Kinematics
MANUSCRIPT

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[1]

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Contents

Chapter 1

Kinematics

1 One-Dimensional Motion

Any system that has one *degree of freedom*, such as a the position of a train on a straight track, is said to be a *one-dimensional* system. To represent location in a one-dimensional system, naturally, we need just a one-dimensional graph. By choosing a point of origin, any position can be represented by some variable x that is a real-number displacement from the origin.

Position Plot

When the position of a one-dimensional system changes with respect to some variable, usually time, it makes sense to represent the states of the system on a two-dimensional graph. For this, it is customary to let displacement be represented vertically, and time represented horizontally as demonstrated in Figure 1.1.

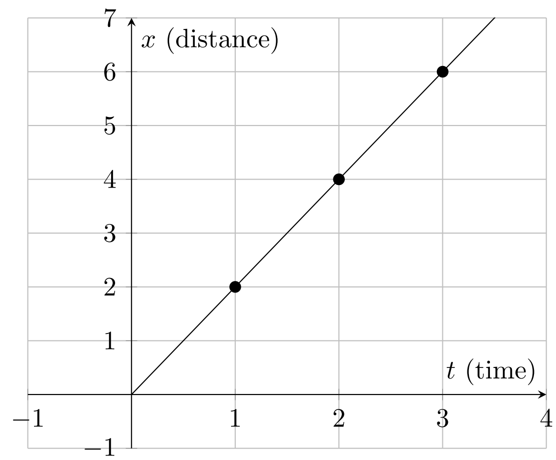

Figure 1.1: Position plot showing uniform one-dimensional motion.

The above Figure depicts the motion of an object that started at the origin at $t = 0$, and then moves two distance units for every time unit. The distance could be centimeters, miles, kilometers, etc. The time unit is most often seconds, hours, or years.

Note that a stationary object with $x(t)$ constant traces a horizontal line in the position plot. Correspondingly, a vertical line would imply that the object's position is undefined except for a small instant of time, during which all possible positions are occupied before becoming undefined again. Such a thing is non-physical at least, and makes very little sense at most.

Slope = Velocity

When one dimensional displacement is represented vertically with time horizontally, then the slope of the line representing the motion is the *velocity*:

$$v(t) = \frac{\Delta x}{\Delta t} \tag{1.1}$$

In one dimension, the velocity can be positive, negative, or zero. At a given moment in time, the quantity $v(t)$ is also known as the *instantaneous velocity*.

Speed

The magnitude $|v|$ of the velocity is the speed *speed*. As an absolute value, the speed is never negative.

Problem 1

If the average speed of an object is zero, what is the average velocity? If the average velocity is zero, why isn't the speed necessarily zero?

1.1 Uniform Velocity

When the velocity of an object throughout its motion does not change, the object is said to have *constant velocity* or *uniform velocity*, denoted v_0. In this case, all of the apparatus for straight-line equations can be used. On the basis of Equation (1.1), we can also write

$$v_0 = \frac{x - x_0}{t - t_0} .$$

Position in Time

One-dimensional motion need not begin at the origin at $t = 0$, as offsets in initial position or initial time don't inherently affect the slope of the line. To this end, use the above to write something like the point-slope form for constant velocity:

$$x - x_0 = v_0 (t - t_0) \tag{1.2}$$

As it turns out, Equation (1.2) contains more freedom than we really need, and it is customary to set $t_0 = 0$ so that x_0 handles the initial (or otherwise given) condition of the motion. Solving for x delivers the first equation of *kinematics* for uniform velocity:

$$x = x_0 + v_0 t$$

Problem 2

A car travels 30 miles at 50 mph and then 90 miles at 30 mph. Show that the average speed is 33.33 mph.

Problem 3

Mr. Smith drives to the store at 55 mph. Mrs. Smith drives half way to the same store at 45 mph, and then drives the other half way at V mph. Solve for V such that Mr. and Mrs. Smith spend the same time driving to the store.

1.2 Uniform Acceleration

Upping the ante, now consider the regime there the velocity $v(t)$ is not a constant, but instead increases or decreases linearly via *acceleration*. The acceleration term, denoted a, is a constant having units

$$[a] = \left[\text{distance/time}^2\right] .$$

Instantaneous Velocity Equation

In terms of the uniform acceleration, the velocity at time t is given by

$$v = v_0 + at , \tag{1.3}$$

where v_0 is the velocity at $t = 0$. As t increases, v's behavior is governed by at.

Velocity Plot

As a linear equation in time, a plot with v on the vertical axis and t on the horizontal axis will trace a straight line with a v-intercept at v_0 as shown in Figure 1.2.

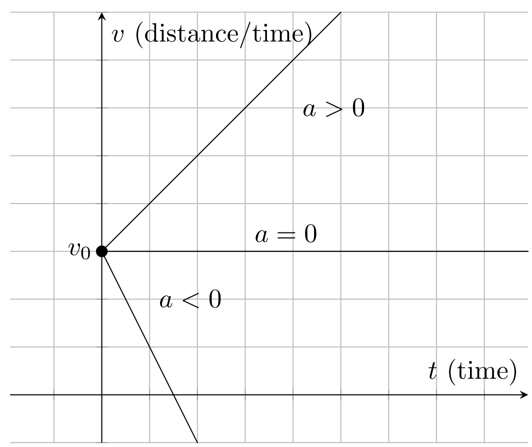

Figure 1.2: Velocity plot showing uniform plot showing uniform acceleration in one dimension.

In the Figure, one sees that the slope of the velocity with respect to time is the constant a. The case $a = 0$ corresponds to constant velocity. The 'negative acceleration' case, also known as *deceleration*, causes v to reduce in time and eventually go negative.

Problem 4

An arrow traveling at 175 ft/sec strikes the ground, coming to rest in 0.1 sec. Assuming the acceleration of the arrow inconstant as it moves through the ground, determine its magnitude in ft/sec^2.

Average Velocity

The *average velocity* $\overline{v}$ is the average of v_0 and v at time t. Calculating this out, we find

$$\overline{v} = \frac{1}{2}(v_0) + \frac{1}{2}(v_0 + at) ,$$

or

$$\overline{v} = v_0 + \frac{1}{2}at . \tag{1.4}$$

Net Displacement

In terms of the average velocity, the net displacement can be calculated via

$$x = x_0 + v_0 t + \frac{1}{2}at^2 . \tag{1.5}$$

in agreement with the previous calculation.

Problem 5

A car stopped at a red light begins accelerating when the light turns green. At the same moment, a truck coasting at a constant 20 m/s passes the car. If the car's acceleration is 7.5 m/s^2, calculate when (t) and where (x) the car catches up to the truck.

Displacement as Area

At a given time t, the total area under the line $v = v_0 + at$ equals the net displacement $x - x_0$ of the motion. (Read that twice!)

From Figure 1.2, one sees the area under the line is the sum of a rectangle and a triangle such that:

$$\text{rectangle area} = v_0 t$$
$$\text{triangle area} = \frac{1}{2}(v - v_0)\,t = \frac{1}{2}at^2$$

If our incantation linking displacement to area is true, then we must have

$$x - x_0 = v_0 t + \frac{1}{2}at^2 .$$

Time-Shift Analysis

One more way to derive the displacement equation for uniform acceleration is by time-shift analysis. For this, introduce an unknown variable A and write the generic form

$$x = x_0 + v_0 t + At^2 ,$$

where the job is to solve for A.

To proceed, introduce the time shift $t \to t + h$, making the position transform via $x \to w$. For this, we write

$$w = x_0 + v_0 (t+h) + A (t+h)^2 .$$

Distribute all products and rearrange in powers of h to get

$$w = \left(x_0 + v_0 t + At^2\right) + h\,(v_0 + 2At) + Ah^2 .$$

Now, suppose t is held constant while h does the job of being the time variable. For this, let $x = x_t$ and $v = v_t$ to remind that these values are 'stuck' at time t. The above becomes

$$w = x_t + h\,(v_0 + 2At) + Ah^2 ,$$

however the parenthesized term multiplying h must have the form of the velocity at time t, i.e.

$$v_0 + 2At = v_0 + at ,$$

revealing finally that

$$A = \frac{1}{2}a .$$

The final form for w reads

$$w = x_t + v_t h + \frac{1}{2}ah^2 .$$

Comparing to

$$x = x_0 + v_0 t + \frac{1}{2}at^2 ,$$

we see that the w-equation is the same as the x-equation up to choice of initial condition.

Position Plot

For uniform acceleration, we established three ways that the position of an object obeys the Equation (1.5). Due to the presence of the t^2-term, the position is quadratic in time as sketched in Figure 1.3.

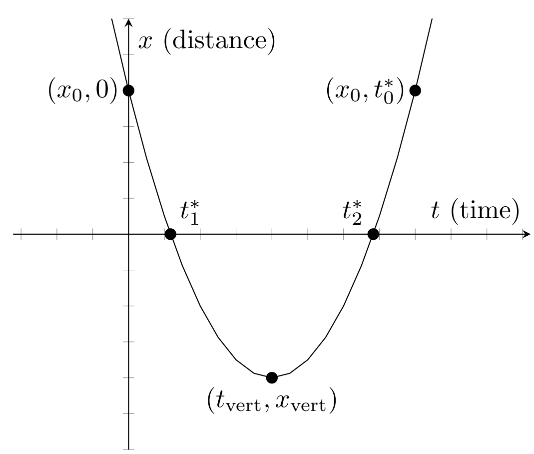

Figure 1.3: Quadratic position curve in one dimension.

Problem 6

Starting with Equation (1.5), complete the square in t to derive

$$x = x_0 - \frac{v_0^2}{2a} + \frac{a}{2}\left(t + \frac{v_0}{a}\right)^2 \tag{1.6}$$

Problem 7

Starting with Equation (1.5), use the quadratic formula to derive

$$t = \frac{-v_0}{a} \pm \frac{v_0}{a}\sqrt{1 + \frac{2a\,(x - x_0)}{v_0^2}} \tag{1.7}$$

Problem 8

Show that the vertex of the parabola is located at

$$(t_{\text{vert}}, x_{\text{vert}}) = \left(\frac{-v_0}{a}, x_0 - \frac{v_0^2}{2a}\right) .$$

Problem 9

Find the time t_0^* at which the position returns to x_0 and discuss its validity. Answer:

$$t_0^* = \frac{-2v_0}{a}$$

Problem 10

Find the times $t_{1,2}^*$ at which the position touches $x = 0$. Answer:

$$t_{1,2}^* = \frac{-v_0}{a}\left(1 \pm \sqrt{1 - \frac{2x_0 a}{v_0^2}}\right)$$

1.3 Kinematic Identities

In addition to Equation (1.3), namely

$$v = v_0 + at ,$$

and Equation (1.5) given by

$$x = x_0 + v_0 t + \frac{1}{2}at^2 ,$$

we can derive a flurry of kinematic identities to quantify one-dimensional motion with constant acceleration.

Recall that in terms of the average velocity via (1.4)

$$\overline{v} = \frac{v_0 + v}{2} = v_0 + \frac{1}{2}at ,$$

the position x can also be written

$$x = x_0 + \overline{v}t . \tag{1.8}$$

Eliminating the v_0 term between the above, one has

$$x = x_0 + vt - \frac{1}{2}at^2 . \tag{1.9}$$

An important kinematic identity is established by eliminating t between Equations (1.3), (1.8). From these, one writes:

$$\begin{aligned} x - x_0 &= \overline{v}t \\ x - x_0 &= \frac{v_0 + v}{2}\left(\frac{v - v_0}{a}\right) \\ 2a\,(x - x_0) &= v^2 - v_0^2 \end{aligned}$$

Solve for v^2 to land at:

$$v^2 = v_0^2 + 2a\,(x - x_0) \tag{1.10}$$

Summary

To summarize, we now have a set of equations describing one-dimensional motion under constant acceleration, each purposefully containing all but one of the variables in play:

Identity:	Missing:
$v = v_0 + at$	$x - x_0$
$x = x_0 + v_0 t + at^2/2$	v
$x = x_0 + (v_0 + v)\,t/2$	a
$x = x_0 + vt - at^2/2$	v_0
$v^2 = v_0^2 + 2a\,(x - x_0)$	t

1.4 Freefall Acceleration

Near Earth's surface, we know from experience that objects tend to fall downward due to gravity. It turns out that the gravitational effect on a freefalling object resolves to a *constant* acceleration, denoted g, having magnitude

$$g = 9.8\,\frac{\text{m}}{\text{s}^2} . \tag{1.11}$$

If we orient $+x$ or $+y$ as the 'upward' direction, the acceleration due to gravity is:

$$a_{\text{freefall}} = -g$$

Problem 11

Suppose a rod of length $L = 1408$ in begins at rest and freefalls downward. How long does it take for the rod to fall a distance equal to its own length? Answer:

$$t = \sqrt{\frac{2L}{g}} = \sqrt{\frac{(2)\,(1408)\,(2.54)}{(100)\,(9.8)}}\ \text{s} = 2.70\,\text{s}$$

Problem 12

On the moon, the acceleration due to gravity is approximately one sixth of that on Earth's surface. Starting from rest, how much longer does it take an object to fall a distance H on the moon than it does on Earth?

Wishing Well Problem

A stone is dropped from rest into a well. The splash is heard 2.059 seconds after release. The speed of sound in air is approximately 330 m/s. Calculate the depth of the well. Refrain from using a machine.

From the information given, we have

$$2.059\text{ s} = \frac{D}{330\text{ m/s}} + \sqrt{\frac{2D}{g}} \ ,$$

which ends up being quadratic in D:

$$(2.059\text{ s})^2 - 2D\frac{2.059\text{ s}}{330\text{ m/s}} + \frac{D^2}{(330\text{ m/s})^2} = \frac{2D}{9.8\text{ m/s}^2}$$

The denominator of the D^2-term overwhelms all others, thus the whole term is close enough to zero to be omitted. Then, the distance D in meters is given by

$$(2.059)^2 \approx D\left(\frac{1}{4.9} + \frac{4.12}{330}\right)$$
$$20.78 \approx D\left(1 + \frac{20.19}{330}\right) \ ,$$

or finally:

$$D \approx \frac{20.78}{1.06} \approx 19.6$$

Two Balls and a Building

A ball is thrown straight up at speed v_0 from the edge of the roof of a building of height H. A second ball is dropped from the roof t_0 seconds later. Ignoring air drag, determine the condition for the two balls hitting the ground simultaneously.

Let y_1 represent the trajectory of the first ball, and y_2 represent the second ball. According to the information given, we may write

$$y_1 = H + v_0 t - \frac{g}{2}t^2$$
$$y_2 = H - \frac{g}{2}(t - t_0)^2 \ ,$$

where H is the height of the building, and v_0 is the initial velocity of the first ball.

Now impose the condition that each ball reaches the ground at via $y_1 = y_2 = 0$ at the same time $t = t^*$ to write

$$0 = H + v_0 t^* - \frac{g}{2}(t^*)^2$$
$$0 = H - \frac{g}{2}(t^* - t_0)^2 \ .$$

The difference of these two results eliminates H to avail a formula for t^*, namely

$$t^* = \frac{g t_0^2/2}{g t_0 - v_0} \ .$$

From this, we see the special case $gt_0 = v_0$ causes t^* to be undefined, and $gt_0 < v_0$ causes t^* to be negative, neither of which is physically possible. Evidently, we must have

$$v_0 < g t_0 = v_{\text{max}}$$

for a valid solution.

Meanwhile, recall there is a time $\tilde{t} = 2v_0/g$ at which the first ball returns to its original height, and this time must be greater than t_0, which means

$$v_0 > \frac{g t_0}{2} = v_{\text{min}} \ .$$

We may solve for t^* in each equation to get

$$t^* = \frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2H}{g}}$$
$$t^* = t_0 \pm \sqrt{\frac{2H}{g}} \ ,$$

combining to give

$$\frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2H}{g}} = t_0 \pm \sqrt{\frac{2H}{g}} \ ,$$

which avails an equation for H, after some simplifying:

$$H = \frac{g t_0^2}{2}\left(\frac{g t_0/2 - v_0}{g t_0 - v_0}\right)^2$$

2 Two-Dimensional Motion

For motion not limited to one dimension, an object free to move in two dimensions occupies some position in the Cartesian plane, thus the description of any position is an ordered pair (x, y). If the x- and y-coordinates are independent (which we may always assume), then the whole kinematics apparatus for one dimension applies to x and y separately.

In particular, Equation (1.5) manifests twice according to

$$x = x_0 + v_{x_0} t + \frac{1}{2}a_x t^2$$
$$y = y_0 + v_{y_0} t + \frac{1}{2}a_y t^2 \ ,$$

where subscripts on initial velocity v_0 and acceleration a have been added accordingly. Similar applies to Equation (1.3), namely

$$v_x = v_{x_0} + a_x t$$
$$v_y = v_{y_0} + a_y t \, .$$

The speed v is constructed using the Pythagorean theorem:

$$v = \sqrt{v_x^2 + v_y^2} \tag{1.12}$$

Freefall Motion

For freefall motion near Earth's surface, the effect of gravity links a_y to the local acceleration constant via

$$a_y = -g$$

in analog to Equation (1.11). By a similar token, $a_x = 0$ unless a gravity-like force acts horizontally (which doesn't exist). For this reason, the motion in the x-direction is given by

$$x = x_0 + v_{x_0} t$$
$$v_x = v_{x_0} \, ,$$

and the motion in the y direction is given by

$$y = y_0 + v_{y_0} t - \frac{1}{2} g t^2$$
$$v_y = v_{y_0} - \frac{1}{2} g t^2 \, .$$

2.1 Deparameterized Equations

For two-dimensional motion, having separate equations for x, v_x, y, v_y all in terms of time t constitutes a parameterized description of the motion. It's often useful to have a set of equations that relates y directly to x, thereby eliminating the parameter t.

Eliminating t between the pair of velocity equations, one finds

$$v_y = v_{y_0} - \frac{g}{2} \left(\frac{x - x_0}{v_{x_0}} \right)^2 \, .$$

More useful is the equation that eliminated t between x and y. Solve for t in the x-equation and replace all instances of t in the y-equation:

$$y = y_0 + v_{y_0} \left(\frac{x - x_0}{v_{x_0}} \right) - \frac{g}{2} \left(\frac{x - x_0}{v_{x_0}} \right)^2 \tag{1.13}$$

Completing the square in x, the above becomes

$$y = y_0 + \frac{v_{y_0}^2}{2g} - \frac{g}{2} \left(\frac{x - x_0}{v_{x_0}} - \frac{v_{y_0}}{g} \right)^2 \, . \tag{1.14}$$

Perhaps not surprisingly, the y-equation is quadratic in the variable x as it was in the variable t. This means the motion of a projectile in uniform gravity free of air drag always follows a parabolic path as shown in Figure 1.4.

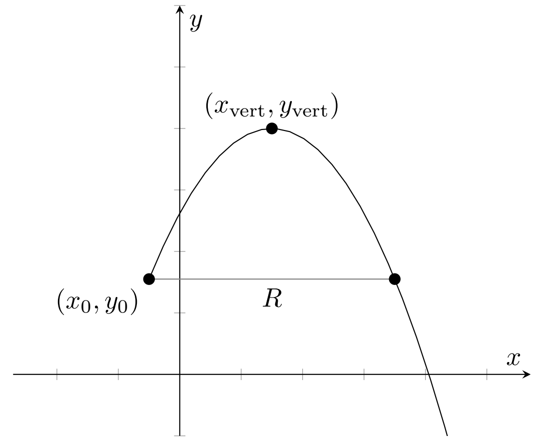

Figure 1.4: Two-dimensional motion in uniform gravity.

Vertex

The above Figure is sketched with $v_{x_0} > 0$ and $v_{y_0} > 0$. The vertex of the parabolic motion is located at $(x_{\text{vert}}, y_{\text{vert}})$, which is easily determined by setting the second term in Equation (1.14) to zero. From this we quickly see

$$y_{\text{vert}} = y_0 + \frac{v_{y_0}^2}{2g}$$

and

$$x_{\text{vert}} = x_0 + \frac{v_{x_0} v_{y_0}}{g} \, .$$

In terms of the vertex location, Equation (1.14) is written more simply:

$$y = y_{\text{vert}} - \frac{g}{2} \left(\frac{x - x_{\text{vert}}}{v_{x_0}} \right)^2$$

Not forgetting time is still relevant, we can write down the time t_{vert} required for the object to reach the vertex. From the x-equation we find

$$t_{\text{vert}} = \frac{x_{\text{vert}} - x_0}{v_{x_0}} \, ,$$

and from the y-equation,

$$t_{\text{vert}} = \frac{v_{y_0}}{g} \, .$$

Horizontal Range

An object moving upward against uniform gravity will eventually reach a maximum height at y_{vert} and fall to its original height y_0. The horizontal distance traversed during this part of the motion is the *horizontal range*, depicted as R in Figure 1.4.

By construction, the horizontal range is two times the distance from x_0 to x_{vert}, or:

$$R = 2\left(x_{\text{vert}} - x_0\right) = \frac{2v_{x_0}v_{y_0}}{g} \tag{1.15}$$

Interestingly, notice that the horizontal range remains the same if mutually swapping $v_{x_0} \leftrightarrow v_{y_0}$. More strongly, notice that the range is the same so long as the product $v_{x_0}v_{y_0}$ remains the same as sketched in Figure 1.5.

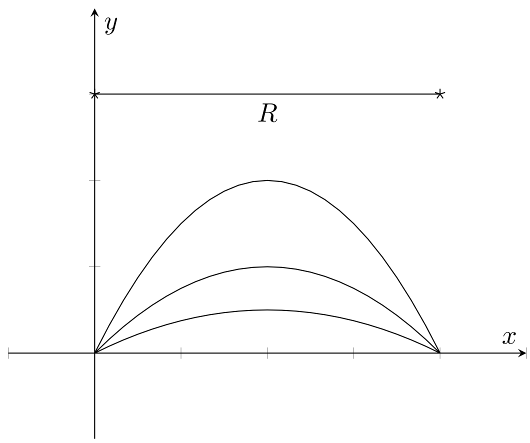

Figure 1.5: Range equivalence of projectile motion.

Flight Time

The time t_R required to traverse the horizontal range is twice the time t_{vert}:

$$t_R = 2\left(\frac{x_{\text{vert}} - x_0}{v_{x_0}}\right) = \frac{2v_{y_0}}{g}$$

Multiplying through by v_{x_0} gives a tidy identity relating the above to R:

$$v_{x_0}t_R = R$$

It's also worth having another equation that relates t directly to y. Adapting Equation (1.7), write

$$t = \frac{v_{y_0}}{g} \mp \frac{v_{y_0}}{g}\sqrt{1 - \frac{2g\left(y - y_0\right)}{v_{y_0}^2}}\,,$$

which simplifies to:

$$t = t_{\text{vert}} \mp \sqrt{\frac{2}{g}\left(y_{\text{vert}} - y\right)}$$

2.2 Envelope of Trajectories

For a given initial speed

$$v_0 = \sqrt{v_{x_0}^2 + v_{y_0}^2}\,,$$

the *envelope of trajectories* is a curve in the Cartesian plane that characterizes all possible paths of motion. For simplicity we'll assume each velocity is greater than zero.

To find the envelope of trajectories for freefall motion, note from the above that

$$\frac{1}{v_{x_0}^2} = \frac{1}{v_0^2}\left(1 + \frac{v_{y_0}^2}{v_{x_0}^2}\right)\,,$$

and substitute into Equation (1.13) for deparameterized kinematics, yielding

$$\begin{aligned} y = y_0 &+ \frac{v_{y_0}}{v_{x_0}}\left(x - x_0\right) \\ &- \frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\left(1 + \frac{v_{y_0}^2}{v_{x_0}^2}\right)\,. \end{aligned}$$

The above is in fact quadratic in the variable v_{y_0}/v_{x_0}. To keep the algebra tame, write

$$0 = A\left(\frac{v_{y_0}}{v_{z_0}}\right)^2 + B\left(\frac{v_{y_0}}{v_{z_0}}\right) + C$$

such that

$$\begin{aligned} A &= -\left(\frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right) \\ B &= \left(x - x_0\right) \\ C &= \left(y_0 - \frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right) - y = -y + y_0 + A\,. \end{aligned}$$

As a quadratic equation, thereby solved by the quadratic formula, the discriminant $B^2 - 4AC$ is the most telling quantity. For the trajectory to exist, the discriminant must be non-negative. The limit case $B^2 = 4AC$ will provide the outline of all allowed trajectories for the given v_{x_0}, v_{y_0}. Setting $B^2 = 4AC$ and simplifying, we find

$$\begin{aligned} \left(x - x_0\right)^2 &= -4A\left(y - y_0\right) + 4A^2 \\ \frac{-2\cancel{A}v_0^2}{g} &= -4\cancel{A}\left(y - y_0\right) + 4A \cdot \cancel{A} \\ y - y_0 &= \frac{v_0^2}{2g} + A\,, \end{aligned}$$

and finally:

$$y - y_0 = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}\left(x - x_0\right)^2 \tag{1.16}$$

Evidently, the envelope of trajectories has parabolic character with intercepts occurring at

$$x_{\text{int}} = x_0 + \frac{v_0^2}{g}$$
$$y_{\text{int}} = y_0 + \frac{v_0^2}{2g} \; .$$

For simplicity, let $x_0 = y_0 = 0$ to visualize the envelope as shown in Figure 1.6.

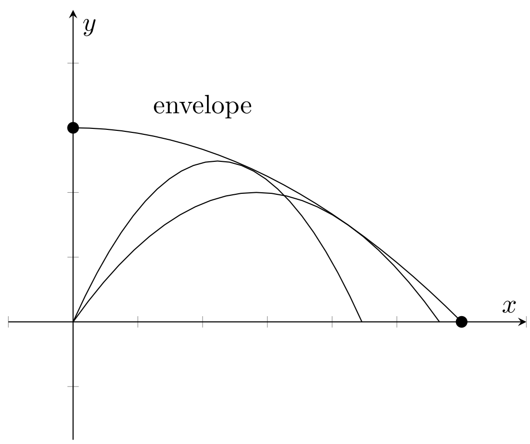

Figure 1.6: Envelope of trajectories.

Maximal Range

Knowing the envelope of trajectories, particularly the x-intercept, we see that the maximum horizontal range for any projectile (assuming $x_0 = 0$) is given by

$$R = \frac{v_0^2}{g} \; .$$

Meanwhile, we already have a formula for R via Equation (1.15), namely

$$R = \frac{2 v_{x_0} v_{y_0}}{g} \; .$$

Equating these, we find a restriction on v_{x_0}, v_{y_0} that correspond to maximal range:

$$0 = v_0^2 - 2 v_{x_0} v_{y_0}$$
$$0 = v_{x_0}^2 + v_{y_0}^2 - 2 v_{x_0} v_{y_0}$$
$$0 = (v_{x_0} + v_{y_0})(v_{x_0} - v_{y_0})$$

For the condition of maximal horizontal range to be satisfied, it must be that:

$$v_{x_0} = v_{y_0} = \frac{v_0}{\sqrt{2}}$$

2.3 Rifle Problems

Identical Rifles

Suppose a pair of identical rifles mounted at the orgin are tilted with unequal angles with respect to the horizon. Let us find the condition for which the projectiles strike the same target located at (x_*, y_*).

To proceed, use Equation (1.13) for each rifle to write

$$y_* = \frac{v_{y_0}}{v_{x_0}} x_* - \frac{g x_*^2}{2} \left(\frac{1 + \left(v_{y_0}/v_{x_0}\right)^2}{v_0^2} \right)$$
$$y_* = \frac{v_{y_0'}}{v_{x_0'}} x_* - \frac{g x_*^2}{2} \left(\frac{1 + \left(v_{y_0'}/v_{x_0'}\right)^2}{v_0^2} \right) \; .$$

Eliminate y_* and simplify to end up with

$$\frac{2 v_0^2}{g x_*} = \frac{v_{y_0}}{v_{x_0}} + \frac{v_{y_0'}}{v_{x_0'}} \; .$$

If the target height y_* is the same height as each rifle, then the horizontal distance x_* is equivalent to the range R, leading to

$$v_{x_0} v_{y_0} = v_{x_0'} v_{y_0'} \; ,$$

and furthermore:

$$v_{y_0'} = v_{x_0}$$
$$v_{x_0'} = v_{y_0}$$

Hunter and the Monkey

A hunter aims his rifle at a monkey at height H in a tree that is horizontal distance L downrange. The instant the hunter shoots his rifle, the monkey lets go of the tree. Does the shot miss the monkey?

The position of the projectile starting from $(0, 0)$ is given by

$$x(t) = v_{x_0} t$$
$$y(t) = v_{y_0} t - \frac{1}{2} g t^2 \; .$$

Meanwhile, the height z of the monkey is given by

$$z(t) = H - \frac{1}{2} g t^2 \; .$$

Eliminate $gt^2/2$ between the y- and z-equations to write the height difference between the monkey and the projectile

$$z(t) - y(t) = H - v_{y_0} t \; ,$$

and evaluate at $t_* = L/v_{x_0}$ to write

$$z(t_*) - y(t_*) = H - \frac{v_{y_0}}{v_{x_0}} L \; .$$

The left side of the above must resolve to zero if the shot hits the monkey. The hunter aims directly at the monkey initially, so we use similar triangles to state

$$\frac{L}{H} = \frac{v_{x_0}}{v_{y_0}} \; .$$

This means the right side of the above cancels out, and

$$z\left(t_*\right) = y\left(t_*\right) \; .$$

Poor monkey.

Index