intersects the x-axis at $(p,0)$.

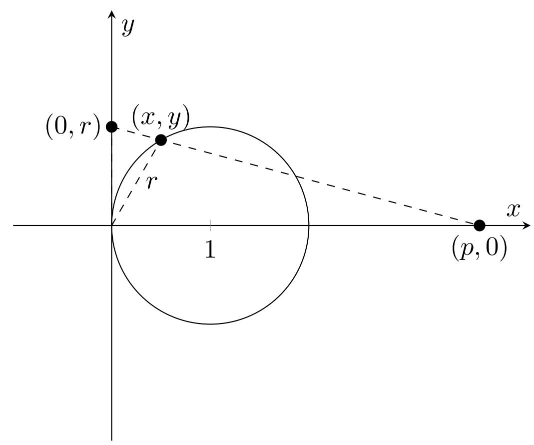

Figure 1.2: Unit circle intersecting straight line.

And now for the problem: Express the distance p in terms of r, and what happens to p as $r \to 0$? Right away, your mind's eye should 'animate' the diagram and watch p slide along the x-axis as (x,y) moves on the circle.

To crack this, we'll need the equation of the offset circle

$$(x-1)^2 + y^2 = 1 \, ,$$

along with the Pythagorean theorem relating x, y, and r

$$x^2 + y^2 = r^2 \, ,$$

which after eliminating x tells us

$$y = \sqrt{r^2 - \frac{1}{4}r^4} \, .$$

Meanwhile, the line from $(0,r)$ to $(p,0)$ has a slope given by $m = -r/p$, but the slope can also be written $m = (y-r)/x$. Eliminating m between these two equations and getting p in terms of r results in

$$p = \frac{r^2}{2 - \sqrt{4 - r^2}} \, .$$

Halfway there, but what happens to p as r goes to zero? Instinct may tell you the answer is $p \to 0$, but this would be *wrong*. Instead, try decreasing values in r and watch for a pattern:

$$\begin{aligned}
p(1) &= \frac{1}{2-\sqrt{3}} \approx 2.73205 \\
p(.5) &= \frac{(.5)^2}{2-\sqrt{4-(.5)^2}} \approx 3.93649 \\
p(.25) &= \frac{(.25)^2}{2-\sqrt{4-(.25)^2}} \approx 3.98431 \\
p(.125) &= \frac{(.125)^2}{2-\sqrt{4-(.125)^2}} \approx 3.99609 \\
p(.0625) &= \frac{(.0625)^2}{2-\sqrt{4-(.0625)^2}} \approx 3.99902
\end{aligned}$$

Evidently, the limit of p as r goes to zero is tending to 4.

2 Functions

2.1 Definition

A *function* is a 'black box' that takes an input value x and returns an output value $f(x)$. In the most general sense, the term x can represent *anything*, and $f(x)$ can be anything else.

Since algebra and calculus deal primarily in real numbers, we'll narrow our idea of functions to those that work with real numbers. In this sense, we can think of a function as something that takes a number and gives back a new, presumably different number.

Domain and Range

The set of all possible x-values that a function can receive is called the *domain*, and the set of all possible outputs $f(x)$ is called the *range*. For instance, the domain of the function $f(x) = \cos(x)$ is all real numbers, as the cosine refuses no inputs. The range, on the other hand, is confined between -1 and 1. For another example, the domain the the square root function $f(x) = \sqrt{x}$ is all non-negative real numbers, and so too is the range.

Uniqueness

For a given function f that takes input x, there can only be one output $f(x)$. For the case of curves $y(x)$ in the Cartesian plane, this amounts to the so-called 'vertical line test'. If a curve $y(x)$ ever has two y-values for a given x, then y is not a function.

A common demonstration of this takes the equation of the unit circle

$$x^2 + y^2 = 1 ,$$

where the entire circle cannot be generated by a single Cartesian function for failing the vertical line test. On the other hand though, breaking the equation into two functions is valid, where we write

$$\begin{aligned}
y_{\text{top}}(x) &= \sqrt{1-x^2} \\
y_{\text{bottom}}(x) &= -\sqrt{1-x^2} ,
\end{aligned}$$

where y_{top} and y_{bottom} are both functions of x.

2.2 Cartesian Functions

Functions of the form $y = f(x)$ that play nicely in Cartesian coordinates are the standard play things of calculus. Every item in the following list qualifies as a function in Cartesian coordinates:

$$\begin{aligned}
\text{Line} \quad & y = mx + b \\
\text{Parabola} \quad & y = ax^2 + bx + c \\
\text{Cubic} \quad & y = ax^3 + bx^2 + cx + d
\end{aligned}$$

$$\begin{aligned}
\text{Polynomial} \quad & y = a_0 + a_1 x + a_2 x^2 + \cdots \\
\text{Square Root} \quad & y = \sqrt{x} \\
\text{Exponential} \quad & y = a^x
\end{aligned}$$

$$\begin{aligned}
\text{Natural Exponential} \quad & y = e^x \\
\text{Hyperbolic Cosine} \quad & y = \left(e^x + e^{-x}\right)/2 \\
\text{Hyperbolic Sine} \quad & y = \left(e^x - e^{-x}\right)/2
\end{aligned}$$

$$\begin{aligned}
\text{Logarithm} \quad & y = \log(x) \\
\text{Natural Log} \quad & y = \ln(x) \\
\text{Factorial} \quad & y = x(x-1)(x-2)\cdots 2 \cdot 1
\end{aligned}$$

$$\begin{aligned}
\text{Cosine} \quad & y = 1 - x^2/2! + x^4/4! - \cdots \\
\text{Sine} \quad & y = x - x^3/3! + x^5/5! - \cdots \\
\text{Tangent} \quad & y = x + x^3/3 + 2x^5/15 + \cdots
\end{aligned}$$

For most of the functions listed above, the valid domain is all real numbers $\mathbb{R}$. Special care is needed with some cases, such as the square root $y = \sqrt{x}$ by rejecting negative inputs.

The curious 'factorial' function, sometimes denoted

$$x! = x(x-1)(x-2)\cdots ,$$

is only valid for whole-number inputs, and has the curious property that

$$0! = 1 \ .$$

This is certainly not reinforced by a traditional limit, but is instead a convention.

Periodicity

Functions that obey

$$f(x) = f(x + nL)$$

where L is a constant and n is an integer are *periodic*, and L is the period. Of the examples listed above, only the cosine and sine are periodic as written. The polynomial expression for tangent only works for one period.

Asymptotes

When a curve $y = f(x)$ 'disappears off' to infinity, whether it be in the horizontal direction or the vertical direction or otherwise, there may be an invisible line that the curve clearly does not cross. Such a line is called an *asymptote*, and the curve is said to be 'asymptotic' to said line.

One curve we've seen exhibiting asymptotic behavior is the reciprocal function $y = 1/x$ depicted in Figure 1.1. The curve never crosses the lines $x = 0$ and $y = 0$, despite being arbitrary close to doing so.

Not every curve that disappears off to infinity has an asymptote, though. For instance the parabolic curve $y = x^2$, despite the enormous U-shape, keeps extending horizontally as it does vertically.

2.3 Classifying Functions

Injective

A function is *one-to-one*, also known as *injective*, when every element in the domain $\{x\}$ maps to a unique element in the range $\{y\}$.

To illustrate quickly, a straight line $y = mx + b$ is injective over all real numbers. However, the parabola $y = x^2$ is not injective unless we confine the domain to say, $x > 0$.

Surjective

A function is *surjective*, also called 'onto', when every point in the range $\{y\}$ can be reached by some input in $\{x\}$.

Many curves we encounter are not surjective. To cook up an easy example anyway, consider the function $y = 2x$ over the domain of natural numbers. The domain is explicitly

$$\{x\} = \{1, 2, 3, 4, 5, 6, 7, 8, \dots\} \ ,$$

and the range is

$$\{y\} = \{2, 4, 6, 8, \dots\} \ .$$

This setup is deemed surjective because every member in $\{y\}$ is present in $\{x\}$.

Bijective

A curve that is both injective and surjective is called *bijective*. That is, a function is bijective if every member in $\{x\}$ uniquely leads to a member in $\{y\}$, and vice versa.

The square root $y = \sqrt{x}$ qualifies as a bijective function, as every x points to a unique y and vice-versa. Conversely, we can square both sides to have $x = y^2$, which is another bijective function $x = f(y)$ in the domain of non-negative real numbers.

2.4 Inverse Functions

Any bijective function $f(x)$ implies the existence of its *inverse function*, denoted $f^{-1}(x)$, also bijective, that interchanges the role of the domain and range. That is, you pretend y is the working variable and $x(y)$ is the inverse to $y(x)$.

The inverse of $y = f(x)$ can be formally defined as

$$f^{-1}(f(x)) = x \tag{1.4}$$

for bijective functions. Applying f to both sides gives a similar statement

$$f\left(f^{-1}(y)\right) = y \ .$$

Symmetry

On the Cartesian plane, it turns out that a function $y = f(x)$ and its inverse $x = f^{-1}(y)$ are symmetric about the line $y = x$.

To show this, suppose we have a function f and its inverse f^{-1} represented as follows:

$$\begin{aligned} y_1 &= f(x) \\ y_2 &= f^{-1}(x) \end{aligned}$$

(To be concrete, one could imagine $y_1 = x^2$ and $y_2 = \sqrt{x}$ in the domain of non-negative reals, but this analysis will stay general.) Next, choose a point x_1 in the domain of y_1 so that

$$y_2 = f^{-1}\left(f\left(x_1\right)\right) = x_1 \; ,$$

and similarly choose a point x_2 in the domain of y_2:

$$y_1 = f\left(f^{-1}\left(x_2\right)\right) = x_2$$

By now we have two locations in the Cartesian plane, namely (x_1, y_1), (x_2, y_2). The line connecting these has slope m and is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \; .$$

Using $y_2 = x_1$ and $y_1 = x_2$, we quickly find

$$m = \frac{x_1 - x_2}{x_2 - x_1} = -1 \; ,$$

which says the slope of the line connecting our two test points is -1. Perpendicular to this line is the line with slope $m_\perp = 1$, and the claim is proven.

2.5 Even and Odd Functions

Any function that obeys

$$f\left(-x\right) = f\left(x\right)$$

is called *even*, and is symmetric about the line $x = 0$ in the Cartesian plane. Any function that obeys

$$f\left(-x\right) = -f\left(x\right)$$

is called *odd*, and is anti-symmetric about the line $x = 0$.

While most functions are not even or odd exclusively, it is true that any function can be conceived as being the sum of an even part plus an odd part:

$$f\left(x\right) = f_{\text{even}}\left(x\right) + f_{\text{odd}}\left(x\right)$$

In terms of $f\left(x\right)$ and $f\left(-x\right)$, the even and odd components of a function are written

$$\begin{aligned} f_{\text{even}} &= \frac{f\left(x\right) + f\left(-x\right)}{2} \\ f_{\text{odd}} &= \frac{f\left(x\right) - f\left(-x\right)}{2} \; . \end{aligned}$$

2.6 Functions and Limits

Functions and limits obey rules somewhat analogous to those of algebra, such as the distributive property. To flesh these out, consider a pair of functions $f\left(x\right)$, $g\left(x\right)$ where, for a special value x_0 in the domain, we also have

$$\begin{aligned} \lim_{x \to x_0} f\left(x\right) &= A \\ \lim_{x \to x_0} g\left(x\right) &= B \; . \end{aligned}$$

Multiplication by a Constant

Consider a constant C. If $f\left(x\right)$ is multiplied by C, it should follow that C can be pulled outside of a limit:

$$\lim_{x \to x_0} \left(C f\left(x\right)\right) = C \lim_{x \to x_0} f\left(x\right) = CA \tag{1.5}$$

The proof for this begins with a mouthful of mathematical jargon. Starting from the definition

$$\lim_{x \to x_0} f\left(x\right) = A \; ,$$

we discern that for any given positive value ϵ (Greek *epsilon*), there exists another positive quantity δ (Greek *delta*), such that if

$$0 < \left|x - x_0\right| < \delta$$

then

$$\left|f\left(x\right) - A\right| < \epsilon \; .$$

All in one line, we write:

$$0 < \left|x - x_0\right| < \delta \implies \left|f\left(x\right) - A\right| < \epsilon$$

Fair enough, but for this proof we need the same thing for $\left|C f\left(x\right) - CA\right|$. Using the language on hand, this means we need to show:

$$0 < \left|x - x_0\right| < \delta \implies \left|C f\left(x\right) - CA\right| < \epsilon$$

Next, choose a different $\epsilon_1 > 0$, which means there exists a different $\delta_1 > 0$. We are free to let $\epsilon_1 = \epsilon / \left|C\right|$ to establish:

$$0 < \left|x - x_0\right| < \delta_1 \implies \left|f\left(x\right) - A\right| < \frac{\epsilon}{\left|C\right|}$$

With this setup, take a look at $\left|C f\left(x\right) - CA\right|$ and factor out the $\left|C\right|$-term:

$$\left|C f\left(x\right) - CA\right| = \left|C\right| \left|f\left(x\right) - A\right|$$

The right-most quantity $\left|f\left(x\right) - A\right|$ can be replaced provided we let $\delta = \delta_1$, which leads to

$$\left|C f\left(x\right) - CA\right| < \cancel{\left|C\right|} \frac{\epsilon}{\cancel{\left|C\right|}} \; ,$$

and the proof is done.

Distribution into Sum

The 'limit' construct distributes freely into the sum of two functions:

$$\lim_{x \to x_0} \left(f\left(x\right) \pm g\left(x\right)\right) = \lim_{x \to x_0} f\left(x\right) \pm \lim_{x \to x_0} g\left(x\right) = A \pm B \tag{1.6}$$

To prove this, define an $\epsilon > 0$ and a pair of terms $\delta_1 > 0$, $\delta_2 > 0$ such that:

$$0 < |x - x_0| < \delta_1 \implies |f\left(x\right) - A| < \frac{\epsilon}{2}$$
$$0 < |x - x_0| < \delta_2 \implies |g\left(x\right) - B| < \frac{\epsilon}{2}$$

Choosing the positive channel in Equation (1.6), take sum of $f\left(x\right)$ and $g\left(x\right)$ and consider the quantity

$$|f\left(x\right) + g\left(x\right) - (A + B)| \ ,$$

and use the triangle inequality to write

$$|f\left(x\right) + g\left(x\right) - (A + B)| = |f\left(x\right) + A| + |g\left(x\right) + B| \ ,$$

and replace the right side to finish:

$$|f\left(x\right) + g\left(x\right) - (A + B)| = \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

For the difference of $f\left(x\right)$ and $g\left(x\right)$, there is no need to introduce new epsilons and deltas. Exploit the previous result along with Equation (1.5) to find

$$\lim_{x \to x_0} \left(f\left(x\right) - g\left(x\right)\right) = A - B \ .$$

Distribution into Product

The 'limit' construct distributes also into the product of two functions:

$$\lim_{x \to x_0} \left(f\left(x\right) \cdot g\left(x\right)\right) = \lim_{x \to x_0} f\left(x\right) \cdot \lim_{x \to x_0} g\left(x\right) = A \cdot B \tag{1.7}$$

The proof of this requires some more epsilon-delta work. Define an $\epsilon > 0$ and a pair of terms $\delta_1 > 0$, $\delta_2 > 0$ such that:

$$0 < |x - x_0| < \delta_1 \implies |f\left(x\right) - A| < \sqrt{\epsilon}$$
$$0 < |x - x_0| < \delta_2 \implies |g\left(x\right) - B| < \sqrt{\epsilon}$$

Choose δ to be the smaller of $\delta_{1,2}$, and then for $0 < |x - x_0| < \delta$, we find

$$|(f\left(x\right) - A) \cdot (g\left(x\right) - B)| = |(f\left(x\right) - A)| \cdot |(g\left(x\right) - B)| \ ,$$

and replace the right side to establish

$$|(f\left(x\right) - A) \cdot (g\left(x\right) - B)| \leq \sqrt{\epsilon}\sqrt{\epsilon} \ .$$

With this, we've proved

$$\lim_{x \to x_0} (f\left(x\right) - A) \cdot (g\left(x\right) - B) = 0 \ ,$$

which helps toward the final result.

To proceed, expand the quantity

$$|(f\left(x\right) - A) \cdot (g\left(x\right) - B)|$$

using the distributive property:

$$|(f\left(x\right) - A) \cdot (g\left(x\right) - B)| = f\left(x\right) g\left(x\right) - Bf\left(x\right) - Ag\left(x\right) + AB$$

Impose the limit $x \to x_0$ on every term. Right away, the left side is zero, and we're left with

$$\begin{aligned} \lim_{x \to x_0} f\left(x\right) g\left(x\right) &= B \lim_{x \to x_0} f\left(x\right) + A \lim_{x \to x_0} g\left(x\right) - AB \\ &= BA + AB - AB \\ &= AB \ , \end{aligned}$$

finishing the proof.

Distribution into Quotient

The 'limit' construct distributes also into the quotient of two functions:

$$\lim_{x \to x_0} \left(\frac{f\left(x\right)}{g\left(x\right)}\right) = \frac{\lim_{x \to x_0} f\left(x\right)}{\lim_{x \to x_0} g\left(x\right)} = \frac{A}{B} \tag{1.8}$$

Note of course that B cannot be zero.

For this proof, we need to first spend some effort to establish

$$\lim_{x \to x_0} \frac{1}{g\left(x\right)} = \frac{1}{B} \ .$$

Whatever B is, there exists a $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \implies |g\left(x\right) - B| < \frac{|B|}{2}$$

Now for a few algebraic manipulations. Start with $|B| = |B|$, and add zero to write

$$|B| = |B - g\left(x\right) + g\left(x\right)| \ .$$

By the triangle equality, we can proceed with

$$|B| < |B - g\left(x\right)| + |g\left(x\right)| \ ,$$

which, assuming the above, is replaced by

$$|B| < \frac{|B|}{2} + |g\left(x\right)| \ .$$

From this we discern

$$\frac{1}{|g(x)|} < \frac{2}{|B|} .$$

To continue, introduce a second $\delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_2 \implies |g(x) - B| < \frac{|B|^2}{2} \epsilon .$$

Choose δ to be the smaller of $\delta_{1,2}$, and then examine the quantity $|1/g(x) - 1/B|$ in the regime

$$0 < |x - x_0| < \delta < \delta_{1,2}$$

so

$$\begin{aligned} \left| \frac{1}{g(x)} - \frac{1}{B} \right| &= \left| \frac{B - g(x)}{B g(x)} \right| \\ &= \frac{1}{|B|} \frac{1}{|g(x)|} |g(x) - B| \\ &< \frac{1}{|B|} \frac{2}{|B|} \frac{|B|^2}{2} \epsilon . \end{aligned}$$

Everything except ϵ cancels on the right, and we have shown

$$\lim_{x \to x_0} \frac{1}{g(x)} = \frac{1}{B} .$$

From here, we have

$$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \to x_0} \left(f(x) \frac{1}{g(x)} \right) ,$$

which by Equation (1.7) is also

$$\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \left(\lim_{x \to x_0} f(x) \right) \left(\lim_{x \to x_0} \frac{1}{g(x)} \right) ,$$

and this resolves to A/B, completing the proof.

Integer Powers

The 'limit' construct distributes also into exponents. We'll establish this for integers n only, however:

$$\lim_{x \to x_0} (f(x))^n = \left(\lim_{x \to x_0} f(x) \right)^n = A^n \tag{1.9}$$

To begin, begin with

$$\lim_{x \to x_0} (f(x))^n = \lim_{x \to x_0} \left(f(x)^{n-1} f(x) \right) ,$$

which decouples by Equation (1.7) to

$$\lim_{x \to x_0} (f(x))^n = A \cdot \lim_{x \to x_0} \left(f(x)^{n-1} \right) .$$

From here, we can use Equation (1.7) again to find

$$\lim_{x \to x_0} (f(x))^n = A^2 \cdot \lim_{x \to x_0} \left(f(x)^{n-2} \right) ,$$

and repeat this recursively for m steps

$$\lim_{x \to x_0} (f(x))^n = A^m \cdot \lim_{x \to x_0} \left(f(x)^{n-m} \right) ,$$

stopping when $n = m$.

Note that it's possible to show that Equation (1.9) also holds when n is any real number.

X to the X

The quantity $y = x^x$ is satisfied by both $x = 1/2$ and $x = 1/4$ (same y). This is interesting because $1/2 \neq 1/4$, but $(1/2)^{1/2} = (1/4)^{1/4}$. Apart from the pair $(1/2, 1/4)$, let us find all pairs that satisfy $a^a = b^b$.

Let either member of the pair be written $1/z$, and the other is $1/z$ divided by a constant λ such that

$$\left(\frac{1}{z} \right)^{1/z} = \left(\frac{1}{\lambda z} \right)^{1/\lambda z} .$$

Take the natural log of each side and simplify to solve for z to get

$$\begin{aligned} z &= \lambda^{1/(\lambda-1)} \\ \lambda z &= \lambda^{\lambda/(\lambda-1)} , \end{aligned}$$

and similarly:

$$\begin{aligned} \frac{1}{z} &= \lambda^{-1/(\lambda-1)} \\ \frac{1}{\lambda z} &= \lambda^{-\lambda/(\lambda-1)} \end{aligned}$$

While the above is a workable solution, proceed by substituting $q = \lambda / (\lambda - 1)$ to find

$$\begin{aligned} \frac{1}{z} &= \lambda^{-q/\lambda} \\ \frac{1}{\lambda z} &= \lambda^{-q} , \end{aligned}$$

also implying

$$z^\lambda = \lambda z .$$

The pair $(1/2, 1/4)$ corresponds to $\lambda = 2$, $z = 2$ satisfied by the above. For another test, let $q = 3$ to find $z = 9/4$, $\lambda = 3/2$, implying:

$$\left(\frac{4}{9} \right)^{4/9} = \left(\frac{8}{27} \right)^{8/27}$$

Let $q = 4$ to get $\lambda = 4/3$, $z = 64/27$ to discover the pair $(27/64, 81/256)$.

Testing a 'large' value such as $\lambda = 100$ gives $z = 100^{1/99} \approx 1.048$. Going further with $\lambda = 1000$

gives $z = 1000^{1/999} \approx 1.0069$. It should be clear that the upper bound on λ is infinity, corresponding to $z = 1$. To prove this, verify

$$z_\infty = \lim_{\lambda\to\infty} \lambda^{1/(\lambda-1)} = 1$$

as expected.

As constructed, we know $\lambda = 2$ is an allowed λ-value, but what is the minimum λ? Trying $\lambda = 1$, calculate

$$z_1 = \lim_{\lambda\to 1} \lambda^{1/(\lambda-1)} = e\,.$$

This is reassuring as $\lambda = 1$ corresponds to the solution $(1/e, 1/e)$, which happens to be the minimum of $y = x^x$.

2.7 Continuity and Smoothness

Piecewise Functions

Consider the curious function

$$f(x) = \begin{cases} 2x - 1 & x < 1 \\ \sqrt{x} + 1 & x \geq 1 \end{cases},$$

which takes different form on either side of the point $x = 1$ as shown in Figure 1.3.

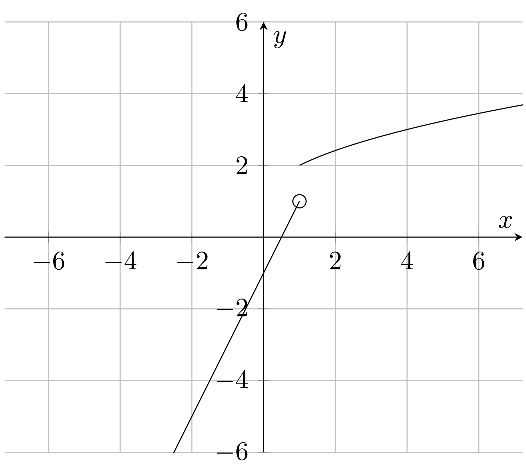

Figure 1.3: Piecewise function.

Continuity

While the above qualifies as a function in every sense, there's still something 'off' about such a piecewise function, namely the abrupt jump at $x = 1$. This type of phenomenon is called a *jump discontinuity*. Or, the function is said to be *discontinuous* at $x = 1$. The remainder of the curve for all points $x \neq 1$ is *continuous*

Smoothness

Modifying the above example, suppose we nixed the $+1$ from the function definition to write

$$f(x) = \begin{cases} 2x - 1 & x < 1 \\ \sqrt{x} & x \geq 1 \end{cases},$$

which has the effect of joining the two separate curves at the point $x = 1$. By doing this, the piecewise function is now continuous, but there is still a sense of something amiss, as there is an abrupt kink in $f(x)$ at $x = 1$. For this reason, the function lacks *smoothness* at $x = 1$. Every other place on the curve is both continuous and smooth.

Essential Singularity

A more severe type of discontinuity is called *essential* or *infinite* singularity, which occurs when one or both results of a two-sided limit around the discontinuity reach toward $\pm\infty$. This is the kind of behavior we see at $x = 0$ in the reciprocal curve $y = 1/x$ depicted in Figure 1.1.

2.8 Removable Singularity

There are plenty of functions in the wild that contain a singularity at first sight, but after some analysis, the singularity can be removed. Naturally, these are called *removable* singularities. A singularity x_0 is formally removable from a function $f(x)$ when the left- and right-sided limits near x_0 are in agreement, but $f(x_0)$ is singular or undefined:

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0} f(x)$$

Factorable Numerator

For an example, consider the function

$$f(x) = \frac{x^2 - 3x + 2}{x - 1},$$

which clearly has a problem at $x = 1$. Looking at a few values around $x = 1$ though, we find something interesting:

$$\begin{aligned} f(0.8) &= -1.2 \\ f(0.9) &= -1.1 \\ f(0.99) &= -1.01 \\ f(0.999) &= -1.001 \\ f(1) &= ? \\ f(1.001) &= -0.999 \\ f(1.01) &= -0.99 \\ f(1.1) &= -0.9 \\ f(1.2) &= -0.8 \end{aligned}$$

Given how $f(x)$ behaves near $x = 1$, it seems that $f(1)$ is tantalizingly close to -1. In the language of limits, this would mean

$$\lim_{x \to x_0^-} f(x) = -1 = \lim_{x \to x_0^+} f(x) \ ,$$

the signature of a removable singularity.

In fact, there is something fishy about the example function, because $x-1$ can be factored out of the numerator as

$$f(x) = \frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}} = x - 2 \ ,$$

like there was never a singularity at all.

Natural Logarithm

For a less trivial example, consider the function

$$y = f(x) = \frac{n^x - 1}{x} \ , \tag{1.10}$$

and let's be interested in the quantity $f(0)$. Immediately we see that $x = 0$ makes Equation (1.10) blow up, but can we remove this point? Consulting a plot of $f(x)$ shown in Figure 1.4, it seems the singularity is removable.

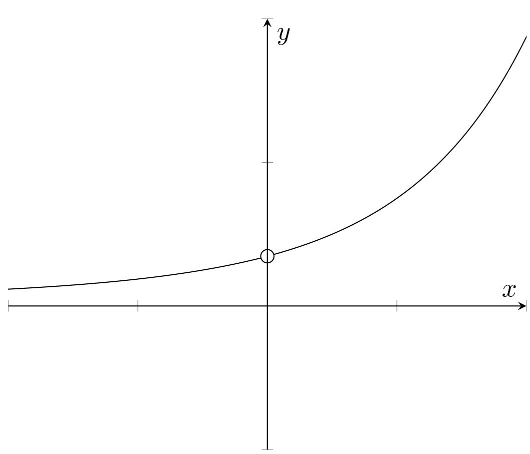

Figure 1.4: Function with removable singularity at $x = 0$.

To proceed, solve Equation (1.10) for n, and set up a limit that pushes x to zero:

$$n = \lim_{x \to 0} (1 + xy)^{1/x}$$

This setup is looking almost familiar, but even more so if we make the substitution

$$h = 1/x \ ,$$

because the limit of x going to zero is the same as the limit of h going to infinity:

$$n = \lim_{h \to \infty} \left(1 + \frac{y}{h}\right)^h$$

By Equation (1.2), the right side of the above is identical to e^y, so we find $n = e^y$. Solving for y, we finally have

$$f(0) = \ln(n) \ .$$

Evidently, the singular point in $f(x)$ is the natural log of n. For completeness, this result can also be stated in a way complimentary to Equation (1.2):

$$\ln(x) = \lim_{h \to 0} \frac{x^h - 1}{h} \tag{1.11}$$

Sinc Function

An interesting singularity arises in the so-called 'sinc' function, which is defined as $\sin(x)$ divided by x,

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} \ , \tag{1.12}$$

sketched in Figure 1.5. As a product of odd functions, the sinc function is even, i.e. symmetric about $x = 0$. The period of oscillation is still 2π like the sine function. The amplitude wiggles pathetically under an envelope of $y = 1/x$.

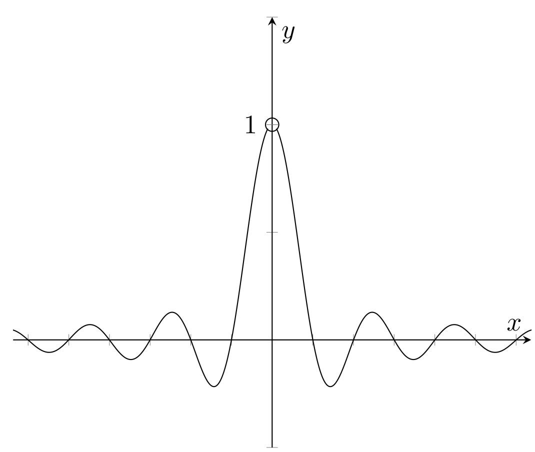

Figure 1.5: Plot of $\sin(x)/x$.

The point $x = 0$ is invokes division by zero, however the plot of the sinc function begs the singularity be removed, as the plot seems to 'want' to pass through the point $(0, 1)$. Framing the result as a limit, this is

$$\operatorname{sinc}(0) = \lim_{x \to 0} \frac{\sin(x)}{x} .$$

Evaluating the right side of the above can be done in a number of ways. One method is to expand $\sin(x)$ as an infinite polynomial, i.e.

$$\sin(x) = x - x^3/3! + x^5/5! - \cdots ,$$

and then $\sin(x)/x$ must be

$$\frac{\sin(x)}{x} = 1 - x^2/3! + x^4/5! - \cdots ,$$

and now the right side has no singularity at all. We can forget about limits and simply set $x = 0$ to find

$$\operatorname{sinc}(0) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1 , \tag{1.13}$$

and the problem is finished.

2.9 Squeeze Theorem

Consider a two functions $f(x)$, $h(x)$ in the Cartesian plane such that

$$f(x) \geq h(x)$$

in the whole domain. Suppose also that there is a point x_0 where the two functions are equal to the same value L. This takes the form of a limit since the curves can never intersect:

$$\lim_{x \to x_0} f(x) = L = \lim_{x \to x_0} h(x)$$

To so-called *squeeze theorem* states that, if we have a third function $g(x)$ defined such that

$$f(x) \geq g(x) \geq h(x) ,$$

which is 'between' the first two curves, then the value of $g(x)$ at x_0 is also approaching L:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$$

<u>Example 1</u>

Use the squeeze theorem to show that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 .$$

For the properties of the sine function, we can write for sure that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for any x. Then, multiply the entire statement through by x^2:

$$-x^2 \leq \sin\left(\frac{1}{x}\right) \leq x^2$$

In the limit $x \to 0$, the terms $-x^2$ and x^2 clearly both go to zero, so the quantity squeezed between them must also go to zero.

2.10 A Strange Beast

Fun Problem

Try to solve for x in the equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\cdots}}}}} = 2 ,$$

where the left side contains an infinite nesting of $\sqrt{x + \cdots}$ as shown.

This is actually easier than it seems (spoiler alert). Square both sides to write

$$x + 2 = 2^2 ,$$

and easily find $x = 2$. (Many who encounter this for the first time can't resist trying it on a calculator.)

Disaster

Now, try to solve for x in this version:

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\cdots}}}}} = 1$$

Doing the same steps, i.e. squaring both sides and so on, the above boils down to

$$x^2 + 1 = 1 ,$$

solved only by $x = 0$. Hang on though, because substituting $x = 0$ tells us

$$\sqrt{0 + \sqrt{0 + \sqrt{0 + \sqrt{0 + \sqrt{\cdots}}}}} = 1 ,$$

but $0 = 1$ *can't* be right, so where is the error?

Analysis

To see what went wrong, generalize the problem by writing an 'open' equation

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\cdots}}}}} \, ,$$

where y surely describes a curve, but hesitate to call y a function just yet. Squaring both sides now gives

$$y^2 = x + y \, ,$$

and solve for y again:

$$y = \frac{1}{2} \pm \frac{\sqrt{1+4x}}{2}$$

From here, we see that setting $x = 0$ gives two results, namely $y_1 = 1$ and $y_2 = 0$. These are two perfectly legal solutions to $y^2 = x + y$.

However, if y is to be a proper function, it follows that one of y_1 or y_2 must be thrown out. To avoid getting $0 = 1$, discard the y_1 solution and keep y_2. The proper way to write $y(x)$ as a closed function must be done in piecewise fashion:

$$y(x) = \begin{cases} 0 & x = 0 \\ 1/2 + \sqrt{1+4x}/2 & x > 0 \end{cases}$$

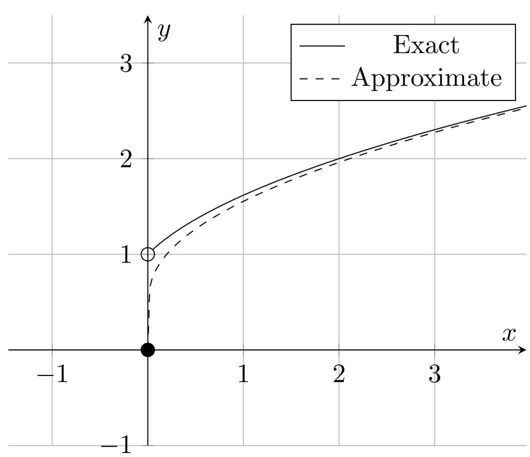

Figure 1.6: A strange beast.

The function $y(x)$ is plotted as a solid curve in Figure 1.6. Note the 'hole' removed removed from $(0, 1)$ is placed instead at $(0, 0)$, and no solution exists for $y = 1$.

Also in the Figure is a dashed-line representation of the approximate answer generated from the 'open' form of $y(x)$ out to three square roots. Truncating the radical this way, one sees the dashed curve starting from $(0, 0)$ and trying to reach $(0, 1)$ in a continuous manner. In the infinite limit of nested roots, the piecewise function needs to take over.

Golden Ratio

An interesting solution to the so-called strange beast equation occurs at $x = 1$. From this we have

$$y(1) = \frac{1+\sqrt{5}}{2} = 1.618034\ldots \, ,$$

a celebrated number called the *golden ratio*, denoted ϕ. Given the open version for $y(x)$, we derive a nifty expression for ϕ:

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{\cdots}}}}} = \phi$$

3 Sequences

3.1 Definition of a Sequence

A *sequence* is an ordered list of numbers or other information. An entire sequence can be represented by a letter or symbol, such as A, or more explicitly, $\{A\}$. Each member of a sequence is assigned a unique whole number index, typically written as a subscript j called an *index*, starting with $j = 1$ unless otherwise specified.

For instance, the numbers 3, 6, 9, in that order, can be assigned

$$\begin{aligned} A_1 &= 3 \\ A_2 &= 6 \\ A_3 &= 9 \, , \end{aligned}$$

and written out via

$$\{A\} = \{3, 6, 9\} \; .$$

A sequence with a countable number of members n is called a *finite* sequence, or *closed* sequence. All finite sequences can be represented by the form

$$\{A\} = \{A_1, A_2, A_3, \ldots, A_n\} \; .$$

Note that the exact representation of a sequence can vary among sources and authors. For instance, the above sometimes appears as:

$$\{A\} = \{A_j\} = \{A_j\}_1^n$$

The starting index need not be $j = 1$.

3.2 Infinite Sequences

A sequence with an infinite number of members $n \to \infty$ can also be conceived, called an *infinite sequence*. The 'last' term L in an infinite sequence can be expressed as the limit

$$L = \lim_{j\to\infty} A_j \, .$$

Convergent Sequence

For finite L, we may require that that for any positive ϵ, there exists some integer m such that

$$j > m \ \to \ |A_j - L| < \epsilon \, .$$

This is only supported by sequences where each A_j is finite. Such a sequence is called *convergent*. The terms of a convergent sequence approach a single value for increasing j.

By this criteria, the following infinite sequences are convergent (assume the patterns go forever):

$$\{A\} = \{10, 20, 1, 2, 0.1, 0.2, 0.01, 0.02, \dots\}$$
$$\{B\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\right\}$$
$$\{C\} = \left\{\frac{9}{10}, \frac{99}{100}, \frac{999}{1000}, \frac{9999}{10000}, \dots\right\}$$

Divergent Sequence

A *divergent* sequence is one that has $L \to \pm\infty$. This is characterized by a growing trend in the late-stage A_j for which there is no finite ceiling (or floor). The following sequences diverge (assume the patterns go forever):

$$\{Q\} = \{1, 10, 100, 1000, 10000, \dots\}$$
$$\{R\} = \{-1, -2, -3, -4, -5, -6, \dots\}$$

Periodic Sequence

A *periodic sequence* is an infinite sequence that neither converges nor diverges, but instead repeats after a certain index. A sequence such as

$$\{A\} = \{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$$

is periodic.

Recursive Sequence

By studying polynomial division, one can argue into existence a sequence $\{F\}$ obeying the the *recursion relation*,

$$F_j = pF_{j-1} + F_{j-2} \, ,$$

where p is a parameter. The case $F_1 = 1$ with $p = 1$ gives the Fibonacci sequence:

$$\{F_{p=1}\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

Nonexistent Limits

Sequences that have a random nature or are otherwise 'not going anywhere' imply a nonexistent limit. For instance, alternating values of ± 1 via

$$\{B\} = \{5, -5, 2, -2, 7, -7, 3, -3, \dots\}$$

implies a nonexistent limit. Even though the terms A_n seem to be oscillating around zero, there is no sense of a strict limit.

Similar comments go for the digits of any irrational number such as *pi*:

$$\{P\} = \{3, 1, 4, 1, 5, 9, 2, 6, 5, \dots\}$$

3.3 Analogy to Functions

A particular comparison between sequences and functions can be framed in terms of a limit. Consider a sequence $\{A\}$, along with a function $f(x)$ such that $f(n) = A_j$. With this, the following is always true:

$$\lim_{x\to\infty} f(x) = L = \lim_{j\to\infty} A_j \tag{1.14}$$

Squeeze Theorem

In the same way that functions obey the so-called squeeze theorem, so too is the case for sequences. Consider three sequences $\{A\}$, $\{B\}$, $\{C\}$. If

$$A_j < B_j < C_j$$

and

$$\lim_{j\to\infty} A_j = L = \lim_{j\to\infty} C_j$$

then

$$L = \lim_{j\to\infty} B_j \, .$$

Absolute Value

Using the fact that any variable x obeys $\pm x \leq |x|$, it follows that member x_j in a sequence obeys

$$-|x_j| \leq x_j \leq |x_j| \, .$$

For an infinite sequence $\{x_j\}$, we can further write (see scalar multiplication below):

$$\lim_{j\to\infty} (-|x_j|) = L = -\lim_{j\to\infty} |x_j|$$

In the special case

$$\lim_{j\to\infty} |x_j| \to 0 \, ,$$

thereby making $L \to 0$, the squeeze theorem leads to a stronger conclusion:

$$\lim_{j\to\infty} x_j \to 0$$

3.4 Algebraic Properties

The members of two convergent sequences $\{A\}$, $\{B\}$, obey algebraic properties you would expect. Each of the following is a consequence of Equation (1.14):

Addition

$$\lim_{j\to\infty} (A_j \pm B_j) = \lim_{j\to\infty} A_j \pm \lim_{j\to\infty} B_j$$

Scalar Multiplication

$$\lim_{j\to\infty} \lambda A_j = \lambda \lim_{j\to\infty} A_j$$

Product

$$\lim_{j\to\infty} (A_j B_j) = \left(\lim_{j\to\infty} A_j\right)\left(\lim_{j\to\infty} B_j\right)$$

Ratio

As long as the denominator is not zero:

$$\lim_{j\to\infty} \frac{A_j}{B_j} = \frac{\lim_{j\to\infty} A_j}{\lim_{j\to\infty} B_j}$$

Exponent

As long as $A_j > 0$:

$$\lim_{j\to\infty} A_j^q = \left(\lim_{j\to\infty} A_j\right)^q$$

3.5 Geometric Sequence

Consider the geometric sequence

$$\{x_j\} = \left\{x^j\right\}_0^\infty \, .$$

Intuitively, it would make sense that the sequence converges only in the interval $-1 < x \leq 1$, but some subtleties arise when proving this. Starting with $f(x) = x^j$, write

$$\lim_{x\to\infty} x^j = L = \lim_{n\to\infty} A_j \, .$$

Handling the easy cases first, we see that $x > 1$ leads to $L \to \infty$, and the sequence diverges. The exact case $x = 1$ gives $L = 1$, which is convergent. The trivial case $x = 0$ gives $L = 0$, also convergent.

For $0 < x < 1$, the limit of $f(x)$ goes to zero, so L is also going to zero, and the sequence converges. The case $-1 < x < 0$ is also convergent. To show this, let $r^j = \left|x^j\right|$ so we're checking the domain $0 < r < 1$. This reproduces the previous case, and we have

$$\lim_{j\to\infty} r^j = 0 = \lim_{j\to\infty} \left|x^j\right| \, ,$$

thus the sequence converges.

The case $x = -1$ attempts a limit that does not exist. Writing this case out, one finds the periodic sequence

$$\left\{(-1)^j\right\} = \{1, -1, 1, -1, 1, -1, \dots\} \, ,$$

exhibiting neither convergence nor divergence.

The ugliest case occurs at $x < -1$. For instance, choosing $x = -2$ leads to

$$\left\{(-2)^j\right\} = \{1, -2, 4, -8, 16, -32, 64 \dots\} \, ,$$

which is not convergent, not divergent, and non-periodic.

In conclusion, we say the geometric sequence $\{x_j\}$ converges only if $-1 < x \leq 1$, and furthermore:

$$\lim_{j\to\infty} x^j = \begin{cases} 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases} \tag{1.15}$$

3.6 Terminology

Now comes some obligatory terminology to be less verbose when talking about sequences. The following comments apply to both finite and infinite sequences.

Increasing

A sequence $\{A\}$ is *increasing* if, for all j:

$$A_{j+1} > A_j$$

Decreasing

A sequence $\{A\}$ is *decreasing* if, for all j:

$$A_{j+1} < A_j$$

Monatonic

A sequence $\{A\}$ that is increasing or decreasing is called *monatonic*.

Bounded Below

For a sequence $\{A\}$, if there exists a *lower bound* M such that $M < A_j$ for all j, the sequence is called *bounded below.*

Bounded Above

For a sequence $\{A\}$, if there exists an *upper bound* M such that $M > A_j$ for all j, the sequence is called *bounded above.*

Bounded

A sequence $\{A\}$ that is bounded below and bounded above is *bounded*. A sequence that converges is bounded and monatonic.

4 Series

4.1 Partial Sum

Given a sequence $\{A_j\}_1^n$, one can imagine packing the members A_j into a *partial sum*:

$$\begin{aligned} s_1 &= A_1 \\ s_2 &= A_1 + A_2 \\ s_3 &= A_1 + A_2 + A_3 \\ s_m &= A_1 + A_2 + A_3 + \cdots + A_m \end{aligned}$$

For the last sum, it's assumed that $m \leq n$.

Finite Series

In the special case $m = n$, i.e. when the partial sum adds all terms in a sequence, the sum is called the *series*. For finite n, the series has a finite number of terms.

4.2 Sigma Notation

Any sum

$$s_m = A_1 + A_2 + A_3 + \cdots + A_m$$

can be written using the so-called sigma notation:

$$s_m = \sum_{j=1}^{m} A_j$$

The variable j, most often an integer, is called the *index*. The initial and final values for j appear as the respective subscript and superscript on the Σ symbol. The index increases by one with each iteration of the sum.

Infinite Series

For an infinite sequence $n \to \infty$, the sum

$$S = \lim_{m \to \infty} s_m = \sum_{j=1}^{\infty} A_j$$

is the *infinite series.*

Perhaps the most accessible infinite series is the infinite geometric series, which converges for $|x| < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j$$

4.3 Converging Series

Some of the groundwork and terminology developed for sequences also applies to infinite series. If $\{A\}$, $\{B\}$ are both convergent sequences, things like addition and scalar multiplication carry directly to the series:

$$\sum_{j=1}^{\infty} (A_j + B_j) = \sum_{j=1}^{\infty} A_j + \sum_{j=1}^{\infty} B_j$$
$$\sum_{j=1}^{\infty} \lambda A_j = \lambda \sum_{j=1}^{\infty} A_j$$

On the other hand, the product of $\{A\}$, $\{B\}$ is not a single series over the product of the components:

$$\left(\sum_{j=1}^{\infty} A_j \right) \left(\sum_{j=1}^{\infty} B_j \right) \neq \sum_{j=1}^{\infty} (A_j B_j)$$

Even worse, it's not clear that the product of two convergent series is itself convergent.

Criteria for Convergence

A sequence $\{A\}$ converges if and only if

$$\lim_{j \to \infty} A_j = 0 \, .$$

To prove this, write two partials sums s_j and s_{j-1}, which only differ by the coefficient A_j:

$$A_j = s_j - s_{j-1}$$

If the sequence converges, both s_j and s_{j-1} converge, namely because j and $j-1$ both tend to infinity. Since the above equation takes their difference, we get zero on the right, finishing the proof.

Absolute Convergence

A sequence $\{A\}$ *converges absolutely* if

$$\sum_{j=1}^{\infty} |A_j|$$

converges.

Conditional Convergence

A sequence $\{A\}$ *converges conditionally* if

$$\sum_{j=1}^{\infty} A_j$$

converges but

$$\sum_{j=1}^{\infty} |A_j|$$

diverges.

Rearranging Terms

When a series is converging, the terms in the series can be rearranged without consequence. This is explicitly untrue for divergent series.

For an example from the geometric series, it's easy to show that

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \cdots ,$$

where the series on the right converges. Grouping positives and negatives together, which should be done with caution in general, gives

$$\frac{1}{2} = \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\right) - \left(\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots\right) ,$$

simplifying to

$$\frac{1}{2} = \left(\frac{1}{4} - \frac{1}{8}\right)\left(\frac{1}{1-1/4}\right) = \left(\frac{3}{8}\right)\left(\frac{4}{3}\right) = \frac{1}{2} ,$$

as expected. The answer will be same regardless of how the terms in the series are arranged.

The story is different for a conditionally converging series. In fact, it turns out that the terms in such a diverging series can be arranged to give *any* real number.

4.4 Finite Sums

Fixed Exponent

Going back to the finite sequence $\{A_j\}_1^n$, there are a few cases that arise often in calculations, so let's understand these ahead of time. Consider the sums:

$$S_1 = \sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n$$

$$S_2 = \sum_{j=1}^{n} j^2 = 1 + 2^2 + 3^{2}" + \cdots + n^2$$

$$S_3 = \sum_{j=1}^{n} j^3 = 1 + 2^3 + 3^3 + \cdots + n^3$$

The sum S_1 can be evaluated by counting matching pairs of numbers that sum to n. The number n itself is trivially paired with zero. The number 1 is paired with $n-1$, summing to n. This continues for any pair j and $n-j$, and there are $n/2$ of these. The 'middle' number in the series, namely $n/2$, gets no matching partner. Therefore we have:

$$S_1 = \frac{n}{2}n + \frac{n}{2} = \frac{n\,(n+1)}{2}$$

In fact, this idenfity is worth memorizing:

$$\sum_{j=1}^{n} j = \frac{n\,(n+1)}{2} \tag{1.16}$$

Calculating S_2 is a bit harder. To get started, guess the solution as being a polynomial involving powers of n and unknown coefficients:

$$S_2\,(n) = \alpha n^3 + \beta n^2 + \gamma n .$$

If Greek characters are unfamiliar, these are 'alpha' (α), 'beta' (β), 'gamma' (γ).

For $n = 1$, the sum is simply 1, and we find

$$S_2\,(1) = 1 = \alpha + \beta + \gamma .$$

For $n = 2$, we may write

$$S_2\,(2) = 1 + 2^2 = 8\alpha + 4\beta + 2\gamma ,$$

and similarly for $n = 3$,

$$S_3\,(2) = 1 + 2^2 + 3^2 = 27\alpha + 9\beta + 3\gamma .$$

What we now have is a system of three equations and three unknowns, which is enough to solve for α, β, γ. Doing so by hand or by using matrix methods, we end up with:

$$\alpha = 1/6$$
$$\beta = 1/2$$
$$\gamma = 1/6$$

Evidently then, we have

$$S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

and te hard work is done. The right side can be factored to deliver the final result:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \tag{1.17}$$

For completeness, it turns out the result for S_3 is:

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{1.18}$$

The proof can be done with the method of unknown coefficients.

Powers of Two

Consider the finite sum

$$I_n = \sum_{j=0}^{n} 2^j = 1 + 2 + 2^2 + 2^3 + \cdots + 2^n$$

for finite integer n.

To evaluate the sum, divide both sides by two to write

$$\begin{aligned} \frac{I_n}{2} &= \frac{1}{2} + 1 + 2 + 2^2 + \cdots 2^{n-1} \\ &= \frac{1}{2} + I_{n-1} \,. \end{aligned}$$

Note from the definition of I_n that

$$I_{n-1} = I_n - 2^n \,,$$

and thus

$$\frac{I_n}{2} = \frac{1}{2} + I_n - 2^n \,.$$

Solving for I_n, one finds

$$I_n = 2^{n+1} - 1 \,.$$

In summary:

$$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \tag{1.19}$$

4.5 Shift of Index

The series

$$S_n = \sum_{j=1}^{n} A_j$$

has one index variable j running over the integers from 1 to n. Sometimes it's useful to perform a shift of index to a new letter k such that

$$k = j + \alpha \,,$$

where α is any integer:

$$S_n = \sum_{k=1+\alpha}^{n+\alpha} A_{k-\alpha}$$

For a striking example of this, consider the infinite series

$$C = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = \sum_{j=1}^{\infty} \frac{j}{(j+1)!} \,.$$

Substitute $k = j + 1$ and the above becomes

$$C = \sum_{k=2}^{\infty} \frac{k-1}{k!} = \sum_{k=2}^{\infty} \frac{1}{(k-1)!} - \sum_{k=2}^{\infty} \frac{1}{k!} \,.$$

Let $m = k - 1$ in the first sum, and simply relabel $k \to m$ in the second:

$$C = \sum_{m=1}^{\infty} \frac{1}{m!} - \sum_{m=2}^{\infty} \frac{1}{m!}$$

Pluck the first term from the first sum and the rest cancels out:

$$C = 1 + \sum_{m=2}^{\infty} \frac{1}{m!} - \sum_{m=2}^{\infty} \frac{1}{m!} = 1$$