# Introductory Algebra

William F. Barnes

May 2, 2023

# Contents

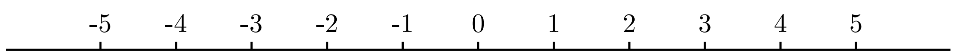
# Chapter 1

# Introductory Algebra

## 1 Numbers and Operations

### 1.1 Numbers

Numbers are symbols used for counting, measuring, or specifying an abstract quantity. Each number is *unique*, meaning no two numbers represent the same value. When two numbers are compared, one must have a *greater* value, and the other will have the *lesser* value. It follows that numbers can be arranged on a *number line*, from lesser (left) to greater (right), as shown:




Note the number line also includes the *negative* numbers, along with *zero*. While we are free to take these for granted, the idea of negative numbers is somewhat 'modern', as it did not occur, for instance, to the ancient Greeks or Romans.

The number line extends beyond the numbers $-5$ on the left, and 5 on the right. By imagining the end-limits (not illustrated for obvious reasons) of the number line, we confront the notion of *infinity*, denoted $\infty$, which is the 'most positive' entry on the number line. It's mirror image, namely $-\infty$, is the 'most negative' number.

All decimals and fractions are well-contained by the number line. For instance, the decimal 2.5 is a perfectly valid number situated midway between 2 and 3. Meanwhile, the fraction $2/3 = 0.66666\ldots$, which would take *forever* to write as a proper decimal, is situated two-thirds between 0 and 1 on the number line.

#### Real Numbers

All members of the number line or implied by the number line fall into a category called *real numbers*, which is the set of all negative, positive, and decimal numbers, including zero:

$$-\infty < \text{Real Numbers} < \infty$$

The symbol used to denote the set of real numbers is $\mathbb{R}$.

#### Rational Numbers

*Rational numbers*, denoted $\mathbb{Q}$, are those that can be expressed as a closed fraction or repeating decimal. These are part of the real numbers, which we express by writing

$$\mathbb{Q} \subset \mathbb{R}\,,$$

translating to '$\mathbb{Q}$ is a *proper set* of $\mathbb{R}$'. For instance, the numbers $1/2$, $-2/3$, and $4/4$ are each rational numbers.

#### Irrational Numbers

A different proper set of the real numbers is the set of *irrational numbers*, denoted

$$\mathbb{Q}' \subset \mathcal{R} \, ,$$

which includes any quantity that cannot be expressed as a closed fraction. All non-repeating decimal numbers, such as $\pi = 3.14159\ldots$, are irrational.

The *union* of the rational numbers and the irrational numbers reconstitutes the set of real numbers:

$$\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$$

#### Integers vs. Fractions

Any rational number is either an integer or a fraction. *Integers* are positive or negative numbers that contain no decimal part. The symbol used to denote the set of integers is $\mathbb{Z} \subset \mathcal{Q}$, particularly

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \; .$$

#### Whole Numbers

The set of *whole numbers* is a subset of the integers that excludes negative numbers, but includes zero:

$$\text{Whole Numbers} = \{0, 1, 2, 3, \ldots\}$$

#### Natural Numbers

The set of *natural numbers*, denoted $\mathbb{N} \subset \mathcal{Q}$, is a subset of the whole numbers without zero:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

## 1.2 Operators and Expressions

Numbers may be manipulated using *arithmetic operators*. Most arithmetic operators, such as 'add' ( $+$ ), 'subtract' ( $-$ ), 'multiply' ( $\times$ ), and 'divide' ( $/$ ), are placed *between* two numbers to form an *expression*. For instance,

$$1+2 \qquad\qquad 3-4 \qquad\qquad 5\times 6 \qquad\qquad 7/8$$

are valid mathematical expressions.

#### Evaluation

Each expression above exhibits a single operator surrounded by a 'left' number and a 'right' number. Since these are rather simple operations, we can easily *evaluate* each expression, which means to calculate each result:

$$1+2=3 \qquad\qquad 3-4=-1 \qquad\qquad 5\times 6=30 \qquad\qquad 7/8=0.875$$

By appending an expression's result to the right of the 'equality' sign ( $=$ ), the expression becomes a mathematical *statement*. (Think of $=$ as the verb in a sentence.)

## 1.3 Parentheses

Bracketing symbols called *parentheses* ( ) can be used to embed expressions within expressions. Parentheses must always 'balance', meaning there can never be dissimilar numbers of opening- and closing-parentheses. For example,

$$4+(3\times 4) \qquad\qquad (4+(3+3))\,/2$$

are valid expressions containing parentheses.

Simplifying expressions with parentheses follows a single rule: *evaluate the most-embedded contents first.* For the first example on hand, we first see that $(3 \times 4)$, being the only parenthesized quantity, simplifies to 12. Thus we write

$$4 + (3 \times 4) = 4 + 12 = 16$$

to finish the first example. For the second example, we must first acknowledge $(3 + 3) = 6$, and the expression can be re-written and simplified:

$$(4 + 6) / 2 = 10/2 = 5$$

## 1.4 Order of Operations

If an expression contains more than one operator, each operation must be evaluated in a specific order, called the *order of operations.* The order is contained in the made-up word 'PEMDAS', which stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction. This can be remembered by the phrase **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally:

| | | |
|---|---|---|
| Parentheses | ( ) | enclose expressions. |
| Exponent | ˆ | results in a *product.* |
| Multiplication | $\times$ | results in a *product.* |
| Division | / | results in a *ratio* or *quotient.* |
| Addition | + | results in a *sum.* |
| Subtraction | − | results in a *difference.* |

Example 1

Evaluate the expression: $4 + 5 \times 9$

| | |
|---|---|
| Look for Parentheses. (None). | |
| Look for Exponents. (None.) | |
| Look for Multiplication. | $5 \times 9 = 45$ |
| Rewrite the expression. | $4 + 45$ |
| Look for Division. (None.) | |
| Look for Addition. | $4 + 45 = 49$ |
| Rewrite the expression. | 49 |
| Look for Subtraction. (None.) | |
| Write the result. | $4 + 5 \times 9 = 49$ |

Example 2

Evaluate the expression: $(4 + 5) \times 9$

| | |
|---|---|
| Look for Parentheses. | $(4 + 5) = 9$ |
| Rewrite the expression. | $9 \times 9$ |
| Look for Exponents. (None.) | |
| Look for Multiplication. | $9 \times 9 = 81$ |
| Rewrite the expression. | 81 |
| Look for Division. (None.) | |
| Look for Addition. (None) | |
| Look for Subtraction. (None.) | |
| Write the result. | $(4 + 5) \times 9 = 81$ |

### Binomials and Trinomials and Polynomials (oh my!)

The simplest kind of mathematical expression is a *binomial*, which contains two numbers, or *terms*, separated by a plus or minus sign. For instance,

$$2 + 4 \qquad\qquad 3 - 8$$

are binomials. On the other hand, expressions such as

$$3 - 4 + 1 \qquad\qquad 7 + 11 - 6$$

are classified as *trinomial* by virtue of having to three terms and two operators. In general, expressions containing multiple terms are called *polynomials*.

**Role of Zero**

You already know what 'zero' does in an intuitive sense, so let us jot down its role in complete statements.

- Adding or subtracting zero to any quantity leaves the quantity unchanged.

$$3 \pm 0 = 3$$

- Multiplying any quantity by zero results in zero.

$$3 \times 0 = 0$$

- Division by zero produces no useful information.

$$\frac{3}{0} = \text{Undefined}$$

**Role of One**

Meanwhile, two more statements can be written about the number one:

- Multiplying any quantity by one leaves the quantity unchanged.

$$3 \times 1 = 3$$

- Dividing any quantity by one leaves the quantity unchanged.

$$3/1 = 3$$

---

Example 3

Insert any combination of operators to make the following statement true:

$$1\,2\,3\,4\,5\,6\,7\,8\,9 = 100$$

$$1 \times 2 - 3 + 4 - 5 + 6 + 7 + 89 = 100$$
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100$$

## 1.5 Prime Numbers

*Prime numbers* are whole numbers that cannot be divided into smaller whole numbers (besides 1). Apart from 2, no even numbers are prime. There is otherwise no simple pattern to the prime numbers, indicated by bold script in the following table. (Prime numbers don't seem to follow a discernible pattern.)

$$\begin{bmatrix} 1 & \mathbf{2} & \mathbf{3} & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 & 9 & 10 \\ \mathbf{11} & 12 & \mathbf{13} & 14 & 15 & 16 & \mathbf{17} & 18 & \mathbf{19} & 20 \\ 21 & 22 & \mathbf{23} & 24 & 25 & 26 & 27 & 28 & \mathbf{29} & 30 \\ \mathbf{31} & 32 & 33 & 34 & 35 & 36 & \mathbf{37} & 38 & 39 & 40 \\ \mathbf{41} & 42 & \mathbf{43} & 44 & 45 & 46 & \mathbf{47} & 48 & 49 & 50 \end{bmatrix}$$

### Prime Decomposition

The *prime decomposition* of a number is defined as a list of prime numbers which when multiplied together, produce the original number. Reading this statement backwards, it follows that any integer can be broken apart or 'decomposed' into prime numbers. To find the prime decomposition of an odd number, try dividing by 2. The result can be one of three things:

- If the result is a prime number, you're done.
- If the result comes out to an integer (no fraction or decimal), then 2 is one of the factors. Proceed by dividing the new result by 2 and repeat.
- If the result was not a round integer, discard the 2 and try dividing by the next prime number, 3 and repeat for 5, 7, and so on.

Example 4

Decompose the number 12 into prime factors.

| | |
|---|---|
| Try dividing 12 by 2: | $12/2 = 6$ |
| Is the result an integer? | Yes, so keep the **2**. |
| Is the result a prime number? | No, so keep going using 6. |
| Try dividing 6 by 2: | $6/2 = 3$ |
| Is the result an integer? | Yes, so keep the **2**. |
| Is the result a prime number? | Yes, so stop at **3**. |
| Write the result and check: | $12 = 2 \cdot 2 \cdot 3$ |

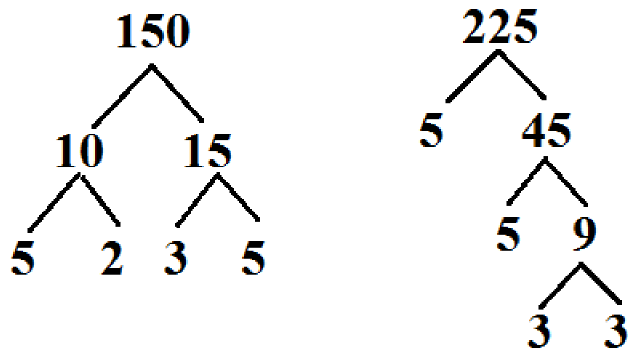
Example 5

Decompose the number 231 into prime factors.

| | |
|---|---|
| Try dividing 231 by 2: | $231/2 = 115.5$ (not an integer) |
| Try dividing 231 by 3: | $231/3 = 77$ |
| Is the result an integer? | Yes, so keep the **3**. |
| Is the result a prime number? | No, so keep going using 77. |
| Try dividing 77 by 2: | $77/2 = 38.5$ (not an integer) |
| Try dividing 77 by 3: | $77/3 = 25.667$ (not an integer) |
| Try dividing 77 by 5: | $77/5 = 15.4$ (not an integer) |
| Try dividing 77 by 7: | $77/7 = 11$ |
| Is the result an integer? | Yes, so keep the **7**. |
| Is the result a prime number? | Yes, so stop at **11**. |
| Write the result and check: | $231 = 3 \cdot 7 \cdot 11$ |

**Tree Method**

A pictorial technique for prime decomposition is the so-called *tree method*, which represents the factors of a number as 'branches'. The final extremity of each branch is the set of prime factors as shown.



Problem 1

Use any method to write the prime factors of:

300 160 99

## 1.6 Properties of Multiplication

### Commutative Property

The *commutative property* contains the idea that two numbers may be multiplied in either order without changing the result. For example:

$$2 \times 3 = 6 \qquad 3 \times 2 = 6$$

### Associative Property

The *associative property* tells us that the grouping of factors in a multiplication problem does not affect the result. For example:

$$2 \times 3 \times 4 = 24 \qquad 2 \times (3 \times 4) = 24 \qquad (2 \times 3) \times 4 = 24$$

### Distributive Property

The *distributive property* dictates how expressions containing sums and differences in parentheses are multiplied. The idea of 'distribution' means that whatever is outside the parentheses is multiplied into *each* term enclosed inside the parentheses. For example we can multiply the number 4 into the binomial $2 - 3$ as:

$$4 \times (2 - 3) = 4 \times 2 - 4 \times 3$$

Supposing the number 4 was instead itself a binomial, such as $4 - 1$, the example becomes:

$$\begin{aligned}(4 - 1) \times (2 - 3) &= (4 - 1) \times 2 - (4 - 1) \times 3 \\ &= 4 \times 2 - 1 \times 2 - 4 \times 3 + 1 \times 3\end{aligned}$$

### FOIL Method for Binomials

A trick for remembering the steps to multiply two binomials is contained in the word 'FOIL', where each respective letter stands for **F**irst, **O**uter, **I**nner, **L**ast. For instance, consider the product

$$(4 + 2)\,(5 - 3)\ .$$

To begin we multiply the **F**irst respective terms to get

$$4 \times 5 = 20\,.$$

Next, multiply the **O**uter terms, resulting in

$$4 \times -3 = -12\,.$$

Next we multiply the **I**nner terms, namely

$$2 \times 5 = 10\,,$$

and finally, the **L**ast terms multiply to give

$$2 \times -3 = -6\,.$$

The final answer is the sum of the four respective products. In our example, we have

$$(4 + 2)\,(5 - 3) = 20 - 12 + 10 - 6 = 12\,.$$

Of course, we this was visible from the outset, as

$$(4 + 2)\,(5 - 3) = (6) \times (2) = 12\,.$$

The FOIL method works on *all* binomials, but *only* works on binomials.

### Greatest Common Factor

Recall form our study of prime decomposition that any whole integer can be decomposed into the product of smaller factors, all the way down to prime numbers. The *greatest common factor* (or GCF) is the largest number that can multiply into another set of numbers. More generally, a *common factor* is any number that can multiply into two other numbers.

Example 6

Find all common factors of 12 and 30. Identify the greatest common factor.

| | |
|---|---|
| List all factors of 12. | 1, 2, 3, 4, 6, 12 |
| List all factors of 30. | 1, 2, 3, 5, 6, 10, 15, 30 |
| Identify the common (shared) factors. | 1, 2, 3, 6 |
| Identify the greatest common factor. | 6 |

Example 7

Find all common factors of 15, 30 and 105. Identify the greatest common factor.

| | |
|---|---|
| List all factors of 15. | 1, 3, 5, 15 |
| List all factors of 30. | 1, 2, 3, 5, 6, 10, 15, 30 |
| List all factors of 105. | 1, 3, 5, 7, 15, 21, 35, 105 |
| Identify the common (shared) factors. | 1, 3, 5, 15 |
| Identify the greatest common factor. | 15 |

Example 8

Use prime decomposition to find the GCF of 24 and 108.

| | |
|---|---|
| Write prime decomposition of 24. | $24 = 2 \times 2 \times 2 \times 3$ |
| Write prime decomposition of 108. | $108 = 2 \times 2 \times 3 \times 3 \times 3$ |
| Identify the greatest common factor. | $2 \times 2 \times 3 = 12$ |

Example 9

Bobby has 160 red marbles and 144 blue marbles. He wants to split the marbles into identical groups (not necessarily the same number of each color in a group). What is the greatest number of groups he can make?

This is solved by finding the GCF of 160 and 144. Working these out, find

$$160 = 2^5 \times 5 \qquad\qquad 144 = 2^4 \times 3^2 \, ,$$

indicating the GCF is $2^4 = 16$, meaning 16 groups of marbles can be made. Each group has $160/16 = 10$ red marbles, and $144/16 = 9$ blue marbles.

Problem 2

Alice has 9 forks and 6 spoons. She intends to lay the utensils out in groups around the dinner table with none left over. How many groups can she make, and how many of each utensil per group?

Problem 3

Cynthia is preparing Halloween treats for her classmates. She has 72 orange candies and 24 red candies in total, and wants to divide them into the greatest number of bags. How many bags can she prepare, and how much of each type of candy should be in each bag?

### Least Common Multiple

The *least common multiple* (or LCM) is the product of prime factors of two or more numbers, with each prime factor taken to the highest power in which it occurs. That is, the LCM is the *lowest* number that can be evenly divided by two or more numbers.

Example 10

Find the least common multiple of 84 and 147.

| | |
|---|---|
| Write prime decomposition of 84. | $84 = 2^2 \times 3 \times 7$ |
| Write prime decomposition of 147. | $147 = 3 \times 7^2$ |
| List each factor to highest power. | $2^2$, $3^1$, $7^2$ |
| The LCM is the product of each factor. | $LCM = 2^2 \times 3 \times 7^2 = 588$ |

Example 11

What is the lowest number that can be evenly divided by 3, 9, and 21?

| | |
|---|---|
| Write prime decomposition of 3. | $3 = 3^1$ |
| Write prime decomposition of 9. | $9 = 3^2$ |
| Write prime decomposition of 21. | $21 = 3^1 \times 7^1$ |
| List each factor to highest power. | $3^2$, $7^1$ |
| The LCM is the product of each factor. | $LCM = 3^2 \times 7 = 63$ |

Example 12

Alice works in the orchard picking peaches, and fits 8 peaches per bag. Bob has the same job, but fits 9 peaches per bag. At the end of the day, they have picked the same number of peaches. What is the smallest number of peaches they each could have picked?

This is solved by finding the LCM of 8 and 9. Note that $8 = 2^3$, and $9 = 3^2$, telling us the LCM is $2^3 \times 2^2 = 72$.

Example 13

Find the greatest common factor (GCF) and lowest common multiple (LCM) of the two numbers:

2940 3150

Decompose each number into primes:

$$2940 = 2 \times 2 \times 3 \times 5 \times 7 \times 7 = 2^2 \times 3 \times 5 \times 7^2$$
$$3150 = 2 \times 3 \times 3 \times 5 \times 5 \times 7 = 2 \times 3^2 \times 5^2 \times 7$$

The GCF is the greatest number that multiplies into 2940 and 3150. This is evidently

$$GCF = 2 \times 3 \times 5 \times 7 = 210\,.$$

Meanwhile, the LCM is the product of each above-listed prime, raised to the highest-occurring power, ignoring the lower-power occurrence. Therefore,

$$LCM = 2^2 \times 3^2 \times 5^2 \times 7^2 = 210^2\,.$$

## 1.7 Properties of Fractions

Any real number that is not an integer is some kind of fraction or decimal. These are numbers such as 1/3, 9.5, −0.6666666, and so on. The conventional symbol for division is the forward-slash ( / ), equivalent to the numerator-over-denominator notation (top number over the bottom number). That is, the following are equivalent:

$$2/3 \qquad \frac{2}{3}$$

### Multiplying Fractions

When fractions are multiplied, first multiply all terms in the numerator, and then separately multiply all terms in the denominator. The resulting fraction is the ratio of the respective products. For example:

$$\frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{5}{6} \qquad\qquad \frac{2}{3} \times \frac{5}{4} \times \frac{7}{9} = \frac{2 \times 5 \times 7}{3 \times 4 \times 9} = \frac{35}{54}$$

### Notion of Reciprocal

The *reciprocal* or *inverse* of a fraction is the result of exchanging the numerator and the denominator. For instance, the reciprocal of 2/7 is 7/2. For another example, the following two fractions are reciprocal of one another:

$$\frac{34}{256} \qquad\qquad \frac{256}{34}$$

In the most general sense, the reciprocal of any value is one divided by that value:

$$\text{reciprocal} = \frac{1}{\text{fraction}}$$

### Dividing Fractions

With a word like 'reciprocal' under our belt, we think about fractions in a more refined fashion. Consider the equivalent fractions

$$2/3 \qquad\qquad \frac{2}{3}$$

that plainly read as 'two over three' or 'two divided by three'. An equivalent interpretation of this fraction would read 'two times one third' because, division by a number is equivalent to multiplying by that number's reciprocal. For our example, this means:

$$\frac{2}{3} = 2 \times \frac{1}{3} \qquad\qquad 2/3 = 2 \times 1/3$$

Example 14

Convert to a single fraction and simplify

$$(3/4)\,/\,(2/5)\ .$$

First, identify the reciprocal of the fraction being divided, namely

$$\frac{5}{2}\,.$$

Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$(3/4)\,/\,(2/5) = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$$

Example 15

Convert to a single fraction and simplify

$$\frac{3/5}{7/8}\,.$$

First, identify the reciprocal of the fraction being divided, namely

$$\frac{8}{7}\,.$$

Next, rewrite the division problem as multiplication by the reciprocal and simplify as necessary:

$$\frac{(3/5)}{(7/8)} = \frac{3}{5} \times \frac{8}{7} = \frac{24}{35}$$

**Change of Denominator**

Fractions can be changed to have a different numerator and denominator, but only if the numerical value of the fraction remains the same. For example, we know that

$$\frac{1}{2} \qquad\qquad \frac{3}{6}$$

are *equivalent*, and evaluate to 0.5, yet the actual numbers used in each fraction are different.

Conversion between equivalent fractions is achieved by multiplying the fraction by a carefully-chosen factor of one. For example, we convert 1/2 to 3/6 by

$$\frac{1}{2} = \frac{1}{2} \times (1) = \frac{1}{2} \times \left(\frac{3}{3}\right) = \frac{1 \times 3}{2 \times 3} = \frac{3}{6}\,.$$

Choosing a different factor of one, we can convert to any fraction that is ultimately equivalent to the original ratio. For instance:

$$\frac{1}{2} = \frac{1}{2} \times (1) = \frac{1}{2} \times \left(\frac{13}{13}\right) = \frac{1 \times 13}{2 \times 13} = \frac{13}{26}$$

Example 16

Consider the ratio 9/3. Change the denominator to 17 without changing the numerical value of the fraction.

$$\frac{9}{3} = \frac{9/3}{1} \times (1) = \frac{9/3}{1} \times \left(\frac{17}{17}\right) = \frac{9/3 \times 17}{17} = \frac{51}{17}$$

Example 17

Consider the two fractions 2/3 and 3/4. Rewrite each fraction such that the denominators are the same.

The LCM of 3 and 4 is 12, thus the target denominator is 12. Thus each fraction becomes:

$$\frac{2}{3} = \frac{2}{3} \times \left(\frac{4}{4}\right) = \frac{8}{12} \qquad\qquad \frac{3}{4} = \frac{3}{4} \times \left(\frac{3}{3}\right) = \frac{9}{12}$$

**Fraction to Decimal Conversion****

**Adding and Subtracting Fractions**

There is one single rule for adding and subtracting fractions, and that is: only fractions having the *same* denominator can be added or subtracted. For instance, consider the true statement:

$$\frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4}$$

The reason the above statement executes without error is the denominator of each fraction being added (subtracted in this case) is the same. On the left side, the two terms 3/4 and 1/4 share a denominator (namely 4), thus the two terms combine easily, which begs the question: what if the denominators not the same? The answer is to convert one (or both) of the fractions such that the denominators match.

Example 18

Evaluate the sum of 2/3 and 3/4.

These fractions can only be summed when the denominators are equal. Borrowing from the Example above, we have

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}\,.$$

Example 19

Evaluate the expression: $1/2 + 1/3 - 1/7$

The LCM of each denominator, namely, 2, 3, and 7 is 42, hence

$$\frac{1}{2} + \frac{1}{3} - \frac{1}{7} = \frac{21}{42} + \frac{14}{3} - \frac{6}{42} = \frac{29}{42}\,.$$

## 1.8 Properties of Exponents

### Exponent Notation

When a number is multiplied by itself more than twice, or many times, we don't want to be burdened by writing things like $6 \times 6 \times 6 \times 6$. This is remedied by exponent notation, demonstrated as follows:

$$\begin{aligned} 4 &= 4^1 \\ 3 \times 3 &= 3^2 \\ 5 \times 5 \times 5 &= 5^3 \\ 2 \times 2 \times 9 \times 9 &= 2^2 \times 9^2 \end{aligned}$$

Exponents obey certain conventions that we shall establish here.

- The *base* number is the 'lower' number to which the exponent applies.
- The *exponent* is written smaller and to the upper-right of the base number.
- When the exponent is 2, the result is the *square* of the base number.
- When the exponent is 3, the result is the *cube* of the base number.

### Negative Base Numbers

Negative numbers with exponents must be treated carefully. For instance, if asked to square the number $-3$, one may write

$$(-3)^2 = -3 \times -3 = (-1) \times (-1) \times 3^2 = 9\,,$$

which has the same result as (positive) $3^2 = 9$. Indeed, raising to any base number *even* power results in a positive number. On the other hand, if a negative base number is raised to an *odd* number, factors of $-1$ occur an odd number of times, as in

$$(-2)^3 = (-1) \times (-1) \times (-1) \times 2^3 = -8\,.$$

Beware the placement of parentheses when handling exponents. For instance the number $-2^4$ translates to $-2 \times 2 \times 2 \times 2 = -16$. However, if we were handed $(-2)^4$, the full statement would be $(-2) \times (-2) \times (-2) \times (-2) = 16$. (The two answers differ by a sign!)

### Multiplying Numbers with Exponents

For two numbers having equal base and unequal exponents, their product is the base number raised to the sum of the exponents. For example, the product $3^2 \times 3^4$ can be immediately translated to $3^6$:

$$(3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^{2+4} = 3^6$$

### Negative Exponents

A number raised to a negative exponent is equivalent to the reciprocal of that number raised to the positive exponent. For example, $3^{-4}$ can be immediately translated to $(1/3)^4$. This is consistent with the notion of adding (or subtracting) exponents when combining numbers of similar base. For instance,

$$\begin{aligned} 3^2 \times 3^4 &= 3^6 \\ \cancel{\left(\frac{1}{3^4}\right)} \times 3^2 \times \cancel{3^4} &= \left(\frac{1}{3^4}\right) \times 3^6 \\ 3^2 &= 3^{6-4} \end{aligned}$$

### Compounded Exponents

A number that is raised to an exponent, and then raised to an exponent *again*, as in

$$\left(2^3\right)^2 = 8^2 = 64$$

is equivalent to *multiplying* the two exponents:

$$\left(2^3\right)^2 = 2^{3\times 2} = 2^6 = 64$$

### Zero Exponent

A number that is raised to an exponent of *zero*, results in precisely *one*. For instance, the product of $3^2$ and $3^{-2}$ resolves to 1:

$$\frac{3^2}{3^2} = 3^2 \times 3^{-2} = 3^0 = 1$$

### Non-Integer Exponents

Numbers raised to non-integer exponents are generally called *radicals*. Valid radical numbers may look like:

$$3^{.5} = \sqrt{3} \qquad 4^{3/2} = \sqrt[3]{4} \qquad 7^{-1.5} = 7^{-3/2} = \frac{1}{7^{3/2}} = \frac{1}{\sqrt[3]{7}}$$

The base number is also denoted the *radicand*, and the exponent is called the *degree* of the radical.

When the exponent is precisely 1/2, the result of the operation is the *square root* of the base number. Similarly, if the exponent is precisely 1/3, the result of the operation is the *cube root* of the base number, and so on. Since the square root is so common, we may as well generate the following reference table using a calculator:

| $x$ | $\sqrt{x}$ | $x$ | $\sqrt{x}$ |
|---|---|---|---|
| 1 | 1 | 26 | 5.0990195135 |
| 2 | 1.4142135620 | 27 | 5.1961524227 |
| 3 | 1.7320508075 | 28 | 5.2915026221 |
| 4 | 2 | 29 | 5.3851648071 |
| 5 | 2.2360679774 | 30 | 5.4772255750 |
| 6 | 2.4494897427 | 31 | 5.5677643628 |
| 7 | 2.6457513110 | 32 | 5.6568542494 |
| 8 | 2.8284271247 | 33 | 5.7445626465 |
| 9 | 3 | 34 | 5.8309518948 |
| 10 | 3.1622776601 | 35 | 5.9160797830 |
| 11 | 3.3166247903 | 36 | 6 |
| 12 | 3.4641016151 | 37 | 6.0827625302 |
| 13 | 3.6055512754 | 38 | 6.1644140029 |
| 14 | 3.7416573867 | 39 | 6.2449979983 |
| 15 | 3.8729833462 | 40 | 6.3245553203 |
| 16 | 4 | 41 | 6.4031242374 |
| 17 | 4.1231056256 | 42 | 6.4807406984 |
| 18 | 4.2426406871 | 43 | 6.5574385243 |
| 19 | 4.3588989435 | 44 | 6.6332495807 |
| 20 | 4.4721359549 | 45 | 6.7082039324 |
| 21 | 4.5825756949 | 46 | 6.7823299831 |
| 22 | 4.6904157598 | 47 | 6.8556546004 |
| 23 | 4.7958315233 | 48 | 6.9282032302 |
| 24 | 4.8989794855 | 49 | 7 |
| 25 | 5 | 50 | 7.0710678118 |

# 2 Variables

A *variable* is any symbol used to represent a number (or anything). Common symbols used are familiar letters $x$, $y$, $z$, $a$, $b$, $c$ - and some unfamiliar (Greek) letters $\alpha$, $\beta$, $\gamma$, etc. Variables behave exactly as numbers do: any place where a number is needed, a variable will work just as well.

## 2.1 Assignment

To *set a variable* is to associate a number with a symbol. For example, the statement $x = 3$ will 'tie' to the value of 3 to the symbol $x$. Any time $x$ occurs, it really means 3. The value of a variable is the number represented by the symbol. That is, we say 3 is the 'value' of $x$.

Variable names such as $x$, $y$, $z$, etc. are heavily recycled. That is, the same letters are prone to show up in different problems. Needless to mention, variables in one problem are not the same in a different problem. (They wouldn't be variables otherwise, would they?)

## 2.2 Coefficients

When a variable is multiplied by a number or another variable, the multiplication symbol is usually ignored. This means $4 \times x$ or $4 \cdot x$ should be written $4x$. When a variable appears alone, there is always an invisible 1 next to it. That is, $x$ is equivalent to $1x$ or $x/1$. The number multiplying the variable is called a *coefficient*. In the following three example terms, anything not precisely $x$ is a coefficient, namely 3, $-1/3$, and $a$, respectively:

$$3x \qquad -x/3 \qquad a \cdot x$$

Problem 1

Show that the GCF and LCM of

$$4a^2b \qquad 6ab \qquad 8ab^2$$

are:

$$GCF = 2 \times a \times b = 2ab$$
$$LCM = 2^3 \times 3 \times a^2 \times b^2 = 24a^2b^2$$

## 2.3 Combining Like Terms

When an expression contains two or more instances of the same variable raised to the same exponent, the coefficients may be added (or subtracted) in a process called *combining like terms*. For example, consider the expression:

$$2x - 5x + 7x^2$$

Notice that $2x$ and $-5x$ each contain a single instance of $x$. It follows that these terms may be added together to give $-3x$. The $7x^2$ term can't be combined with anything, because there are no other $x^2$ terms in play. The simplified expression is then:

$$-3x + 7x^2$$

## 2.4 Adding Polynomials

The addition and subtraction of binomials, trinomials, and polynomials is a straightforward application of combining like terms.

Example 1

Add the two trinomials:

$$\left(2x^2 + 2y^2 - 3\right) + \left(y^2 - 9x^2 - 2\right)$$

Step 1: Remove unnecessary parentheses:

$$2x^2 + 2y^2 - 3 + y^2 - 9x^2 - 2$$

Step 2: Rearrange to group like terms:

$$2x^2 - 9x^2 + 2y^2 + y^2 - 3 - 2$$

Step 3: Combine like terms:

$$-7x^2 + 4y^2 - 5$$

Example 2

Add the two polynomials:

$$\left(2x^2 + 2y^2 - 3\right) - \left(2y^2 - 4x^2\right)$$

Step 1: Distribute the minus sign into the second term:

$$\left(2x^2 + 2y^2 - 3\right) + \left(-2y^2 + 4x^2\right)$$

Step 2: Remove unnecessary parentheses:

$$2x^2 + 2y^2 - 3 - 2y^2 + 4x^2$$

Step 3: Rearrange to group like terms:

$$2x^2 + 4x^2 + 2y^2 - 2y^2 - 3$$

Step 4: Combine like terms:

$$6x^2 - 3$$

## 2.5 Multiplying Polynomials

Multiplying two polynomials is an application of the distributive property, which entails multiplying *every* term in the first polynomial into *every* term in the second polynomial, and adding all results. For instance, in the product

$$(y-4)\left(y^2 - 3x - 2\right) \ ,$$

we may proceed by distributing $y$, and then $-4$ separately into the second polynomial

$$\begin{aligned}(y-4)\left(y^2 - 3x - 2\right) &= y\left(y^2 - 3x - 2\right) - 4\left(y^2 - 3x - 2\right) \\ &= y^3 - 3xy - 2y - 4y^2 + 12x + 8 \ ,\end{aligned}$$

and we're done. Alternatively, we may instead distribute the terms of the second polynomial into the first to get the same result:

$$\begin{aligned}\left(y^2 - 3x - 2\right)(y-4) &= y^2(y-4) - 3x(y-4) - 2(y-4) \\ &= y^3 - 4y^2 - 3xy + 12x - 2y + 8\end{aligned}$$

## 2.6 Pascal's Triangle

An interesting pattern can be discovered by expanding each power of $(a+b)^n$, where $n = 0, 1, 2, 3, \ldots$ . Starting with $n = 0$ and going upward, we find:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\
(a+b)^7 &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \\
(a+b)^8 &= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8
\end{aligned}$$

Plucking out only the coefficients on the right side, we can arrange them in *Pascal's triangle*:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

If we are instead interested in expansions of $(a-b)^n$, the triangle becomes:

$$
\begin{array}{c}
+1 \\
+1 \quad -1 \\
+1 \quad -2 \quad +1 \\
+1 \quad -3 \quad +3 \quad -1 \\
+1 \quad -4 \quad +6 \quad -4 \quad +1 \\
+1 \quad -5 \quad +10 \quad -10 \quad +5 \quad -1 \\
+1 \quad -6 \quad +15 \quad -20 \quad +15 \quad -6 \quad +1 \\
+1 \quad -7 \quad +21 \quad -35 \quad +35 \quad -21 \quad +7 \quad -1 \\
+1 \quad -8 \quad +28 \quad -56 \quad +70 \quad -56 \quad +28 \quad -8 \quad +1
\end{array}
$$

# 3 Equations

Formally, an *equation* is made of two expressions joined by an equal sign. However 'ugly' an equation may appear, the left side and right side are equal. Examples of equations are:

$$\begin{aligned} 5+5 &= 10 \\ 3-x &= 2 \\ x^2 &= 24+25 \end{aligned}$$

## 3.1 Manipulating an Equation

Whatever manipulations are performed on an equation, the information it contains must not change: the left side must always equal the right side. This constrains us to not use equations to produce nonsense. The number of allowed manipulations boil down to just *two*: you may add zero to an equation, or multiply one into an equation, that's it. Below we explore a few special cases of either multiplying by one or adding zero.

### Adding or Subtracting a Number

Adding or subtracting the same number on both sides of an equation is one allowed operation. For example, consider the equation

$$x - 5 = 3\,.$$

If we simply add 5 to each side, the equation becomes

$$x - 5 + 5 = 3 + 5\,,$$

which simplifies very nicely:

$$x = 8$$

By isolating $x$, we have successfully 'solved for' its value, plugging $x - 8$ into the original equation, we find no mistakes:

$$8 - 5 = 3$$

### Multiplying or Dividing a Number

Multiplying or dividing the same number on both sides of an equation is another allowed operation. For example, let us isolate $x$ in the equation

$$2x = 24\,.$$

To do so, multiply both sides by a factor of $1/2$ to get

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 24\,,$$

and the coefficients cancel out on the left:

$$x = 12$$

### Raising to a Power

Raising each side to a power is yet another allowed operation on an equation. For example, consider the equation. For example, consider the equation:

$$\sqrt{x} = 3$$

If we raise each side to the power 2, the equation becomes

$$\left(\sqrt{x}\right)^2 = 3^2\,,$$

where the square root and the 2-power cancel, giving

$$\left(\sqrt{x}\right)^2 = \left(x^{1/2}\right)^2 = x^1 = x = 9\,.$$

### Taking a Root

Taking the square root (or cube root, etc.) of each side is one more allowed operation on an equation. For example, consider the equation:

$$x^2 = 49$$

Putting the square root symbol around the entire left side and the entire right side, the equation becomes

$$\sqrt{x^2} = \sqrt{49}\,,$$

where the square root and the 2-power cancel, giving

$$\sqrt{x^2} = \left(x^2\right)^{1/2} = x^1 = x = 7\,.$$

Before moving on however, note that $x = -7$ also satisfies the original equation $x^2 = 49$. This in fact true for all square roots: whenever we take the square root of a number, there are *two* results, one positive and one negative. In the most general case, we will always have

$$\sqrt{a^2} = \begin{cases} +a \\ -a \end{cases}\,,$$

usually condensed using the 'plus-or-minus' symbol ( $\pm$ ):

$$\sqrt{a^2} = \pm a$$

### Strategy

When performing operations that change an equation (on both sides!), make sure to properly apply multiplied terms, divided terms, and exponents/roots across the whole expression. To illustrate, consider the equation

$$3x + 6 = 12y + 18\,.$$

There are many ways to manipulate the equation above, but for no particular reason, let us try the square root operation. It would be WRONG to try:

$$\sqrt{3x} + \sqrt{6} = \sqrt{12y} + \sqrt{18}\,,$$

as this approach forgets to treat the left side and right side equally. To proceed correctly, we must write

$$\sqrt{3x+6} = \sqrt{12y+18}\,.$$

If instead we wanted to square each side of the equation, it would be WRONG to try

$$3x^2 + 6^2 = 12y^2 + 18\,,$$

which again forgets to treat each side equally. Instead, we must have

$$(3x+6)^2 = (12y+18)^2\,.$$

<u>Problem 1</u>

Given $a = b$, spot the error in the following 'proof' that $2 = 1$:

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= b(a-b) \\ \cancel{(a-b)}(a+b) &= b\cancel{(a-b)} \\ a+b &= b \\ b+b &= b \\ 2b &= b \\ 2 &= 1 \end{aligned}$$

## 3.2 Solving for a Variable

When a variable is embedded in an equation, we often need to determine its exact value to solve a problem. This is done by manipulating the equation so as to 'get the variable by itself' on the left side or the right side, a process called *solving for a variable*.

Example 1

Solve for $x$:

$$\frac{4x}{5} + \frac{7x}{9} = 4$$

Step 1: Rewrite fractions with a common denominator:

$$\frac{36x}{45} + \frac{35x}{45} = 4$$

Step 2: Combine like terms:

$$\frac{71x}{45} = 4$$

Step 3: Multiply both sides by 45:

$$\frac{71x}{\cancel{45}} \times \cancel{45} = 4 \times 45$$

Step 4: Divide both sides by 71 and simplify:

$$x = \frac{4 \times 45}{71} = \frac{180}{71}$$

Example 2

Find two solutions for $x$:

$$x^2 + 3 = 12$$

Step 1: Subtract 3 from both sides:

$$x^2 = 9$$

Step 2: Take the square root of each side:

$$\sqrt{x^2} = \sqrt{9}$$

Step 3: Simplify to write the result:

$$x = \pm 3$$

Example 3

Solve for $x$:

$$6 = 2\sqrt{x+3}$$

Step 1: Divide by 2 on both sides:

$$3 = \sqrt{x+3}$$

Step 2: Raise each side to the power 2:

$$3^2 = \left(\sqrt{x+3}\right)^2$$

Step 3: Simplify:

$$9 = x + 3$$

Step 4: Subtract 3 from both sides:

$$6 = x$$

Example 4

If $y = 2$, determine all solutions for $x$:

$$x^2 - 10y - 5 = 0$$

Step 1: Add $10y + 5$ to both sides:

$$x^2 = 10y + 5$$

Step 2: Take the square root of both sides:

$$x = \pm\sqrt{10y + 5}$$

Step 3: Substitute $y = 2$ and simplify:

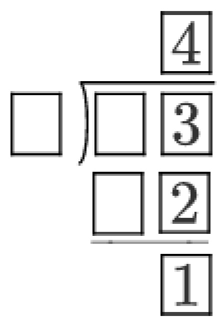
$$x = \pm\sqrt{25}$$

Step 4: Use the a table, calculator, or memory to write the final answer:

$$x = \pm 5$$

Example 5

Shown is a long division with some digits hidden. Which of $(5, 6, 7, 8)$ could represent one of the empty boxes?



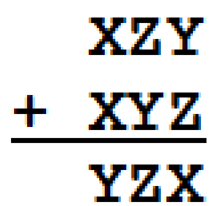

Label the divisor as $x$, where $y$ and $z$ are integer digits. Observe $10x + 3 - (10z + 2) = 1$ to conclude $y = z$. Observe next that

$$\frac{10y + 3}{x} = 4 + \frac{1}{x}\,,$$

which only delivers integer $y = 3$ if $x = 8$.

Example 6

What three digits are represented by $X$, $Y$, and $Z$ in the addition problem shown?




The first column indicates $Y \neq 0$. The second column suggests $Z + Y \geq 10$, and furthermore $10 + 10Z + 10Y = 10Z + 100$, giving $Y = 9$. Thus, the first column is only satisfied by $X = 4$. Finally, the third column tells us $Y + Z = 10 + X$, or $Z = 5$.

Problem 2

At temperature $T$ in Celsius, the speed of sound $v$ in air (in meters per second) is given by

$$v \approx 331.1 \times \sqrt{1 + \frac{T}{273.15}}\,.$$

If the speed of sound in a certain place is measure to be exactly $350\,m/s$, calculate the required temperature.

Problem 3

The area of a circle is $\pi R^2$, where $\pi \approx 3.14$ and $R$ is the radius (center-to-edge distance). If the area of a certain pizza is $100\,\text{in}^2$, calculate the length of a slice (same as $R$).

## 3.3 Systems of Equations

When handed multiple equations containing multiple variables, this is called a system of equations. The number of equations/variables is called the *order* of the system. For example, below we have an order-two system, having two equations and two variables:

$$\begin{aligned} 4x + 2y &= 20 \\ x + 3y &= 15 \end{aligned}$$

There is a reliable technique for solving a system of two equations and two variables. Using the above system as an example, the following procedure generally works:

- Solve either equation for $x$ or $y$, whichever is easier.

$$\begin{aligned} y &= 10 - 2x \\ x + 3y &= 15 \end{aligned}$$

- Substitute the previous result into the unused equation.

$$x + 3\,(10 - 2x) = 15$$

- The resulting equation should contain only one variable:

$$x = 3$$

- Substitute the known variable value into either of the original equations.

$$3 + 3y = 15$$

- Solve for the last unknown:

$$y = 4$$

If the number of equations/variables does not match, the system is either *underdetermined* (not enough information), or *overdetermined* (conflicting information). In either case, there is no clear solution to the system.

Example 7

In standard conditions, water freezes at $0^\circ\ C = 32^\circ\ F$, and it boils at $100^\circ\ C = 212^\circ\ F$. The two temperature scales (Fahrenheit and Centigrade) obey a linear relation

$$T_F = m \cdot T_C + b\,.$$

Determine $m$ and $b$ (as in $y = mx + b$).

Generate two equations and two unknowns:

$$32^\circ F = m \cdot 0 + b \qquad\qquad 212^\circ F = m \cdot 100^\circ C + b$$

First eliminate $b$ to solve for $m$:

$$212^\circ F = m \cdot 100^\circ C + 32^\circ F \qquad \rightarrow \qquad m = \frac{180^\circ F}{100^\circ C}$$

Next solve for $b$, and write the final answer:

$$b = 32^\circ F \qquad \rightarrow \qquad T_F = \frac{9}{5} \cdot T_C + 32^\circ F$$

## 3.4 Linear vs. Nonlinear Equations

A(n) (system of) equation(s) is considered *linear* if all variables $x$, $y$, $z$, etc. have an exponent of exactly *one*. A linear equation should never have terms such $\sqrt{x}$, or $x \cdot y$, or $z^2$, as these would be considered *nonlinear* terms. For example, the set of equations

$$\begin{aligned} 4x + 2y &= 20 \\ x + 3y &= 15 \end{aligned}$$

qualifies as an order-two linear system, and can always be solved by the procedure outlined above.

On the other hand the system

$$\begin{aligned} x^2 + y &= 21 \\ xy &= 20 \end{aligned}$$

contains two nonlinear equations due to the $x^2$- any $xy$-terms. Nonetheless, the solution can be attempted by reducing the system to having one equation and one variable. In this case, substitute $y = 20/x$ into the first equation and simplify to get

$$x^3 - 21x + 20 = 0 ,$$

which is now even *more* nonlinear, but *does* have a solution (it happens to be $x = 4$).

## 3.5 Transcendental Equations

An equation is *transcendental* if the variable(s) being solved for cannot be isolated by the usual tools of algebra. For instance, the equation

$$2^x = x^2$$

qualifies as transcendental, as no ordinary process can be used to isolate $x$. Many transcendental equations can still be solved by analytic means, or when that fails, approximated by computer.

# 4 Logarithm Operator

In the same sense that subtraction is the inverse of addition, and that division is the inverse of multiplication, the exponent operator can also be inverted by an operator called the *logarithm*.

## 4.1 Base-Ten Logarithm

Consider the perfect power-of-ten numbers

$$100 = 10^2 \qquad 1000 = 10^3 \qquad 10000 = 10^4 \; ,$$

and so on. Introducing the *base-ten logarithm*, we can 'solve for' the exponent that brings the base number 10 to higher powers. In practice, this appears as

$$\log_{10}(100) = \log_{10}\left(10^2\right) = 2 \qquad \log_{10}(1000) = \log_{10}\left(10^3\right) = 3 \; ,$$

where we clearly have

$$\log_{10}\left(10^N\right) = N \; .$$

The logarithm operator applies to base numbers apart from 10. Supposing we start with

$$49 = 7^2 \qquad 343 = 7^3 \; ,$$

we can apply a base-7 logarithm to dig out the exponents:

$$\log_7(49) = \log_7\left(7^2\right) = 2 \qquad \log_7(343) = \log_7\left(7^3\right) = 3$$

## 4.2 Arbitrary-Base Logarithm

The whole apparatus generalizes to any base number. If we have a quantity

$$Y = X^N \; ,$$

we may apply a base-$X$ logarithm operation as

$$\log_X(Y) = \log_X\left(X^N\right) = N \; .$$

#### Unit Exponent

For the special case $N = 1$, we establish an important identity:

$$\log_X\left(X^1\right) = \log_X(X) = 1$$

#### Inverse Logarithm

The operation that 'undoes' the logarithm is simply the exponent operator. To see this plainly, note first that $X^A = X^A$ always holds, and then use $A = \log_X\left(X^N\right) = N$ to write

$$X^{\log_X\left(X^N\right)} = X^N \; ,$$

and then substitute $Y = X^N$ to get

$$X^{\log_X(Y)} = Y \; .$$

Evidently, the combination $X^{\log_X(Y)}$ leaves the number $Y$ unchanged.

#### Isolating the Base

Starting from $Y = X^N$, we easily isolate the base number $X$ by raising each side to the power $1/N$:

$$Y^{1/N} = X^{N/N} = X$$

## 4.3 Change of Base

Any quantity $Y = X^N$ has an equivalent representation using a different base number $q$ raised to a different exponent $r$, i.e. $Y = q^r$. To establish this, apply the base-$X$ logarithm to both sides of $X^N$

$$\log_X(Y) = N \ ,$$

so we may eliminate $N$ to write

$$Y = X^{\log_X(Y)} \ .$$

In this form, we see that $X$ can easily be swapped with a different arbitrary number $q$ such that

$$Y = q^{\log_q(Y)} = q^r \ ,$$

where the exponent term is

$$r = \log_q(Y) \ .$$

On the other hand, if $r$ is given $q$ is unknown, we simply use

$$q = Y^{1/r} \ .$$

## 4.4 Identities

### Addition Identity

Suppose we have two logarithmic quantities $\log_X(a)$, $\log_X(b)$ of the same base number $X$, and we are interested in the sum

$$S = \log_X(a) + \log_X(b) \ .$$

To proceed, raise each side as an exponent with $X$ as the base number

$$X^S = X^{\log_X(a)+\log_X(b)} = X^{\log_X(a)} \cdot X^{\log_X(b)} \ ,$$

where the rule $z^{c+d} = z^c z^d$ has been used. Next, note that the right side reduces to the product $ab$, or

$$X^S = ab \ .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^S\right) = S$, and arrive at

$$\log_X(ab) = \log_X(a) + \log_X(b) \ .$$

Of course, this result generalizes to more than two terms:

$$\log_X(abc\cdots) = \log_X(a) + \log_X(b) + \log_X(c) + \cdots \ .$$

As a corollary to the addition identity, we notice that multiplying the product $abc\cdots$ by a factor of one, thus not changing the product, is equivalent to adding zero to the right side, namely

$$\log_X(1) = 0 \ ,$$

for any base number.

### Subtraction Identity

Suppose we have two logarithmic quantities $\log_X(a)$, $\log_X(b)$ of the same base number $X$, and we are interested in the difference

$$D = \log_X(a) - \log_X(b) \ .$$

To proceed, raise each side as an exponent with $X$ as the base number

$$X^D = X^{\log_X(a)-\log_X(b)} = X^{\log_X(a)} \cdot \frac{1}{X^{\log_X(b)}} \ ,$$

where the rule $z^{c-d} = z^c/z^d$ has been used. Next, note that the right side reduces to the ratio $a/b$, or

$$X^D = a/b \ .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^D\right) = D$, and arrive at

$$\log_X\left(\frac{a}{b}\right) = \log_X(a) - \log_X(b) \ .$$

### Product Identity

For a base number $X$, consider a logarithmic quantity $\log_X(a)$, along with an arbitrary number $b$, and let us take the product

$$P = b \cdot \log_X(a) \ .$$

To proceed, raise each side as an exponent with $X$ as the base number

$$X^P = X^{b \cdot \log_x(a)} = \left(X^{\log_x(a)}\right)^b \ ,$$

where the rule $z^{cd} = (z^c)^d$ has been used. Next, note that the right side reduces to the exponent $a^b$, or

$$X^P = a^b \ .$$

Finally, apply the $\log_X$-operator to each side to find $\log_X\left(X^P\right) = P$, and arrive at

$$\log_X\left(a^b\right) = b \cdot \log_X(a) \ .$$

This result is also evident from the addition identity. In the special case $a = b = c \dots$ with $q$ total variables, we find

$$\log_X(a \cdot a \cdot a \cdots) = \log_X(a) + \log_X(a) + \log_X(a) + \cdots \ ,$$

or, more compactly:

$$\log_X(a^q) = q \cdot \log_X(a)$$

# 5 Rate, Interval, and Work

## 5.1 Rates

A *rate* is defined as some quantity, such as a cost, weight, or area, measured against another quantity, such as time or distance. Rates are *always* ratios, containing two pieces of information, hence the similarity in name. Common examples of rates may occur as:

$$14\,\frac{dollar}{hour} \qquad 15\,\frac{meter}{second} \qquad 55\,\frac{pound}{inch^2}$$

In general, we shall denote a rate with the variable letter $R$. For instance, for a car traveling at 40 miles per hour (abbreviated $mi/h$), we may write

$$R = 40\,\frac{mi}{h}\ .$$

## 5.2 Units and Dimension

Quantities that*modify* numbers, such as dollars, seconds, or pounds are called *units*, also called the *dimension* of a number. Unit quantities can be treated identically as variables, and are subject to the same rules. For instance, we may add numbers carrying units by combining like terms as in

$$2\,meter + 3\,meter = 5\,meter\ ,$$

or multiply dimensional numbers such as

$$2\,second \times 3\,second = 6\,second^2\ .$$

As long as we obey the rules for combining fractions, rates can also be combined, as in

$$14\,\frac{dollar}{hour} - 3\,\frac{dollar}{hour} = 11\,\frac{dollar}{hour}\ .$$

To remove any ambiguity, the above calculations are identical to

$$\begin{aligned} 2x + 3x &= 5x \\ 2y \times 3y &= 6y^2 \\ 14z - 3z &= 11z\ , \end{aligned}$$

where $x$ replaces *meter*, $y$ replaces *second*, and $z$ replaces $dollar/hour$. Any quantities that have no units, i.e. plain real numbers, are called *dimensionless.*

## 5.3 Interval

Looking again at the definition of a rate, each has a numerator of a certain dimension (dollars, meters, etc.), along with a denominator of a different dimension (usually *time*, but not always.)

For any given rate $R$, we can create a new number called an *interval*, denoted $I$, having units matching those in the denominator of $R$. For instance, consider an example rate

$$R = 11\,\frac{dollar}{hour}\ .$$

To construct an interval, choose *some* quantity $I$ measured specifically in hours, perhaps

$$I = 3\,hour\ .$$

By computing the product $R \times I$, we find

$$R \times I = 11\,\frac{dollar}{\cancel{hour}} \times 3\,\cancel{hour} = 33\,dollar\ ,$$

which cancels the 'hours' unit altogether, leaving the result in only dollars. That is, the product $R \times I$ can have any numerical value, but must have units matching those in the numerator of $R$.

### 5.4 Work

The product of a rate and an interval (of proper dimension) shall be called *work*, denoted $W$:

$$W = R \times I \,.$$

Reiterating the previous example, we say that a rate of $R = 11\,dollar/hour$ multiplied by an interval of $I = 3\,hour$ amounts to the work value of $W = 33\,dollar$.

The so-called 'work equation' can be solved for $R$, namely

$$R = \frac{W}{I}\,,$$

telling us that a rate can be interpreted as the work value divided by the interval. Similarly, the interval

$$I = \frac{W}{R}$$

is the ratio of the work value to the rate.

Example 1

Joseph works as a journeyman electrician for a pay rate of 30 dollars per hour. After a 9-hour workday, what are his daily earnings?

Step 1: Identify the rate, the interval, and the work value:

$$R = 30\,\frac{dollar}{hour} \qquad I = 9\,hour \qquad W = ?$$

Step 2: Apply the version of the work equation with $W$ as the unknown:

$$W = R \times I = 30\,\frac{dollar}{hour} \times 9\,hour$$

Step 3: Cancel the *hour* units, and simplify the fraction:

$$W = R \times I = 30\,\frac{dollar}{\cancel{hour}} \times 9\,\cancel{hour} = 30 \times 9\,dollar = 270\,dollar$$

Example 2

Joseph keeps working as a journeyman electrician for a pay rate of 30 dollars per hour. After a long workday, his earnings were 330 dollars. How many hours did he work?

Step 1: Identify the rate, the interval, and the work value:

$$R = 30\,\frac{dollar}{hour} \qquad I = ? \qquad W = 330\,dollar$$

Step 2: Apply the version of the work equation with $I$ as the unknown:

$$I = \frac{W}{R} = \frac{330\,dollar}{30\,dollar/hour}$$

Step 3: Cancel the *dollar* units, and simplify the fraction:

$$I = \frac{W}{R} = \frac{330\,\cancel{dollar}}{30\,\cancel{dollar}/hour} = \frac{330}{30}\,hour = 11\,hour$$

Example 3

Joseph agrees to work on a special project for 10 hours. His earnings for the project were 350 dollars. Calculate the pay rate for the special project.

Step 1: Identify the rate, the interval, and the work value:

$$R = ? \qquad I = 10\,hour \qquad W = 350\,dollar$$

Step 2: Apply the version of the work equation with $R$ as the unknown:

$$R = \frac{W}{I} = \frac{350\,dollar}{10\,hour}$$

Step 3: Simplify to write the answer:

$$R = 35\,\frac{dollar}{hour}$$

Example 4

A bee can visit 644 flowers in 7 hours. How many flowers can the bee visit in 9 hours?

Step 1: Set up a work equation containing known information

$$W = R \times I$$
$$644\,flower = R \times 7\,hour$$

Step 2: Solve for the rate $R$:

$$R = \frac{644}{7}\frac{flower}{hour} = 92\,\frac{flower}{hour}$$

Step 3: Step up a work equation with $R$ known and $I = 9\,hour$ to get the result:

$$W = R \times I$$
$$W = 92\,\frac{flower}{\cancel{hour}} \times 9\,\cancel{hour}$$
$$W = 92 \times 9\,flower = 828\,flower$$

**Mixed Units**

Certain problems call for a rate-like quantity to have mixed units. Suppose a certain brand of concentrated iced tea advises that $2\,oz$ of product mixed with $12\,oz$ of water makes the perfect serving. One may wonder, how much of each ingredient is needed to fill a $64\,oz$ container at the proper ratio?

Proceed by writing an equation containing known information:

$$14\,oz\,tea = 2\,oz\,product + 12\,oz\,water\;,$$

and then divide each side side by the number 14 to write a 'unit work' equation:

$$1\,oz\,tea = \frac{1}{7}\,oz\,product + \frac{6}{7}\,oz\,water$$

We can interpret the work value as $1\,oz\,tea$, but the quantity $I \times R$ is tangled up in the sum on the right. The natural choice for the $I$-variable is the number of ounces of each ingredient, so we write the above in equivalent form:

$$(1\,oz)\;\;tea = \left(\frac{1}{7}\,product + \frac{6}{7}\,water\right) \times (1\,oz)$$

Now, it's clear that $1\,oz$ can be replaced with any 'work' quantity, not restricted to ounces. Choosing $64\,oz$, we find

$$(64\,oz)\;\;tea = \left(\frac{1}{7}\,product + \frac{6}{7}\,water\right) \times (64\,oz)$$
$$= 9.14\,oz\,product + 54.86\,oz\,water$$

## 5.5 Adding Rates

Certain situations call for individual rates to be combined into an *effective rate*. That is, if we have a handful of compatible rates $R_1$, $R_2$, etc., the effective rate is

$$R_{eff} = R_1 + R_2 + \cdots ,$$

obeying the same work equation

$$W = R_{eff} \times I .$$

The obvious assumption is that all rates being combined share the same units. It makes no sense (in any context) to add rates with dissimilar units.

Example 5

An inlet pipe can fill a swimming pool in 5 hours, while an outlet pipe can drain the pool in 8 hours. By mistake, a maintenance worker left both pipes open. Will the pool overflow?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R = R_{eff} &= R_{fill} + R_{drain} \\ I &= ? \\ W &= 1\,pool \end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 5, and 8.)

$$\begin{aligned} R_{eff} &= \frac{1\,pool}{5\,hour} - \frac{1\,pool}{8\,hour} \\ R_{eff} &= \frac{8\,pool}{40\,hour} - \frac{5\,pool}{40\,hour} \\ R_{eff} &= \frac{3\,pool}{40\,hour} = 0.075\,\frac{pool}{hour} \end{aligned}$$

Step 3: Observe that $R_{eff}$ came out positive, meaning the pool will overflow.

Example 6

A cold water faucet can fill a bathtub in 12 minutes, and a hot water faucet can fill the bathtub in 18 minutes. The drain can empty the bathtub in 24 minutes. If both faucets are on and the drain is open, how long would it take to fill the bathtub?

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned} R = R_{eff} &= R_{hot} + R_{cold} + R_{drain} \\ I &= ? \\ W &= 1\,tub \end{aligned}$$

Step 2: Determine the effective rate: (Hint: Find the LCM of 12, 18, and 24.)

$$\begin{aligned} R_{eff} &= \frac{1\,tub}{12\,minute} + \frac{1\,tub}{18\,minute} - \frac{1\,tub}{24\,minute} \\ R_{eff} &= \frac{6\,tub}{72\,minute} + \frac{4\,tub}{72\,minute} - \frac{3\,tub}{24\,minute} \\ R_{eff} &= \frac{7\,tub}{72\,minute} \end{aligned}$$

Step 3: Apply the version of the work equation with $I$ as the unknown:

$$I = \frac{W}{R_{eff}} = W \times \frac{1}{R_{eff}} = 1\,tub \times \frac{72\,minute}{7\,tub}$$

Step 4: Cancel the *tub* units, and simplify the fraction:

$$I = 1\,\cancel{tub} \times \frac{72\,minute}{7\,\cancel{tub}} = \frac{72}{7}\,minute = 10.29\,minute$$

Example 7

It takes Tom 4 hours to build a fence. If he hires Jack to help him, together they can do the job in just 3 hours. If Jack built the same fence alone, how long would it take him?

Step 1: Discern the following equations from the problem statement:

$$\begin{aligned}
1\,fence &= R_{Tom} \times 4\,hour \\
1\,fence &= (R_{Tom} + R_{Jack}) \times 3\,hour \\
1\,fence &= R_{Jack} \times I_{Jack}
\end{aligned}$$

Step 2: Solve the middle equation for $R_{Jack}$:

$$R_{Jack} = \frac{1\,fence}{3\,hour} - R_{Tom} = \frac{1\,fence}{3\,hour} - \frac{1\,fence}{4\,hour} = \frac{1\,fence}{12\,hour}$$

Step 3: Solve the third equation for $I_{Jack}$:

$$I_{Jack} = \frac{1\,fence}{R_{Jack}} = 1\,\cancel{fence} \times \frac{12\,hour}{1\,\cancel{fence}} = 12\,hour$$

Example 8

A woodworking shop is open eight hours per day. Tony can build three birdhouses in the eight-hour workday. Joe is new, and accidentally destroys one birdhouse every four hours. If the shop hires a third builder for six hours per day, how many birdhouses must he build during his shift in order for the shop to produce five birdhouses in an eight-hour day? Hint: Let $x$ be the number of birdhouses the new worker builds per six hours.

Step 1: Identify the rate, the interval, and the work value:

$$\begin{aligned}
R &= R_{eff} = R_{Tony} + R_{Joe} + \frac{x\,house}{6\,hour} \\
I &= 8\,hour \\
W &= 5\,house
\end{aligned}$$

Step 2: Write the work equation using the information provided and simplify:

$$\begin{aligned}
W &= R_{eff} \times I \\
5\,house &= \left(\frac{3\,house}{8\,hour} - \frac{1\,house}{4\,hour} + \frac{x\,house}{6\,hour}\right) \times 8\,hour \\
5\,\cancel{house} &= \left(\frac{3\,\cancel{house}}{8\,\cancel{hour}} - \frac{1\,\cancel{house}}{4\,\cancel{hour}} + \frac{x\,\cancel{house}}{6\,\cancel{hour}}\right) \times 8\,\cancel{hour} \\
5 &= \left(\frac{3}{8} - \frac{1}{4} + \frac{x}{6}\right) \times 8
\end{aligned}$$

Step 3: Solve the above for $x$:

$$x = 6 \cdot \left(\frac{5}{8} - \frac{3}{8} + \frac{1}{4}\right) = 3$$

## 5.6 Multiplying Rates

Quantities carrying dimension can be converted from one set of units to another using *dimensional analysis*, also known as *unit conversion*. One may convert inches to centimeters, or seconds to hours, or liters to gallons because these conversions preserve the fundamental dimension of the quantity. On the other hand, trying to convert microseconds to pounds is a nonsensical attempt to equate a time to a weight. Common unit equivalanes are given in the following table:

$$\begin{aligned}
1\text{ hour} &= 60\text{ minute} = 3600\text{ second}\\
1\text{ inch} &= 2.54\text{ centimeter}\\
1\text{ foot} &= 12\text{ inch} = (1/3)\text{ yard}\\
1\text{ meter} &= 1.094\text{ yard}\\
1\text{ kilometer} &= 1000\text{ meter}\\
1\text{ mile} &= 5280\text{ foot} = 1.609\text{ kilometer}\\
1\text{ gallon} &= 128\text{ fluid ounce} = 3.785\text{ liter}\\
1\text{ millileter} &= (1/1000)\text{ liter}\\
1\ cm^3 &= 3.785\text{ milliliter}\\
1\ kg &= 2.2\text{ pound (only at sea level)}\\
360^\circ &= 2\pi\text{ radian}
\end{aligned}$$

**Unit Conversion**

Starting with any entry from the table above, for instance, $1\ in = 2.54\ cm$, note this can be converted into a *unit rate* by dividing through by either side:

$$1 = \frac{2.54\ cm}{1\ in} \qquad\qquad 1 = \frac{1\ in}{2.54\ cm}$$

In each case, we have a dimensionless number one on the left side of the equation. On the right, we have a rate with definite units.

By now, we know that multiplying any quantity by a factor of one leaves the quantity unchanged, so let's see what happens if we multiply some number of inches, say twelve, by a carefully chosen factor of one:

$$12\ in = 12\ in \times (1) = 12\,\cancel{in} \times \left(\frac{2.54\ cm}{1\,\cancel{in}}\right) = 12 \times 2.54\ cm = 30.48\ cm$$

The new result must be the same as $12\ in$, but the 'inch' unit has canceled, replaced by centimeters. This operation must work in reverse. Start with with $30.48\ cm$, and multiply by another carefully-chosen factor of one:

$$30.48\ cm = 30.48\ cm \times (1) = 30.48\,\cancel{cm} \times \left(\frac{1\ in}{2.54\,\cancel{cm}}\right) = 12\ in$$

**Areas and Volumes**

Of course, $1^2$ is still 1, so we multiply by $1^2$ or $1^3$, etc., for converting areas, volumes, and so on. For instance:

$$25\ in^2 = 25\ in^2 \times \left(1^2\right) = 25\,\cancel{in^2}\left(\frac{2.54^2\ cm^2}{1^2\,\cancel{in^2}}\right) = 25 \cdot 2.54^2\ cm^2 \approx 161\ cm^2$$

---

Example 9

The speed limit on a Canadian highway is 120 kilometers per hour. Convert this speed to miles per hour.

$$120\,\frac{km}{hr} = 120\frac{\cancel{km}}{hr}\left(\frac{1\ mi}{1.609\,\cancel{km}}\right) = \frac{120}{1.609}\,\frac{mi}{hr} = 74.6\ mph$$

Example 10

Convert 70 meters per second to milers per hour.

Multiply by factors of 1 until the correct units come out:

$$\begin{aligned} 70\,m/s =& 70\left(\frac{\cancel{m}}{\cancel{s}}\right)\left(\frac{1\,\cancel{km}}{1000\,\cancel{m}}\right)\left(\frac{1\,mi}{1.609\,\cancel{km}}\right)\left(\frac{3600\,\cancel{s}}{1\,hr}\right) \\ =& \frac{70\times 3600}{1000\times 1.609}\left(\frac{mi}{hr}\right) = 156.6\,mph \end{aligned}$$

Example 11

Convert 4500 gallons to cubic meters.

Multiply by factors of 1 and $1^3$ until the correct units come out:

$$\begin{aligned} 4500\,gal =& 4500\,\cancel{gal}\left(\frac{3.785\,\cancel{liter}}{1\,\cancel{gal}}\right)\left(\frac{1000\,\cancel{mL}}{1\,\cancel{liter}}\right)\left(\frac{1\,\cancel{cm^3}}{1\,\cancel{mL}}\right)\left(\frac{1\,m}{100\,\cancel{cm}}\right)^3 \\ =& \frac{4500\times 3.785\times 1000}{100^3}\,m^3 = 17.03\,m^3 \end{aligned}$$

Example 12

Three gallons of paint are uniformly spread over a wall having area 600 $ft^2$. Calculate the thickness of the paint in millimeters.

Need two ways to express the volume of paint. We have $V = 3.00gal$, or equivalently, $V = \text{thickness}\times\text{area}$. Eliminating $V$ and solving for thickness $T$, find:

$$\begin{aligned} T =& \frac{3.00\,gal}{600\,ft^2} = \frac{3.00\,gal}{600\,ft^2}\left(\frac{3.785\,L}{1\,gal}\right)\left(\frac{1000\,mL}{1\,L}\right)\left(\frac{1\,cm^3}{1\,mL}\right) = 18.925\left(\frac{cm^3}{ft^2}\right) \\ =& 18.925\left(\frac{cm^3}{ft^2}\right)\left(\frac{1\,ft}{12\,in}\right)^2\left(\frac{1\,in}{2.54\,cm}\right)^2\left(\frac{10\,mm}{1\,cm}\right) = 0.204\,mm \end{aligned}$$

# 6 Scientific Notation

In *scientific notation*, any number $X$ should be reported in the form

$$X = A \times 10^N ,$$

where $A$ is a number confined to $|A| < 10$, and $N$ is an integer called the *order of magnitude*. Any quantity $X$ can often be quickly converted to scientific notation by multiplying and dividing by the required factor of ten, provided ten is its base number. On the other hand, if we are given a number such as $55^{99}$, it's not trivial to work out the same number in scientific notation by inspection.

To proceed, let us recast the variable $A = 10^C$ or $C = \log_{10}(A)$. With this, the variable $C$ is confined to the interval $0 < C < 1$, and the above becomes

$$X = 10^C \times 10^N = 10^{C+N} .$$

Apply the base-ten logarithm operator to each to side to get

$$\log_{10}(X) = \log_{10}(C + N) = C + N .$$

In practice, this means we take the result $\log_{10}(X)$, assigning the fractional component to $C$, and the integer component to $N$.

### Integer Operator

Let us define an *integer operator* that simply 'shaves off' any decimals from a number. For instance if we start with $N = 5.74$, then $\text{int}(N) = \text{int}(5.74) = 7$. The integer operator does *no* rounding. That is, $\text{int}(3.99999) = 3$. Using the integer operator, we can solve for $C$ and $N$ in the above, giving:

$$N = \text{int}(\log_{10}(X))$$

$$C = \log_{10}(X) - N$$
$$C = \log_{10}(X) - \text{int}(\log_{10}(X))$$

---

Now, we can write any number $X$ in scientific notation:

$$X = A \times 10^N$$
$$X = 10^{\log_{10}(X) - \text{int}(\log_{10}(X))} \times 10^{\text{int}(\log_{10}(X))}$$

To tighten things up, let us substitute

$$Y = \log_{10}(X) = C + N ,$$

so we finally have

$$X = 10^{Y - \text{int}(Y)} \times 10^{\text{int}(Y)} = 10^C \times 10^N .$$

Example 1

Convert the number $X = 55^{99}$ to scientific notation.

Step 1: Compute $\log_{10}\left(55^{99}\right)$:

$$\log_{10}\left(55^{99}\right) = 99 \log_{10}(55) = 99 \times 1.7404 = 172.296$$

Step 2: Identify $Y = 172.296$, so then:

$$C = 0.296 \qquad N = 172$$

Step 3: Assemble the result using $X = 10^C \times 10^N$:

$$55^{99} = 10^{0.296} \times 10^{172} = 1.977 \times 10^{172}$$

## 6.1 Order of Magnitude

Every number carries an order of magnitude, which tells us the 'size of the number in how many powers of ten are present. For instance, the number four thousand is written 4000, equivalently written $4 \times 10^3$, where 3 is the order of magnitude. The order of magnitude can also be negative for numbers smaller than one. For instance, nine microseconds is equal to 0.000009 *sec*, or equivalently, $9.0 \times 10^{-6}$ *sec*.

The advantage to scientific notation is we don't have to waste time writing zeros. Many of the powers of ten have a special name, listed below from small to large. Common entries are emphasized in bold.

| **Power** | **Prefix** | **Abbreviation** |
|---|---|---|
| $10^{-24}$ | yocto | y |
| $10^{-21}$ | zepto | z |
| $10^{-18}$ | atto | a |
| $10^{-15}$ | femto | f |
| $10^{-12}$ | pico | p |
| $10^{-9}$ | **nano** | **n** |
| $10^{-6}$ | **micro** | $\mu$ **(Greek 'myu')** |
| $10^{-3}$ | **milli** | **m** |
| $10^{-2}$ | **centi** | **c** |
| $10^{-1}$ | **deci** | **m** |
| $10^{3}$ | **kilo** | **k** |
| $10^{6}$ | **mega** | **M** |
| $10^{9}$ | **giga** | **G** |
| $10^{12}$ | **tera** | **T** |
| $10^{15}$ | peta | P |
| $10^{18}$ | exa | E |
| $10^{21}$ | zetta | Z |
| $10^{24}$ | yotta | Y |

| **Order** | **Example** |
|---|---|
| $10^{-30}$ | Mass of an electron in kilograms. |
| $10^{-27}$ | Mass of a hydrogen atom in kilograms. |
| $10^{-15}$ | Diameter of a hydrogen atom in meters. |
| $10^{-11}$ | Radius of a hydrogen atom in meters. |
| $10^{-9}$ | Volume of a flea in cubic meters. |
| $10^{-6}$ | Lifetime of a muon particle in seconds. |
| $10^{-3}$ | Oscillation period of a guitar string in seconds. |
| $10^{0}$ | Height of a human child in meters. |
| $10^{3}$ | Mass of a car in kilograms. |
| $10^{6}$ | Radius of Earth in meters. |
| $10^{9}$ | Duration of a century in seconds. |
| $10^{12}$ | Number of cells in human body. |
| $10^{16}$ | Radius of the solar system in meters. |
| $10^{27}$ | Mass of Jupiter in kilograms. |
| $10^{30}$ | Mass of the Sun in kilograms. |

Example 2

Convert the Massachusetts highway speed limit of 65 *mph* into lightyears per second. (A lightyear is a *distance*, equal to the distance that a beam of light will travel in one year.)

Multiply by factors of 1 and $1^3$ until the correct units come out, using the information below:

$$c = \text{speed of light} = 3.00 \times 10^8 \; m/s$$
$$1 \text{ lightyear} = c \; \times 365.25 \text{ day}$$

$$\begin{aligned}65\,mph &= 65\left(\frac{\cancel{mi}}{\cancel{hr}}\right)\left(\frac{\cancel{hr}}{3600\,\cancel{sec}}\right)\left(\frac{1609\,\cancel{m}}{1\,\cancel{mi}}\right)\\ &\times\left(\frac{c\,\cancel{sec}}{3.00\times10^{8}\,\cancel{m}}\right)\left(\frac{365.25\,day}{365.25\,\cancel{day}}\right)\left(\frac{\cancel{day}}{24\times3600\,sec}\right)\\ &= \frac{65\times1609}{3600^{2}\times3.00\times10^{8}\times365.25}\,ly/s\\ &= 3.07\times10^{-15}\,ly/s\end{aligned}$$

## 6.2 Measurement and Significant Figures

### Insignificant Figures

In any laboratory, engineering, or other real-world setting, a number's precision is subject to the methods or apparatus used to attain that number. For instance, a length measured by a ruler might be precise to the one millimeter range, which means two things: (i) Any object smaller than one millimeter won't be 'visible' to the ruler. (ii) Any length measured by the ruler cannot be known to greater precision than 1 $mm$. To illustrate, suppose your lab assistant measures his shoelace with the above-mentioned ruler and reports the result $L = 26.5912\,cm$. You should immediately round this to $L = 26.6\,cm$, because the 'junk' digits 912 occupy decimal places that are smaller than the precision of the ruler.

A way to justify throwing away the 'junk' digits is to consider what happens when the measurement is repeated. While it's likely that the length $L$ will be close to 26.6 $cm$ again, it's unlikely that the digits 912 would follow a second time. In fact, we aren't confident that the shoelace really is 26.6 $cm$ until the measurement is repeated.

### Significant Figures

The precision in a number is represented by how many decimal places are *confidently* known, where each decimal place is called a *significant figure* or *significant digit*. Supposing a quantity is handed to us *without* any junk digits, there are several rules for counting the number of significant figures. These are:

1. Zeros to the left of the number are not significant. (For example, \$00237.66 is the same as \$237.66.)
2. Zeros to the right of the number are significant. (More zeros mean more precision.)
3. All nonzero digits in the number are significant.
4. Any zeros between significant digits are significant.

The table below lists a few numbers with varying significant figures as expressed in scientific notation:

| **Quantity** | **Sig. Figs.** | **Sci. Not.** |
|---|---|---|
| 7 | 1 | $7\times10^{0}$ |
| $10^{-3}$ | 1 | $1\times10^{-3}$ |
| $3.0\times10^{8}$ | 2 | $3.0\times10^{8}$ |
| 000.0000010 | 2 | $1.0\times10^{-6}$ |
| $2.998\times10^{8}$ | 4 | $2.998\times10^{8}$ |
| 0623.53 | 5 | $6.2353\times10^{2}$ |
| 12.340 | 5 | $1.2340\times10^{1}$ |
| 90000 | 5 | $9.0000\times10^{4}$ |
| 90000.00 | 7 | $9.000000\times10^{4}$ |

Sometimes the precision of a number is not obvious as given. If someone says they earned \$4500 last month, does this precisely mean \$4500.00 dollars and zero cents, or was \$4500 rounded from some nearby number? Should we trust only the 4 and the 5? This issue is resolved by the following convention: *When the precision is ambiguous, do intermediate calculations to four significant figures, and report answers to three significant figures.* For the example on hand, we can agree that an honest representation of \$4500 is $\$4.50\times10^{3}$.

## 6.3 Propagation of Significant Figures

When two numbers of finite precision are combined, the natural question arises: what's the precision of the result? The answer depends on the operation being performed, but there are two main ways to proceed.

**Multiplication, Division, Roots, Powers**

Be pessimistic: In your calculation, seek the quantity with the lowest precision, and adopt *that* precision for the final answer, as illustrated in the following examples:

$$
\begin{aligned}
2.345 \times 5.4 &\approx 13 \\
55555 \times 0.0100 &\approx 556 \\
\sqrt{\pi \times 2.56 \boldsymbol{\nabla} \cdot 5.567} &\approx 1.20 \\
70 \times 23.846251 &\approx 1.7 \times 10^3 \\
\left[(2/3) \times \left(6.984 \times 10^{-8}\right)\right]^2 &\approx 2.170 \times 10^{-15}
\end{aligned}
$$

Notice that purely mathematical factors like 2 or $\pi = 3.14159265358979323846264\ldots$ or $2/3 = 0.666\bar{6}$ are more-or-less infinitely precise numbers, causing no penalty in the overall precision of a calculation. Calculators assume all numbers are pure, so watch out!

**Addition and Subtraction**

If the quantities being added or subtracted do not involve any decimals, you're off the hook, and may proceed as usual. When there *are* decimals though, be pessimistic again: the number of decimal places carried by the answer is determined by the contributing factor with the *least* number of decimal places. See the examples that follow.

$$
\begin{aligned}
12551 + 3114 &\approx 15665 \\
1000 - 3 &\approx 997 \\
2.345 + 5.4 &\approx 7.7 \\
100.2 + 15.438 &\approx 115.6 \\
1000 - 0.3 &\approx 1000
\end{aligned}
$$

Example 3

The speed of light in vacuum is $3.00 \times 10^8\ m/s$. Convert this speed to miles per second and also miles per hour.

$$
\begin{aligned}
3.00 \times 10^8 \, \frac{m}{s} &= 3.00 \times 10^8 \, \frac{\cancel{m}}{s} \left(\frac{100 \, \cancel{cm}}{\cancel{m}}\right) \left(\frac{1 \, \cancel{in}}{2.54 \, \cancel{cm}}\right) \left(\frac{1 \, \cancel{ft}}{12 \, \cancel{in}}\right) \left(\frac{1 \, mi}{5280 \, \cancel{ft}}\right) \\
&= 1.86 \times 10^5 \, \frac{mi}{s} \\
&= 1.86 \times 10^5 \, \frac{mi}{\cancel{sec}} \left(\frac{3600 \, \cancel{sec}}{1 \, hour}\right) = 6.71 \times 10^8 \, mph
\end{aligned}
$$

Example 4

Estimate your age in seconds by starting with years (rounded to the nearest tenth) and multiplying that number by factors of 1 until you have the answer.

Assuming your age is twenty years exactly:

$$
Age = 20.0 \, yr = 20.0 \, \cancel{yr} \left(\frac{365.25 \, \cancel{day}}{\cancel{yr}}\right) \left(\frac{24 \, \cancel{hr}}{1 \, \cancel{day}}\right) \left(\frac{3600 \, sec}{1 \, \cancel{hr}}\right) = 6.31 \times 10^8 \, s
$$

Example 5

The thickness of paper is closest to (choose one):

$10^{-4}\,m$ $\qquad 10^{-1}\,m$ $\qquad 10^{1}\,cm$ $\qquad 10^{-7}\,m$

Paper thickness is closest to one tenth of a millimeter, or $10^{-4}\,m$. The rest are way off.

Example 6

The length of a football field (100 yards) is closest to (choose two):

$10^{4}\,m$ $\qquad 10^{-1}\,km$ $\qquad 10^{5}\,mm$ $\qquad 10^{3}\,foot$

The two reasonable answers (are actually equivalent):

$$10^{-1}\,km = 10^{6}\,mm$$

Example 7

The radius of Earth is about $10^6$ meters. If the volume of a sphere is given by $V = (4/3)\pi R^3$, roughly estimate Earth's volume in cubic meters.

Earth is actually a bit bigger than this, but the rough estimation is still valid: $V \approx (4/3)\pi(10^6\,m)^3 \approx 4\left(10^6\,m\right)^3 \approx 10^{18}\,m^3$.

Example 8

The mass of Earth is approximately $M = 5.972 \times 10^{24}\,kg$. Using radius $R = 6371\,km$, calculate the average density (total mass over total volume) of Earth in grams per cubic centimeter.

Density is mass over volume:

$$\begin{aligned}\rho = \frac{M}{V} = \frac{M}{(4/3)\,\pi R^3} &= \frac{5.972 \times 10^{24}\,kg}{(4/3)\,\pi\,(6371\,km)^3} = 5.513 \times 10^{12}\,\frac{kg}{km^3} \\ &= 5.513 \times 10^{12}\frac{kg}{km^3}\left(\frac{1\,km}{1000\,m}\right)^3\left(\frac{1\,m}{100\,cm}\right)^3\left(\frac{1000\,g}{kg}\right) = 5.513\,\frac{g}{cm^3}\end{aligned}$$

Example 9

At sea level, a 2.205 pound rock weighs the same as a 1 kilogram object. On Mt. Everest, a 2.199 pound rock weighs the same as a 1 kilogram object. What is the weight in pounds of an 80 $kg$ person at each location?

At sea level, $80\,kg \sim 176.4\,lb$. On Mt. Everest, $80\,kg \sim 175.9\,lb$. There's about a half-pound difference.

Example 10

Suppose a very wide sheet of paper has thickness $0.1\,mm$. If the paper is folded once, the thickness doubles to $0.2\,mm$. If the resulting sheet is folded again, the total thickness doubles again to $0.4\,mm$. A third fold makes $0.8\,mm$, a fourth fold makes $1.6\,mm$, and so on. How thick is the paper after 20 folds?

If the number of folds is $N$, the total thickness $T$ is

$$T = 2^N \times 0.1\,mm\,.$$

Inserting $N = 20$, we have

$$\begin{aligned}T &= 2^{20} \times 0.1\,mm \\ &= 1048576 \times 0.1\,mm \\ &= 104.9\,m\end{aligned}$$

Example 11

If a very wide sheet of paper with thickness $0.1mm$ is folded $N$ times, determine $N$ such that the resultant thickness is $10^{16}\ m$, the radius of the solar system.

Starting with

$$T = 2^N \times 0.1\ mm\ ,$$

solve for $N$ to get

$$N = \log_2 \left( \frac{10^{16}\ m}{0.1\ mm} \right) = \log_2 \left( \frac{10^{16}\ m}{10^{-3}\ m} \right) = \log_2 \left( 10^{19} \right) \approx 63$$

# 7 Graphing

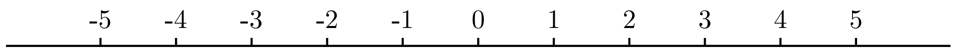
Graphing is the 'art' of projecting mathematical information onto a visual or other sensory medium.

## 7.1 One-Dimensional Graphs

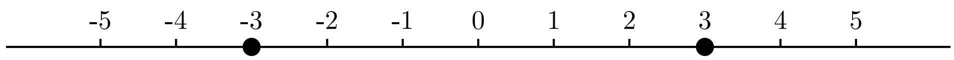
By now, we are familiar with solving a single equation containing a single variable. Taking the example equation

$$x^2 = 9\,,$$

we may easily verify two solutions $x = 3$ and $x = -3$. In order to *visualize* these solutions, we prepare a blank number line as follows:




On the number line, the solutions $x = \pm 3$ are easily marked, or *plotted* at their respective locations:



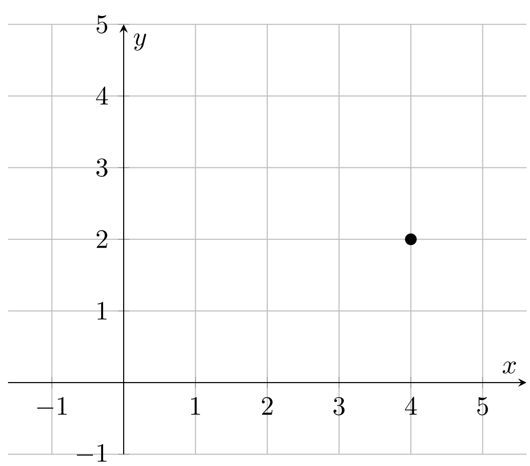

Used for this purpose, the real number line is a *one-dimensional* graph. Any particular location on the graph is called a *point*.

## 7.2 Two-Dimensional Graphs

We can also solve multiple equations containing multiple variables. For instance, the order-two system

$$\begin{aligned} 2x + 3y &= 14 \\ x - y &= 2 \end{aligned}$$

is only satisfied by $x = 4$, $y = 2$. To visualize these two solutions though, it only makes sense to have a separate graph for the $x$-solution, and another graph for the $y$-solution. To do so, we draw a second number line perpendicular to the familiar one-dimensional number line. Labeling each axis as $x$ and $y$ respectively, the solution to the above system appears as a single point in the following graph:




In the above, the marked location $x = 2$, $y = 4$ denotes the solution to the solution to the system of equations.

### 7.3 Ordered-Pair Solutions

Suppose you are given *one* equation containing *two* variables, such as

$$4y - 3x = -2 \, .$$

To 'solve' the (underdetermined) equation, surely we can solve for $x$, but that still leaves $y$ unknown (and vice-versa). Doing so *anyway*, we have

$$x = \frac{4y+2}{3} \, .$$

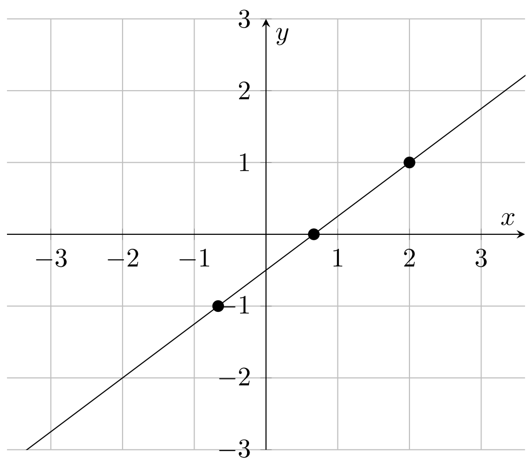
Since $x$ and $y$ are both variable, we have freedom to play with *one* of their values, which will strictly determine the other. For instance, setting $y = 0$, we quickly find that $x = 2/3$. Choosing instead $y = -1$, the corresponding $x$-value must be $x = -2/3$. Similarly, $y = 1$ gives $x = 2$, and so on.

The groups of numbers $(x = 2/3, y = 0)$, $(x = -2/3, y = -1)$, $(x = 2, y = 1)$, etc. are called *ordered pair solutions* to the equation. Often, the $x =$, $y =$ nomenclature is omitted by convention unless explicitly needed. Apart from

$$(2/3, 0) \qquad (-2/3, -1) \qquad (2, 1) \; ,$$

it turns out there are an infinite number of ordered-pair solutions to the equation $4y - 3x = -2$. For any $x$ we could possibly choose, there is some corresponding $y$ that satisfies the equation. This is not to say that *any* pair of numbers is a solution to the equation. For instance, the choice $x = 0$, $y = 0$ leads to the untrue statement $0 = 2/3$.

Now we shall plot the known ordered-pair solutions to $4y-3x = -2$. Plotted on a two-dimensional graph, the three points seem to adhere to a straight line. Extending that line through the points and beyond, the result comes out as follows:




The straight line connecting the known points in fact runs through *all* solutions to the equation $4y-3x = -2$. This is precisely why such an equation is classified as 'linear', as its solution 'looks like' a straight line on a two-dimensional graph.

---

For another example, consider the equation

$$x^2 + y = 3 \, .$$

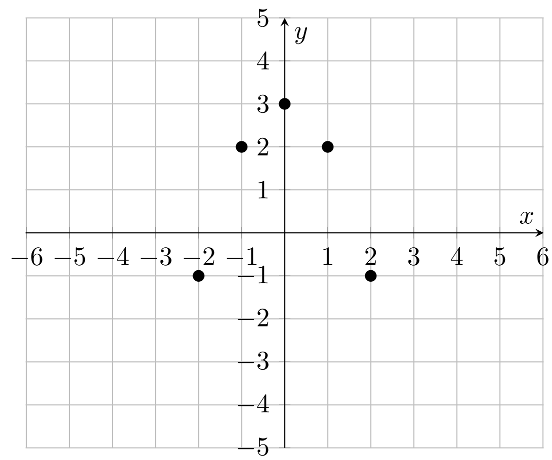
Solutions are most easily found by solving for $y$:

$$y = 3 - x^2$$

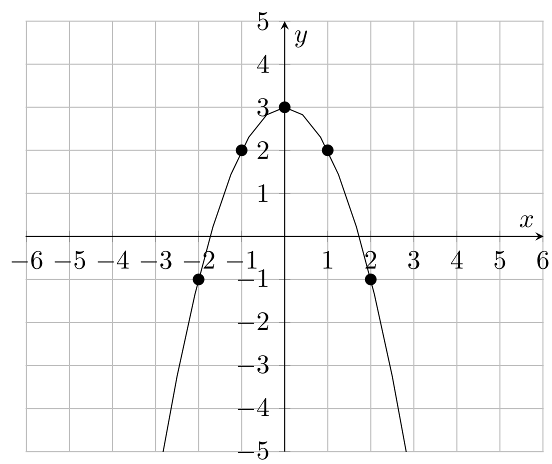
Choosing a few $x$-values, we easily generate the following ordered-pairs:

$$(-2, -1) \qquad (-1, 2) \qquad (0, 3) \qquad (1, 2) \qquad (2, -1)$$

Plotting these on a two-dimensional graph, we find:




Clearly, a straight line cannot be used to connect the points as drawn, thus $x^2 + y = 3$ is a nonlinear equation. The shape that *does* connect the points is called a *parabola*. Indeed, solutions to the equation on hand are 'parabolic' in form as shown:

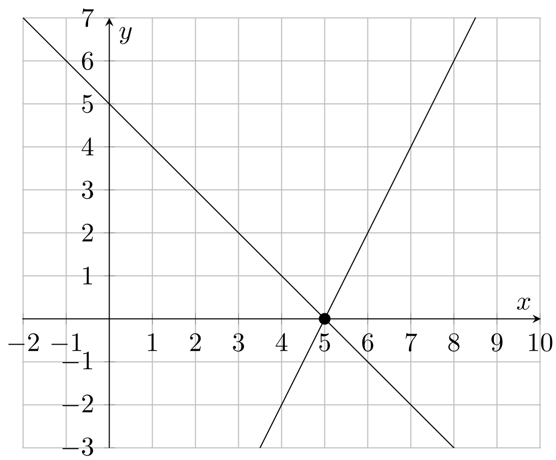



## 7.4 Graphical Solution Methods

Consider the system of linear equations of two equations and two unknowns:

$$x + y - 5 = 0 \qquad\qquad y - 2x + 10 = 0$$

While we may proceed analytically to solve the system, here we instead develop a 'shortcut' that involves plotting solutions to each equation on the same graph as follows:



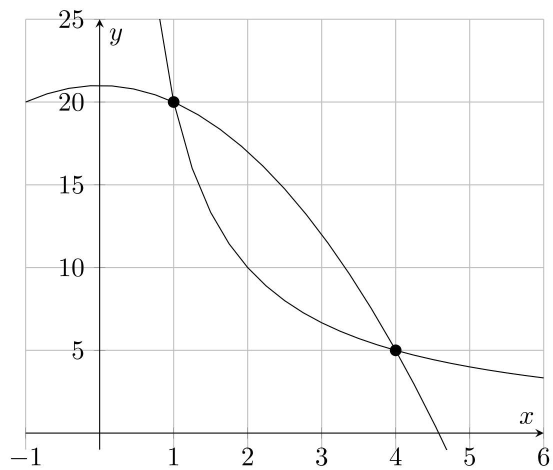

The two lines intersect at the point $(5, 0)$, which is precisely the same result one would attain analytically. Indeed, the original set of equations is satisfied only by $x = 5$, $y = 0$.

---

The graphical method also helps to solve nonlinear equations. Taking the system

$$x^2 + y = 21$$
$$xy = 20\ ,$$

one quickly notices that attaining an analytic solution becomes a lengthy endeavor. Supposing we generate ordered-pair solutions to each equation, their combined plot appears as:




In the above, we observe the the intersection of a parabola, along with a special case of the general shape $xy =$ constant, called a *hyperbola*. Evidently, the system of equations is solved by two ordered pairs

$$(1, 20) \qquad\qquad (4, 5)\ .$$

In general, the number of solutions to a nonlinear system is not obvious. It's possible for one, two, zero, or even infinite solutions to exist.

---

Finally, we can apply the graphical method to a *single* equation, or even a transcendental equation. To illustrate, consider the nonlinear equation

$$x^2 - x - 6 = 0\ ,$$

which in general may or may not have valid solutions. To proceed, move the $x^2$-term to the other side of the equation, giving

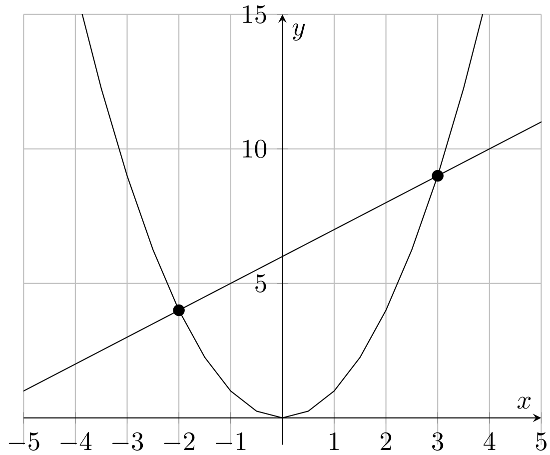
$$x^2 = x + 6 .$$

Since the left side and the right side are equal, we are allowed to break the equation into two equal parts:

$$y = x^2 \qquad\qquad y = x + 6$$

If in doubt, simply eliminate $y$ to restore the original equation.

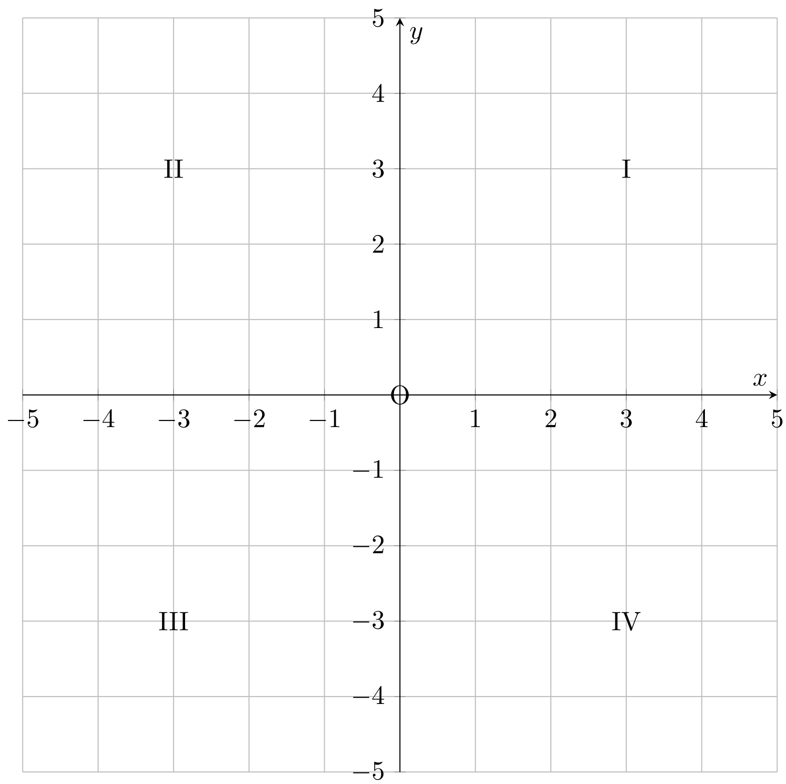
Next, plot the ordered-pair solutions to each (individual) equation, generating the following graph:




The two plots intersect at $x = -2$, and also at $x = 3$. Evidently, the equation $x^2 - x - 6 = 0$ has two solutions at these points.

## 7.5 Cartesian Plane

The two-dimensional grid formed by perpendicular number lines is called the *Cartesian plane*. By convention, the number lines intersect at position $x = 0, y = 0$ as shown:




**Quadrants**

- The quarter-plane with $x > 0$, $y > 0$ is the *first quadrant* (I).
- The quarter-plane with $x < 0$, $y > 0$ is the *second quadrant* (II).
- The quarter-plane with $x < 0$, $y < 0$ is the *third quadrant* (III).
- The quarter-plane with $x > 0$, $y < 0$ is the *fourth quadrant* (IV).

**Terminology**

- The horizontal number line is called the *x-axis*.
- The vertical number line is called the *y-axis*.
- The point $x = 0$, $y = 0$ is called the *origin*.
- The Cartesian plane extends to $x = \pm\infty$ and $y = \pm\infty$.

# 8 Straight Line Analysis

## 8.1 General Equation

Equations containing only linear terms, i.e. nothing like $x^2$, $\sqrt{y}$, etc., follow a general form

$$A + Bx + Cy + Dz + \cdots = 0\,,$$

where $A$, $B$, $C$, etc., are numerical coefficients. Taking the special case of two dimensions, the above reduces to

$$A + Bx + Cy = 0\,,$$

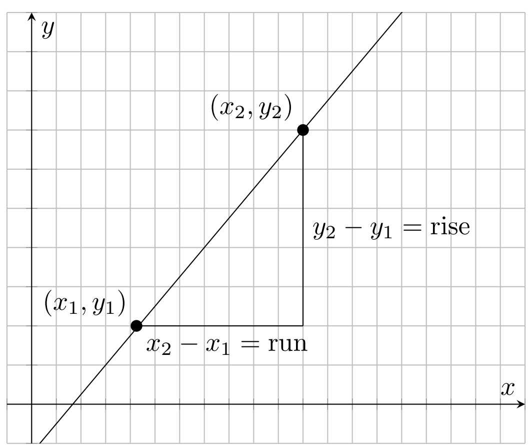
which we call the 'equation of a line' in the Cartesian plane. The position and orientation of the line is contained in the coefficients $A$, $B$, and $C$. All ordered pairs $(x, y)$ that solve the above equation are points on the line.

## 8.2 Slope

The formal name for the 'orientation' of a line is called the *slope*. Taking any two ordered-pair solutions on the line, call them $(x_1, y_1)$, $(x_2, y_2)$, the slope of the line is defined as

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

That is, the slope of a straight line is the change in $y$ divided by the change in $x$ between any two points as shown:




A line with zero slope is strictly horizontal, as there is no change in $y$ for any given $x$. When the slope is positive, the height $y$ of the line is increasing for increasing $x$. By contrast, a negative slope means the height $y$ is ramping downward for increasing $x$. Any vertical line has *infinite* slope, as all possible $y$-values correspond to the same $x$, and the slope equation gives division by zero.

### Slope Analysis

Suppose we are given two points $(x_1, y_1)$, $(x_2, y_2)$ in the Cartesian plane that are known to fall onto a straight line. Let us take the general equation of a line, and apply it each point:

$$A + Bx_1 + Cy_1 = 0 \qquad\qquad A + Bx_2 + Cy_2 = 0$$

What we have is two equations and three unknowns, which is an overdetermined system - there is one variable too many. To proceed, divide each equation by $C$ to get

$$\frac{A}{C} + \frac{B}{C}\,x_1 + y_1 = 0 \qquad\qquad \frac{A}{C} + \frac{B}{C}\,x_2 + y_2 = 0\,.$$

Since $A$, $B$ and $C$ are arbitrarily-valued, it does no hard to define

$$\frac{A}{C} = \tilde{A} \qquad\qquad \frac{B}{C} = \tilde{B}\,,$$

which effectively reduces the number of unknowns to two. (Don't forget that each $x$ and $y$ is known in this case.)

$$\tilde{A} + \tilde{B}\,x_1 + y_1 = 0 \qquad\qquad \tilde{A} + \tilde{B}\,x_2 + y_2 = 0\,.$$

Next, solve each equation for $\tilde{A}$ and set them equal:

$$\tilde{B}\,x_1 + y_1 = \tilde{B}\,x_2 + y_2$$

Condense all $x$-terms on one side, and all $y$-terms on the other:

$$\tilde{B}\,(x_1 - x_2) = y_2 - y_1$$

Finally, divide each side by $x_2 - x_1$ to land at

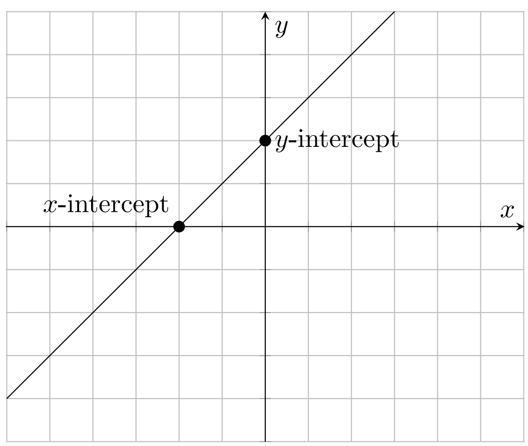
$$-\tilde{B} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

Comparing this result to the definition of the slope of a line, we find that $-\tilde{B}$ is precisely equal to the slope of the line. Restoring the original variables, we find

$$m = -\tilde{B} = -\frac{B}{C}\,.$$

## 8.3 Intercepts

Any straight line that is not purely vertical or not purely horizontal eventually hits both the $x$-axis and the $y$-axis. The location $(x_{int}, 0)$ where the line hits the $x$-axis is called the *$x$-intercept*. Similarly, the location $(0, y_{int})$ where the line hits the $y$-axis is called the *$y$-intercept* as shown:




Starting with the general equation $A+Bx+Cy = 0$, the $x$- and $y$-intercepts, respectively, are determined by setting $y = 0$ and $x = 0$, in that order, giving

$$A + B\cdot 0 + C\cdot y_{int} = 0 \qquad\qquad A + B\cdot x_{int} + C\cdot 0 = 0\,,$$

telling us

$$y_{int} = -\frac{A}{C} \qquad\qquad x_{int} = -\frac{A}{B}\,.$$

The values $y_{int}$, $x_{int}$ combine to tell us the slope of the line by eliminating $-A$ between both equations:

$$C\cdot y_{int} = B\cdot x_{int} \qquad \rightarrow \qquad \frac{y_{int}}{x_{int}} = \frac{B}{C} = -m$$

## 8.4 Slope-Intercept Equation

Let us return to the general equation of a line in the Cartesian plane,

$$A + Bx + Cy = 0 \,,$$

and solve the equation for $y$:

$$y = \frac{-A - Bx}{C} = -\frac{A}{C} - \frac{B}{C}\,x \,.$$

Next, replace $-A/C$ and $-B/C$ with the equivalent terms $y_{int}$, $m$, respectively:

$$y = y_{int} + mx$$

By convention, the term $y_{int}$ is almost always denoted as $b$, and written after the $mx$ term. Finally, we have the *slope-intercept* form of the equation of a line:

$$y = mx + b$$

The variables $m$ (slope) and $b$ ($y$-intercept) contain *all* of the information needed to plot the entire line (that is, to find all ordered-pair solutions to $y = mx + b$).

## 8.5 Point-Slope Equation

Let us return to the equation that defines the slope of a line, namely

$$m = \frac{y_2 - y_1}{x_2 - x_1} \,.$$

Next, suppose that $m$ has a known value, and that some point $(x_1, y_1)$ on the line is also known. Letting $(x, y)$ denote *any* other point on the line, we arrive at the *point-slope* equation of a line:

$$m\,(x - x_1) = y - y_1$$

As a sanity check, we can make sure that the point-slope form and the slope-intercept forms are equivalent. Solving the above for $y$, we have

$$y = mx - mx_1 + y_1 \,,$$

which is only true if $-mx_1 + y_1$ equals the $y$-intercept, or

$$y_1 = mx_1 + b \,,$$

which is precisely the slope-intercept equation as applied to $(x_1, y_1)$.

## 8.6 Summary

In summary, we have developed three equations of a straight line:

- General form:
$$A + Bx + Cy = 0$$
- Slope-Intercept form:
$$y = mx + b$$
- Point-Slope form:
$$m\,(x - x_1) = y - y_1$$

The coefficients $A$, $B$, $C$ contain information on the slope, the $x$-intercept, and the $y$-intercept. Explicitly, we found:

$$m = -\frac{B}{C} = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_{int} = -\frac{A}{B} \qquad b = y_{int} = -\frac{A}{C}$$

Problem 1

Determine the slope of a line connecting the points $(2, 3)$ and $(7, 4)$.

Problem 2

Write the point-slope equation of a line connecting the points $(-1, 10)$ and $(5, 2)$.

Problem 3

Write the slope-intercept equation of a line connecting the points $(-8, -8)$ and $(-7, 9)$.

Problem 4

Find the $y$-intercept of a line having $m = 2$ that passes through $(2.5, 0)$.

Problem 5

Write the equation of a line with an $x$-intercept of 3, and a $y$-intercept of 6.

# 9 Quadratic Equations

## 9.1 Building the Form

Recall that the equation of a straight line in the Cartesian plane is given by

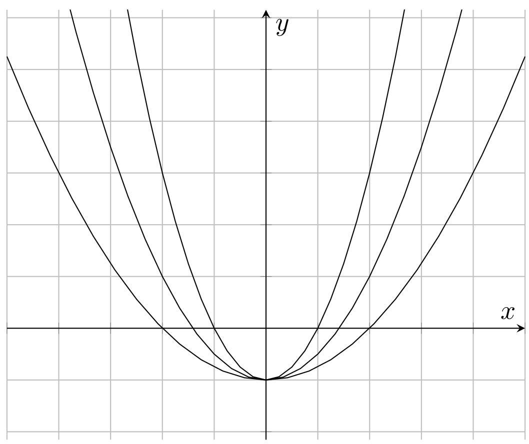
$$y = mx + b\,.$$

Any point $(x, y)$ on the line is an ordered-pair solution to the equation $y = mx + b$, and the character of the line is given by its slope $m$ and $y$-intercept $b$.

Let us now modify the straight line equation to replace $x$ with a *nonlinear* term, particularly $x^2$. The $y$-intercept plays the same role, but by convention, we relabel $b \to k$. The notion of 'slope' becomes ambiguous, so let $m$ be replaced by a general 'scaling factor' called $a$. To visualize this, we plot ordered-pair solutions to the equation

$$y = ax^2 + k$$

and momentarily fix $k$ while allowing $a$ to vary, generating a *family of plots*:




In the above, observe that larger $a$-values lead to a 'skinny' parabolic curve, whereas smaller $a$-values widen the plot into a 'bowl' shape. Negative values of $a$ would flip the plots upside-down.

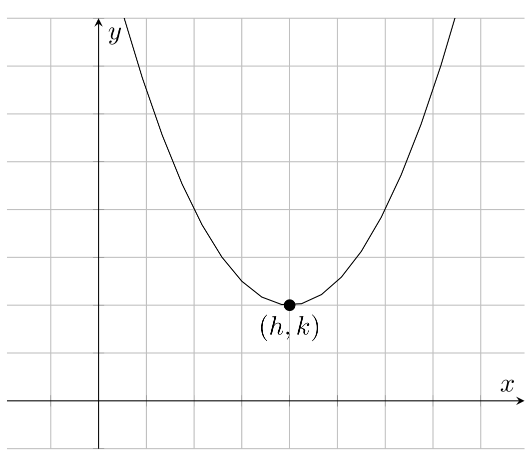
We can also translate the entire plot in the right/left direction by shifting the $x$-variable via

$$x \to x - h\,,$$

leading to the general equation of a shifted parabola

$$y = a\,(x - h)^2 + k\,,$$

sketched below:



In the above, observe that positive values of $k$ shift the plot to the right of the $y$-axis. The point $(h, k)$ is called the *vertex* of the parabola. The vertical line passing through the vertex is called the *axis*.

## 9.2 General Form

Another name for a scaled-and-shifted parabola is a *quadratic* equation. Proceeding by analogy to straight line analysis, observe that our quadratic equation

$$y = a\,(x - h)^2 + k$$

can also be written in general form, namely

$$y = Ax^2 + Bx + C\ ,$$

where the task on hand is to relate the new $A$, $B$, $C$ to the scaling and shifting variables $a$, $h$, $k$. This is started by multiplying out the $(x - h)^2$-term in the top equation, and then combining like terms:

$$y = (a)\,x^2 + (-2ah)\,x + \left(ah^2 + k\right)$$

By comparing the coefficients on the $x^2$, $x$, and dimensionless terms, we find:

$$\begin{aligned} A &= a \\ B &= -2ah \\ C &= ah^2 + k \end{aligned}$$

Solving the system for $a$, $h$, and $k$, we have:

$$\begin{aligned} a &= A \\ h &= -\frac{B}{2A} \\ k &= C - \frac{B^2}{4A} \end{aligned}$$

## 9.3 Solving Quadratic Equations

The quadratic equation

$$y = a\,(x - h)^2 + k$$

can be solved for $x$ with relative ease, remembering that square roots have both positive and negative solutions:

$$x = h \pm \sqrt{\frac{y - k}{a}}$$

It's important to notice that any $y$-value less than $k$ cannot possibly be a valid solution, as no part of the parabola exists for $y < k$ (for positive $a$). If we try inserting $y < k$ anyway, the square root term contains a negative number, which is in fact *imaginary* (technical term). In the special case $y = k$, the only solution corresponds to $x = h$.

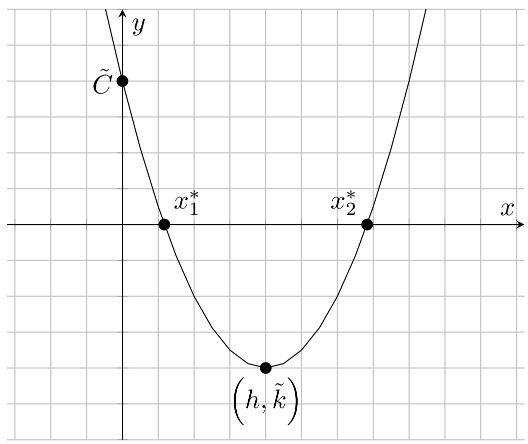
For a fixed 'height' $y^*$ on the parabola, having solution(s) $x^*$, the governing equation is

$$y^* = a\,(x^* - h)^2 + k\ .$$

Since $y^*$ and $k$ are like terms (not multiplied by $x$ or $x^2$), it's harmless to combine them into a new dimensionless constant $\tilde{k} = k - y^*$ such that

$$0 = a\,(x^* - h)^2 + \tilde{k}\ ,$$

which transforms the problem into finding the $x$-intercept(s) of a shifted parabola with a vertex at $x = h$, $y = \tilde{k}$:




The substitution $\tilde{k} = k - y^*$ applies similarly in the general equation

$$y^* = A\left(x^*\right)^2 + Bx^* + C\ ,$$

where by letting

$$\tilde{C} = C - y^* = ah^2 + k - y^* = ah^2 + \tilde{k}\ ,$$

the problem similarly reduces to finding the $x$-intercept(s) of

$$0 = A\left(x^*\right)^2 + Bx^* + \tilde{C}\ .$$

We've shown that finding solutions to the quadratic equation can be reduced to the task of finding the $x$-intercept(s) of a shifted version of the equation. Since it would be burdensome to constantly maintain the superscripts in $x^*$, $y^*$, along with the tilde marks on $\tilde{k}$, $\tilde{C}$, let us for now on ignore these special marks, knowing they can be restored if needed. Thus, we can move forward by seeking solutions to:

$$0 = a\left(x-h\right)^2 + k \qquad \leftrightarrow \qquad 0 = Ax^2 + Bx + C$$

## 9.4 Quadratic Formula

Solving for $x$ in the equation $0 = a\left(x-h\right)^2 + k$, we quickly find

$$x = h \pm \sqrt{-\frac{k}{a}}\ .$$

Replacing $a$, $h$, and $k$ with their equivalent representations in terms of $A$, $B$ and $C$, we have

$$x = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}\ ,$$

known as the (infamous) *quadratic formula*.

Explicitly, the quadratic formula gives solutions to

$$0 = Ax^2 + Bx + C\ .$$

In order for solutions to make sense, the quantity $B^2 - 4AC$ must evaluate to a positive number, otherwise the square root receives a negative input, which mean there are no real solutions to the equation. In the special case $B^2 = 4AC$, the only surviving solution is $x = -B/2A$.

## 9.5 Features of Quadratic Equations

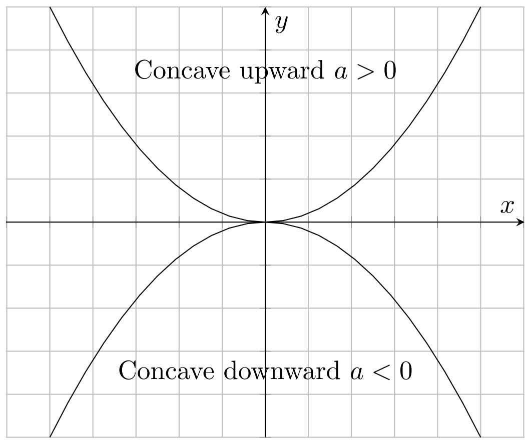
Supposing you are handed either of

$$y = a\left(x-h\right)^2 + k \qquad\qquad y = Ax^2 + Bx + C\ ,$$

a plot of ordered-pair solutions will appear as a parabola centered *somewhere* in the Cartesian plane, and will be *concave upward* (a bowl), or *concave downward* (a mountain).

**Concavity**

The concavity of a parabola is given directly by the $A$-term, or equivalently, the $a$-term. Upward concavity corresponds to positive values, where downward concavity corresponds to negative vales as shown:




**Vertex**

Regardless of its concavity, the vertex of the parabola is the location $(h, k)$. The particular values $h$, $k$ are especially obvious in $y = a\,(x - h)^2 + k$, but we dealing with the general form $y = Ax^2 + Bx + C$ , we have to remember

$$h = -\frac{B}{2A} \qquad\qquad k = C - \frac{B^2}{4A}\ .$$

**Axis**

The axis of a parabola is the vertical line that slices down the middle, positioned at

$$x_h = h = -\frac{B}{2A}\ .$$

**y-Intercept**

A concave upward or concave downward parabola will *always* have a $y$-intercept, which is given by $x = 0$ in either equation

$$y_{int} = a\,(0 - h)^2 + k = ah^2 + k \qquad\qquad y_{int} = A \cdot 0^2 + B \cdot 0 + C = C\ ,$$

telling us

$$y_{int} = C = ah^2 + k\ .$$

**x-Intercept(s)**

In general, a quadratic equation may have up to two $x$-intercepts, but as few as zero. Setting $y = 0$ in either version of the quadratic equation, we have

$$0 = a\,(x - h)^2 + k \qquad\qquad 0 = Ax^2 + Bx + C\ ,$$

with each being solved by the quadratic formula:

$$x_{int} = h \pm \sqrt{-\frac{k}{a}} \qquad\qquad x_{int} = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

If the quantity in the square root is positive, there are two solutions for $x_{int}$, indicating two $x$-intercepts. These are symmetric about the axis of the parabola. If it happens that $k = 0$, or equivalently if $B^2 - 4AC = 0$, the square root disappears, and the $x$-intercept is precisely at the vertex

$$x_h = h = -\frac{B}{2A} .$$

If the square root ultimately contains a negative number, the parabola does not touch the $x$-axis, and there are no $x$-intercepts.

**Symmetry**

Any parabola is mirror-symmetric about its axis, i.e. the line $x = h$. This means if we choose any location $(h - x^*, y^*)$ on the parabola, and then invert the sign on $x^*$ such that $x^* \to -x^*$, then the height $y^*$ of the mirror-image point $(h + x^*, y^*)$ remains the same.

## 9.6 Approximating Roots

**Approximating Square Roots**

Suppose we are tasked with finding the square root $Q$ of a number $Q^2$ that is not a perfect square, i.e. not $\sqrt{25} = 5$, $\sqrt{36} = 6$, or $\sqrt{100} = 10$, etc. Since $Q$ is not an integer, it follows that there exist two integers $N$ and $M = N + 1$ such that $N < Q < M$, where

$$N + x = Q \qquad\qquad M - y = Q ,$$

such that

$$x + y = 1 ,$$

and

$$(N + x)^2 = Q^2 \qquad\qquad (M - y)^2 = Q^2 .$$

Use the distributive property on the left side of each equation to get

$$N^2 + 2Nx + x^2 = Q^2 \qquad\qquad M^2 - 2My + y^2 = Q^2 .$$

Note that the restriction $x + y = 1$ restricts $x^2$ and $y^2$ to being small, and a tempting path forward is to ignore these terms altogether, or at least ignore one of them. A more symmetric pursuit would have us assume, on average, that $\langle x \rangle = \langle y \rangle = 0.5$ such that $\langle x^2 \rangle = \langle y^2 \rangle = 0.25$ while $x$ and $y$ are still unknowns to be determined. Doing this, we have

$$N^2 + 2Nx + 0.25 \approx Q^2 \qquad\qquad M^2 - 2My + 0.25 \approx Q^2 ,$$

allowing $x$ and $y$ to be isolated:

$$x \approx \frac{Q^2 - N^2 - 0.25}{2N} \qquad\qquad y \approx \frac{Q^2 - M^2 - 0.25}{-2M}$$

Putting the solutions together, we have

$$Q \approx \frac{N}{2} + \frac{Q^2 - 0.25}{2N} \qquad\qquad Q \approx \frac{M}{2} + \frac{Q^2 - 0.25}{2M} .$$

Of course, we still have two different answers for $Q$, and the average of these is likely to be more accurate than either:

$$Q_{ave} = \frac{N + M}{4} \left( 1 + \frac{Q^2 - 0.25}{NM} \right)$$

A more rigorous approach allows $\langle x^2 \rangle$ and $\langle y^2 \rangle$ to be separate, where we instead get

$$Q \approx \frac{N}{2} + \frac{Q^2 - \langle x^2 \rangle}{2N} \qquad\qquad Q \approx \frac{M}{2} + \frac{Q^2 - \langle y^2 \rangle}{2M} ,$$

and the average becomes

$$Q_{ave} = \frac{N+M}{4}\left(1+\frac{Q^2}{NM}\right) - \frac{1}{4}\left(\frac{\langle x^2\rangle}{N} + \frac{\langle y^2\rangle}{M}\right) .$$

Now, even though we are 'solving' for $x$ and $y$, we can still guess their averages based on the information given. We can see that if $x$ is very small then $N \approx Q$ and $y \approx 1$. On the other hand, if $y$ is very small then $M \approx Q$, and $x \approx 1$. To capture this, let us write

$$\langle x\rangle = \frac{Q^2 - N^2}{M^2 - N^2} \qquad \langle y\rangle = \frac{M^2 - Q^2}{M^2 - N^2}$$

so that

$$\langle x^2\rangle = \left(\frac{Q^2 - N^2}{M^2 - N^2}\right)^2 \qquad \langle y^2\rangle = \left(\frac{M^2 - Q^2}{M^2 - N^2}\right)^2 .$$

It turns out that the $Q_{ave}$-equation, with $\langle x^2\rangle$, $\langle y^2\rangle$ substituted, produces the result within a few thousands of the true answer (i.e. accuracy is $\approx 99.9\%$). Note this is a non-iterative formula.

**Approximating Cube Roots**

The above exercise can be repeated for cube roots. Supposing we are given $Q^3$, there are two integers $N$ and $M = N+1$ such that $N < Q < M$, where

$$N + x = Q \qquad M - y = Q ,$$

such that

$$x + y = 1 ,$$

and

$$(N+x)^3 = Q^3 \qquad (M-y)^3 = Q^3 .$$

Use the distribute property on the left side of each equation to get

$$N^3 + 3N^2x + 3Nx^2 + x^3 = Q^2 \qquad M^3 - 3M^2y + 3My^2 - y^3 = Q^2 .$$

By similar arguments used above, let

$$\langle x^2\rangle = \left(\frac{Q^3 - N^3}{M^3 - N^3}\right)^2 \qquad \langle y^2\rangle = \left(\frac{M^3 - Q^3}{M^3 - N^3}\right)^2$$

and

$$\langle x^3\rangle = \left(\frac{Q^3 - N^3}{M^3 - N^3}\right)^3 \qquad \langle y^3\rangle = \left(\frac{M^3 - Q^3}{M^3 - N^3}\right)^3 ,$$

and then solve for $x$ and $y$:

$$x \approx \frac{Q^3 - N^3 - 3N\langle x^2\rangle - \langle x^3\rangle}{3N^2} \qquad y \approx \frac{Q^3 - M^3 - 3M\langle y^2\rangle + \langle y^3\rangle}{-3M^2}$$

Restoring $Q$ via $Q = N + y$ and $Q = M - y$, we have

$$Q \approx \left(1 - \frac{1}{3}\right) N + \frac{Q^3 - 3N\langle x^2\rangle - \langle x^3\rangle}{3N^2}$$

$$Q \approx \left(1 - \frac{1}{3}\right) M + \frac{Q^3 - 3M\langle y^2\rangle + \langle y^3\rangle}{3M^2} ,$$

whose average is

$$Q_{ave} = \left(1 - \frac{1}{3}\right)\left(\frac{N+M}{2}\right) + \frac{1}{2\cdot 3}\left(\frac{Q^3 - \langle x^3\rangle}{N^2} + \frac{Q^3 + \langle y^3\rangle}{M^2}\right) - \frac{1}{2}\left(\frac{\langle x^2\rangle}{N} + \frac{\langle y^2\rangle}{M}\right) .$$

**Babylonian Method**

There are numerous ways to approximate square roots, and the previous calculation came tantalizingly close to a rather ancient technique called the *Babylonian method*. In particular, we reasoned that the square root $Q$ of a known number $Q^2$ is the sum $N + x$, where $x < 1$ and $N$ is the integer just less than $Q$, meaning

$$(N + x)^2 = Q^2 ,$$

which expands to

$$N^2 + 2Nx + x^2 = Q^2 .$$

Now, let us modify the approach by allowing $N$ to stray from the integers. In this case, we can look for solutions where $x$ is exceedingly small such that $x^2 \approx 0$. With this assumption, solve for $x$ to get

$$x \approx \frac{Q^2 - N^2}{2N} ,$$

telling us, not surprisingly that

$$Q \approx \frac{N}{2} + \frac{Q^2}{2N} .$$

This is the same as a previous result, without the factor of $-0.25$ in the numerator.

The Babylonians realized this result can be used *recursively* to calculate square roots to arbitrary precision. The idea is to let $N$ be an initial guess near the answer, from which $Q$ is calculated as the refined guess. Then replace $N$ with the contents of $Q$, and repeat until satisfied.

**Babylonian Cube Roots**

The Babylonian method can also work for cube roots (and beyond). That is, if we are handed $Q^3$, we may still write $Q = N + x$ where $N$ is a number near $Q$ and $x$ is a small correction. Then,

$$(N + x)^3 = Q^3$$

expands into

$$N^3 + 3N^2x + 3x^2N + x^3 = Q^3 ,$$

and we assume that the $x^2$- and $x^3$-terms are much smaller than the others in the equation. Solving for $x$ then, we find

$$x \approx \frac{Q^3 - N^3}{3N^2} ,$$

resulting in a recursive formula:

$$Q \approx \frac{2}{3}N + \frac{Q^3}{3N^2}$$

**Babylonian Nth Roots**

Generalizing the problem to finding the $k$th root of $Q^k$, the result becomes:

$$Q \approx \left(1 - \frac{1}{k}\right) N + \frac{Q^k}{k \cdot N^{k-1}}$$

# 10 Factoring Quadratics

## 10.1 Reverse Distribution

The distributive property of multiplication involves multiplying an expression *into* another expression, for instance

$$a(b+c) = ab + ac\,.$$

This operation can be 'undone' by a move called *factoring*. In general, factoring an expression means to transform a sum or difference into a product. To undo the distribution above, we would write

$$ab + ac = a(b+c)\ ,$$

and the expression is now 'factored'.

Consider next the square of a sum, $(x+y)^2$. Using the distributive property (or FOIL method), we find

$$(x+y)^2 = x^2 + 2xy + y^2\ .$$

Reading this equation in reverse, we have that whenever the form $x^2 + 2xy + y^2$ is encountered, the factored version must be $(x+y)^2$. Similarly, the square of a difference reads simply swaps $y$ for $-y$ in the above:

$$x^2 - 2xy + y^2 = (x-y)^2$$

Example 1

Find the GCF and LCM of:

$$x^2y - xy^2 \qquad\qquad 3x - 3y \qquad\qquad x^2 - 2xy + y^2$$

Step 1: Factor each expression:

$$\begin{aligned} x^2 - xy^2 &= xy(x-y) \\ 3x - 2y &= 3(x-y) \\ x^2 - 2xy + y^2 &= (x-y)^2 \end{aligned}$$

Step 2: Identify the GCF:

$$GCF = (x-y)$$

Step 3: Identify the LCM:

$$LCM = 3xy(x-y)^2$$

## 10.2 Solutions as Zeros

Consider the quadratic equation

$$0 = Ax^2 + Bx + C\ .$$

Supposing for the moment that two solutions $x_1$ and $x_2$ exist, it follows that the same quadratic equation can be written

$$0 = A(x-x_1)(x-x_2)\ .$$

This is checked easily by setting $x$ to either $x_1$ or $x_2$, giving zero on the right. For this reason, solutions to an equation are often called *zeros*. For any value $x^*$ that solves the quadratic equation, the quantity $(x - x^*)$ is a factor. As a reality check, recall that we already have solutions $x_1$, $x_2$ from the quadratic formula

$$x_1 = -\frac{B}{2A} + \frac{\sqrt{B^2-4AC}}{2A} \qquad\qquad x_2 = -\frac{B}{2A} - \frac{\sqrt{B^2-4AC}}{2A}\ ,$$

which can be substituted into the above to recover the original equation:

$$\begin{aligned} 0 &= A(x-x_1)(x-x_2) \\ 0 &= Ax^2 - A(x_1+x_2)x + Ax_1x_2 \\ 0 &= Ax^2 - A\left(-\frac{B}{A}\right)x + A\left(\frac{B^2}{4A^2} - \frac{B^2-4AC}{4A^2}\right) \\ 0 &= Ax^2 + Bx + C \end{aligned}$$

## 10.3 Completing the Square

It's possible to solve for $x$ directly in a quadratic equation by a move called *completing the square*. Begin with the quadratic equation

$$0 = Ax^2 + Bx + C \,,$$

and divide through by $A$ to isolate the $x^2$-term:

$$0 = x^2 + \frac{B}{A}x + \frac{C}{A}$$

The goal is to write

$$0 = (x - p)^2 + q \,,$$

where $p$, $q$ are determined by $A$, $B$, $C$.

Expanding out the square term and comparing coefficients to the above, we must have

$$-2p = \frac{B}{A} \qquad\qquad p^2 + q = \frac{C}{A} \,,$$

or

$$p = -\frac{B}{2A} \qquad\qquad q = \frac{C}{A} - \frac{B^2}{4A^2} \,.$$

Now the hard work is finished. Solve for $x$ in the above to get

$$x = p \pm q = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \,,$$

none other than the quadratic formula.

## 10.4 Special Coefficients

Factoring a quadratic expression is easier when the leading coefficient on the $x^2$-term is precisely one. For the case $A = 1$, the expression $x^2 + Bx + C$ factors as

$$x^2 + Bx + C = (x - x_1)(x - x_2) \,,$$

where

$$x_1 = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right) \qquad\qquad x_2 = \frac{1}{2}\left(-B - \sqrt{B^2 - 4C}\right) \,.$$

Taking a moment to calculate the sum $x_1 + x_2$, we find

$$x_1 + x_2 = \frac{1}{2}\left(-B + \cancel{\sqrt{B^2 - 4C}}\right) + \frac{1}{2}\left(-B - \cancel{\sqrt{B^2 - 4C}}\right) = -B \,,$$

and the result is simply $-B$. Calculating the product $x_1 \cdot x_2$ gives

$$x_1 \cdot x_2 = \frac{1}{4}\left(\cancel{B^2} - \cancel{B^2} + 4C\right) = C \,,$$

which could hardly be simpler.

---

If the leading coefficient is not one ($A \neq 1$), we may factor $A$ out of the expression such that

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right) = A\left(x^2 + \tilde{B}x + \tilde{C}\right) \,,$$

effectively removing the leading coefficient, so we instead solve

$$x^2 + \tilde{B}x + \tilde{C} = (x - x_1)(x - x_2) \,,$$

where $\tilde{B} = B/A$, and $\tilde{C} = C/A$. Similar to the above, the sum of $x_1$, $x_2$ must equal $-\tilde{B}$, and the product must equal $\tilde{C}$. While this trick is handy, it often turns out that $\tilde{B}$ and $\tilde{C}$ compute to 'ugly' numbers.

In the special case that the first and third coefficients are the same ($A = C$), we can divide through by $A$ to get

$$Ax^2 + Bx + A = A\left(x^2 + \tilde{B}x + 1\right) ,$$

where $\tilde{B} = B/A$. Supposing $x_1$, $x_2$ are solutions to the equation, the factoring problem reduces to solving

$$x^2 + \tilde{B}x + 1 = 0 ,$$

a special application of the quadratic formula:

$$x = \frac{1}{2}\left(-\tilde{B} \pm \sqrt{\tilde{B}^2 - 4}\right)$$

Next, let us make the curious change of variables such that $z \to 1/x$, where the above becomes

$$\left(\frac{1}{z}\right)^2 + \frac{\tilde{B}}{z} + 1 = 0 ,$$

and multiplying through by $z^2$ results in the same equation we started with, with $z$ in place of $x$:

$$1 + \tilde{B}z + z^2 = 0$$

Then, it must be true that if we have a solution $x_1$ to the first equation, then the second solution $x_2$ must be the reciprocal of $x_1$. Checking this carefully, we find:

$$x_1 = \frac{1}{2}\left(-\tilde{B} - \sqrt{\tilde{B}^2 - 4}\right)$$

$$\frac{1}{x_1} = \frac{-2}{\tilde{B} + \sqrt{\tilde{B}^2 - 4}} \cdot \frac{\tilde{B} - \sqrt{\tilde{B}^2 - 4}}{\tilde{B} - \sqrt{\tilde{B}^2 - 4}} = \frac{1}{2}\left(-\tilde{B} + \sqrt{\tilde{B}^2 - 4}\right) = x_2$$

## 10.5 Method of Transform

Now we develop a wholly general method for factoring quadratic expressions. Starting with the general form $Ax^2 + Bx + C$, write the expression in an already-factored form, necessitating two new variables $f$, $g$ such that

$$\begin{aligned} Ax^2 + Bx + C &= Ax^2 + fx + gx + C \\ &= \frac{1}{C}\left(fx + C\right)\left(gx + C\right) , \end{aligned}$$

where we must have

$$\begin{aligned} f + g &= B \\ f \cdot g &= AC . \end{aligned}$$

The above constitutes a system of two equations and two unknowns. Solving the system for $f$, we find a 'transformed' equation

$$f^2 - Bf + AC = 0 ,$$

which is a quadratic equation with $f$ as the variable, and the leading coefficient is *one*. We could also solve the system for $g$ and write an identical equation, which is in fact not necessary, as the $f$-equation alone produces both solutions:

$$f = \frac{1}{2}\left(B + \sqrt{B^2 - 4AC}\right) \qquad g = \frac{1}{2}\left(B - \sqrt{B^2 - 4AC}\right)$$

Of course, the idea is to *avoid* calculating anything like $\sqrt{B^2 - 4AC}$, as it's often easier to solve $f^2 - Bf + AC = 0$ 'by eye'. In the special case $B = 0$, we readily have $g = -f = \sqrt{-AC}$.

Example 2
Factor:

$$6x^2 + 11x + 4$$

Step 1: Identify coefficients:

$$A = 6 \qquad B = 11 \qquad C = 4 \qquad AC = 24$$

Step 2: Write the transformed equation:

$$f^2 - 11f + 24 = 0$$

Step 3: Factor the transformed equation:

$$(f - 8)(f - 3) = 0$$

Step 4: Identify $f = 8$ and $g = 3$, and rewrite the original expression:

$$6x^2 + 11x + 4 = \frac{1}{4}(8x + 4)(3x + 4)$$

Step 5: Simplify to get the result:

$$6x^2 + 11x + 4 = (2x + 1)(3x + 4)$$

Example 3
Factor:

$$15x^2 + 14x - 8$$

Step 1: Identify coefficients:

$$A = 15 \qquad B = 14 \qquad C = -8 \qquad AC = -120$$

Step 2: Write the transformed equation:

$$f^2 - 14f - 120 = 0$$

Step 3: Factor the transformed equation:

$$(f - 20)(f + 6) = 0$$

Step 4: Identify $f = 20$ and $g = -6$, and rewrite the original expression:

$$15x^2 + 14x - 8 = -\frac{1}{8}(20x - 8)(-6x - 8)$$

Step 5: Simplify to get the result:

$$\begin{aligned} 15x^2 + 14x - 8 &= \frac{(20x - 8)}{4}\frac{(6x + 8)}{2} \\ &= (5x - 2)(3x + 4) \end{aligned}$$

Example 4
Factor:

$$6x^2 - 4x - 16$$

Step 1: Identify coefficients:

$$A = 6 \qquad B = -4 \qquad C = -16 \qquad AC = -96$$

Step 2: Write the transformed equation:

$$f^2 + 4f - 96 = 0$$

Step 3: Factor the transformed equation:

$$(f + 12)(f - 8) = 0$$

Step 4: Identify $f = -12$ and $g = 8$, and rewrite the original expression:

$$6x^2 - 4x + 16 = -\frac{1}{16}(-12x - 16)(8x - 16)$$

Step 5: Simplify to get the result:

$$\begin{aligned} 6x^2 - 4x - 16 &= \frac{(12x + 16)}{2}\frac{(8x - 16)}{8} \\ &= (6x + 8)(x - 2) \end{aligned}$$

Example 5

Factor:

$$-2x^2 - 6x + 56$$

Step 1: Identify coefficients:

$$A = -2 \qquad B = -6 \qquad C = 56 \qquad AC = -112$$

Step 2: Write the transformed equation:

$$f^2 - 6f - 112 = 0$$

Step 3: Factor the transformed equation:

$$(f + 14)(f - 8) = 0$$

Step 4: Identify $f = -14$ and $g = 8$, and rewrite the original expression:

$$-2x^2 - 6x + 56 = \frac{1}{56}(-14x + 56)(8x + 56)$$

Step 5: Simplify to get the result:

$$\begin{aligned} -2x^2 - 6x + 56 &= \frac{(-14x + 56)}{7}\frac{(8x + 56)}{8} \\ &= (-2x + 8)(x + 7) \\ &= -2(x - 4)(x + 7) \end{aligned}$$

Example 6

Factor:

$$24x^2 - 6xy - 9y^2$$

Step 1: Factor a 3 out of the expression:

$$24x^2 - 6xy - 9y^2 = 3\left(8x^2 - 2xy - 3y^2\right)$$

Step 2: Identify coefficients:

$$A = 8 \qquad B = -2y \qquad C = -3y^2 \qquad AC = -24y^2$$

Step 3: Write the transformed equation:

$$f^2 + 2yf - 24y^2 = 0$$

Step 4: Factor the transformed equation:

$$(f+6y)(f-4y)=0$$

Step 5: Identify $f=-6y$ and $g=4y$, and rewrite the original expression:

$$\begin{aligned}24x^2-6xy-9y^2 &= 3\left(8x^2-2xy-3y^2\right)\\ &= 3\times\frac{1}{-3y^2}\left(-6xy-3y^2\right)\left(4xy-3y^2\right)\end{aligned}$$

Step 6: Simplify to get the result:

$$\begin{aligned}24x^2-6xy-9y^2 &= -\frac{\left(-6xy-3y^2\right)}{y}\frac{\left(4xy-3y^2\right)}{y}\\ &= 3(2x+y)(4x-3y)\end{aligned}$$

Example 7

Factor:

$$36x^2-121y^2$$

Step 1: Identify coefficients:

$$A=36 \qquad B=0 \qquad C=-121y^2$$

Step 2: Write the transformed equation:

$$f^2-36\cdot 121y^2=0$$

Step 3: Factor the transformed equation:

$$f=\pm 6\cdot 11y=\pm 66y$$

Step 4: Identify $f=66y$ and $g=-66y$, and rewrite the original expression:

$$36x^2-121y^2=-\frac{1}{121y^2}\left(66xy-121y^2\right)\left(-66xy-121y^2\right)$$

Step 5: Simplify to get the result:

$$\begin{aligned}36x^2-121y^2 &= \frac{(66x-121y)}{11}\frac{(-66x-121y)}{11}\\ &= (6x-11y)(6x+11y)\end{aligned}$$

## 10.6 Nonlinear System

Consider the following nonlinear system of two equations and two unknowns:

$$2x^2 + 3y = 19$$
$$4x - y = 3$$

To solve for $x$ and $y$ analytically, we first need to combine the two equations to eliminate one of the variables. Multiplying the second equation through by a factor of three, and then adding the result to the first equation, we find

$$x^2 + 6x = 14 \ ,$$

which is clearly a quadratic equation, having two solutions:

$$x = -3 \pm \frac{1}{2}\sqrt{6^2 + 4 \cdot 14} \qquad \rightarrow \qquad \begin{cases} x_1 = -3 + \sqrt{23} \approx 1.796 \\ x_2 = -3 - \sqrt{23} \approx -7.796 \end{cases}$$

Since there are *two* $x$-values on hand, it's not clear which one we should be used to determine the proper $y$-solution. Repeating the process to eliminate $x$ instead, we find another quadratic equation

$$y^2 + 30y = 143 \ ,$$

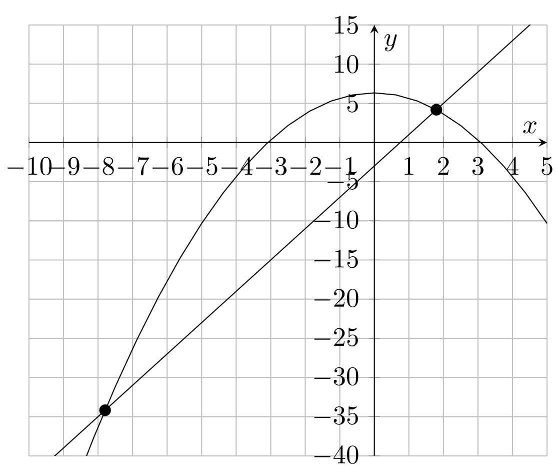
having two solutions:

$$y = -15 \pm \frac{1}{2}\sqrt{30^2 + 4 \cdot 143} \qquad \rightarrow \qquad \begin{cases} y_1 = -15 + \sqrt{368} \approx 4.183 \\ y_2 = -15 - \sqrt{368} \approx -34.183 \end{cases}$$

At this stage, we have four possible solutions to the system

$$(x_1, y_1) \qquad (x_1, y_2) \qquad (x_2, y_1) \qquad (x_2, y_2) \ ,$$

however not all are valid. Each can easily be checked against the original equations, which is equivalent to checking with a graphical method. Solving each equation for $y$ and plotting each on the same graph, we see the intersection of a parabola and a line:




Evidently, intersections occur at $(1.796, 4.183)$ and $(-7.796, -34.183)$, thus the valid solutions to the system are $(x_1, y_1)$ and $(x_2, y_2)$.

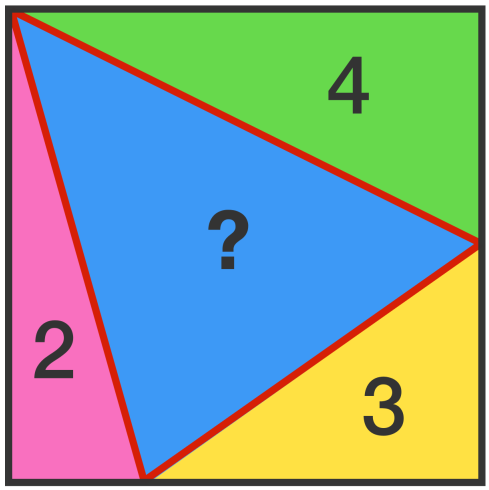
# 11 Problems

## 11.1 Inscribed Triangle in Square

Source: Brilliant
`https://brilliant.org/`

Problem 1

Determine the unknown blue area contained in the square as shown.




Solution 1

Starting on the left side and reading counter-clockwise, the four perimeter segments are $(A)$, $(B+C)$, $(D+E)$, $(A)$. Note also that

$$AB = 4 \qquad CD = 6 \qquad AE = 8 \, .$$

Use $(A-B)(A-E) = CD$ to arrive at

$$A^2 + \frac{32}{A^2} = 18 \, ,$$

which resolves to $A = 4$, making the unknown area equal to 7.

## 11.2 Ladder Touching Cube

Source: stirlingsouth.com
`http://www.stirlingsouth.com/richard/trig9.htm`

Problem 1

A 20-foot ladder leans on a perpendicular wall such that it touches the edge of a $6 \times 6 \times 6$-foot cube flatly pushed against the wall, as seen in the figure. Find the vertical height of the ladder above the cube.

Solution 1

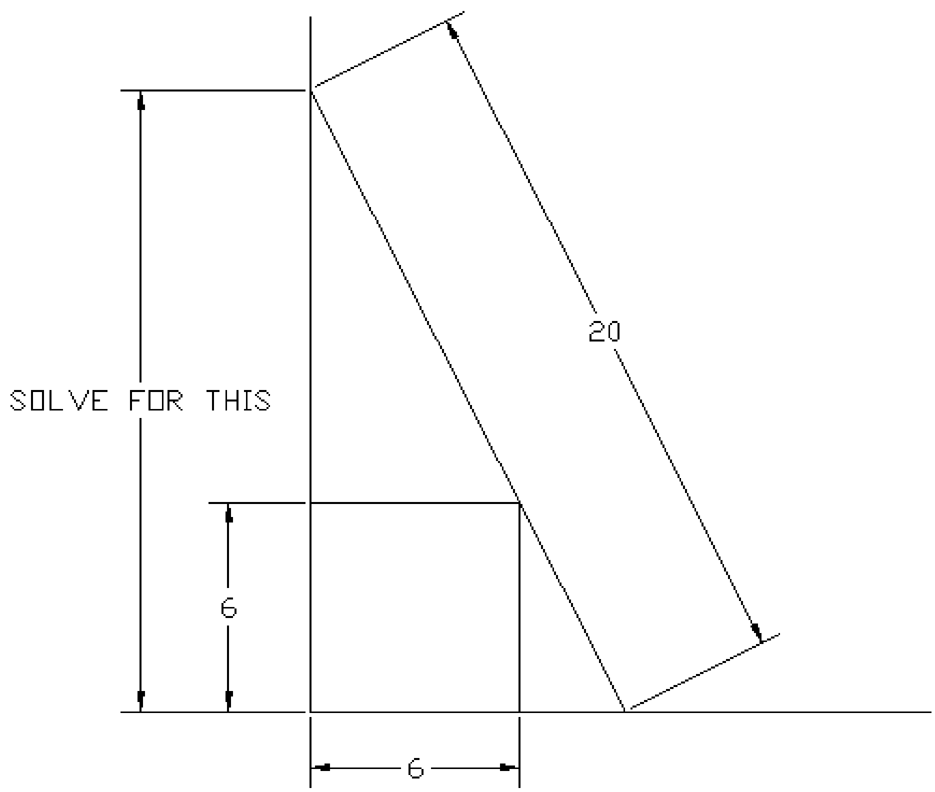
Denote $x$ as the unknown vertical component of the ladder's projection, and $y$ as the unknown horizontal component. By area arguments, or by considering similar triangles, observe that $xy = 36$. Further, the Pythagorean theorem dictates

$$(x+6)^2 + (y+6)^2 = 20^2 \, ,$$

where completing the square in the variable $x+y$ eventually gives

$$x + y = -6 + 2\sqrt{109} \, ,$$

resolving to $x \approx 11.840$, $y \approx 3.0405$.


20
SOLVE FOR THIS
6
6

# 12 Cubic Equations

As a generalization to the quadratic expression, a *cubic expression* includes a third-order term multiplied by a coefficient:

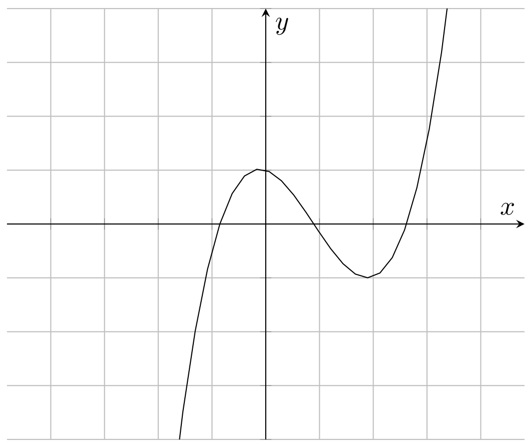
$$Ax^3 + Bx^2 + Cx + D$$

## 12.1 Features of Cubic Equations

The formula

$$y = Ax^3 + Bx^2 + Cx + D$$

constitutes a *cubic equation*, generally appearing as follows:




In the above, the coefficients are chosen such that

$$A = \frac{1}{2} \qquad B = -\frac{4}{3} \qquad C = -\frac{1}{3} \qquad D = 1\,.$$

#### Coefficients

The leading coefficient $A$ determines the overall 'climb' of the plot. For $A > 0$, the plot grows upward for increasing $x$, while dipping (very) negative for decreasing $x$. This is true regardless of $B$, $C$, $D$, as the $x^3$-term grows much faster than the lower-order terms. For $A < 0$, the plot grows downward as $x$ increases, whereas largely-negative $x$-values lead to very large $y$-values.

The coefficients $B$ and $C$ are responsible for the overall structure of the plot near the origin (for not-so-large values of $x$). The coefficient $D$ plays the role of the $y$-intercept, controlling the vertical placement of the plot on the Cartesian graph.

#### x-Intercept(s)

In the general case, a cubic equation flaunts two vertex points and up to three $x$-intercepts. Depending on the values of $B$, $C$, $D$, the number of $x$-intercepts can vary, but there are *never* zero.

## 12.2 Solving Cubic Equations

We now develop a method to solve cubic equations, which amounts to looking for $x$-intercepts in the cubic equation

$$0 = Ax^3 + Bx^2 + Cx + D\,.$$

### Depressed Cubic

To begin, we effectively do away with the $x^2$-term by making the substitution

$$x = z - \frac{B}{3A}$$

as follows:

$$\begin{aligned}
0 &= A\left(z - \frac{B}{3A}\right)^3 + B\left(z - \frac{B}{3A}\right)^2 + C\left(z - \frac{B}{3A}\right) + D \\
0 &= A\left(z^3 - \cancel{3z^2\frac{B}{3A}} + 3z\left(\frac{B}{3A}\right)^2 - \left(\frac{B}{3A}\right)^3\right) \\
&\quad + B\left(\cancel{z^2} - 2z\frac{B}{3A} + \left(\frac{B}{3A}\right)^2\right) + C\left(z - \frac{B}{3A}\right) + D \\
0 &= Az^3 + z\left(-\frac{B^2}{3A} + C\right) - A\left(\frac{B}{3A}\right)^3 + B\left(\frac{B}{3A}\right)^2 - C\left(\frac{B}{3A}\right) + D \\
0 &= z^3 + z\left(-\frac{B^2}{3A^2} + \frac{C}{A}\right) + 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A}
\end{aligned}$$

Proceed by setting

$$a = -\frac{B^2}{3A^2} + \frac{C}{A} \qquad\qquad -b = 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A}$$

to arrive at the equation of the *depressed cubic*:

$$z^3 + az = b$$

Since we're operating in a general scheme, it's clear that any cubic equation can be 'depressed' into the form above, and then the problem becomes finding solutions for $z$ in terms of $a$ and $b$. With this understood, let us make switch of notation $z \to x$ for convenience, knowing the true $x$ can be restored by adding $B/3A$.

### Geometric Interpretation

The depressed cubic equation can be solved by a trick attributed to Gerolamo Cardano in 1545. Tracing Cardano's steps, begin with a cube of side $p$, and then introduce three planes inside the cube, parallel to the top, right, and back faces. Set each plane length $q$ from the cube's respective sides as shown.

The total volume illustrated consists of the 'main' cube of side $p - q$, three slabs of volume $q\,(p-q)^2$, three bars of volume $q^2\,(p-q)$, and a small cube of side $q$. Meanwhile, the total volume is simply $p^3$, allowing us to write

$$p^3 = (p-q)^3 + 3q^2\,(p-q) + 3\,(p-q)^2\,q + q^3\ ,$$

readily simplifying to

$$(p-q)^3 + 3pq\,(p-q) = p^3 - q^3\ ,$$

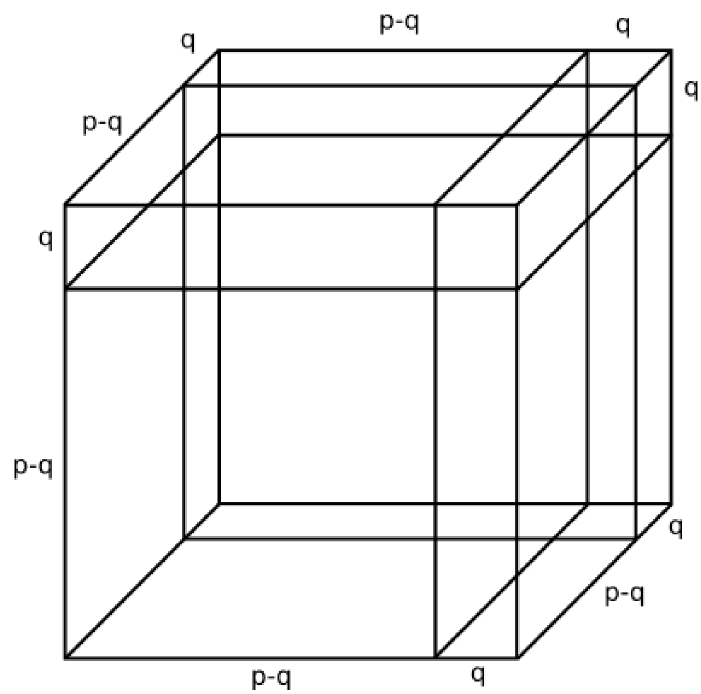
which is indeed another equation for the depressed cubic equation. Identifying

$$p - q = x \qquad\qquad 3pq = a \qquad\qquad p^3 - q^3 = b\ ,$$

we recover the form $x^3 + ax = b$.

Eliminating $q$ between the latter two equations, we end up with

$$p^6 - bp^3 - \left(\frac{a}{3}\right)^3 = 0\ ,$$




which is in fact a quadratic equation in the variable $p^3$, easily isolated by the quadratic formula:

$$p^3 = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}$$

This automatically gives us $q^3$, specifically

$$q^3 = p^3 - b = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3},$$

allowing a solution for $x$ to be written:

$$x = \left(\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}\right)^{1/3} - \left(-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}\right)^{1/3}$$

Recall though we are working in the shifted variable $x \to x - B/3A$, so the factor $B/3A$ must be added to the above to account for this.

Example 1

Find one solution to:

$$x^3 - \frac{x}{3} - \frac{2}{27} = 0$$

Step 1: Identify the above as a depressed cubic equation and pick out coefficients:

$$3pq = a = -\frac{1}{3} \qquad\qquad p^3 - q^3 = b = \frac{2}{27}$$

Step 2: Solve for $p^3$, and write $p$ and $q$:

$$p^3 = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3} = \frac{1}{27} \pm \sqrt{\left(\frac{1}{27}\right)^2 - \left(\frac{1}{9}\right)^3} = \frac{1}{27}$$

$$p = \frac{1}{3} \qquad\qquad q = -\frac{1}{3}$$

Step 3: Write the solution for $x$ in terms of $p$ and $q$:

$$x = p - q = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

**Two More Solutions**

In the general case, the geometric approach to solving a depressed cubic equation

$$x^3 + ax = b$$

produces a solution, however there should exist (up to) three total solutions to the equation. Labeling the known solution as $w$, it follows that $(x - w)$ can be factored out of the depressed cubic equation according to

$$x^3 + ax - b = (x - w)\left(x^2 + tx + u\right) ,$$

where coefficients $t$, $u$ are determined by $a$, $b$. By comparing same-order terms in $x$, we find

$$t = w \qquad\qquad u = \frac{b}{w}$$

to get

$$x^3 + ax - b = (x - w)\left(x^2 + wx + \frac{b}{w}\right) .$$

Remaining solutions to the depressed cubic equation are given by the zeros of the expression

$$x^2 + wx + \frac{b}{w} ,$$

an application of the quadratic formula:

$$x = \frac{w}{2}\left(-1 \pm \sqrt{1 - \frac{4b}{w^3}}\right)$$

Example 2

Use the known solution $w = 2/3$ to continue factoring the expression:

$$x^3 - \frac{x}{3} - \frac{2}{27}$$

Step 1: Substitute $w = 2/3$ and $b = -2/27$ into the equation for $x$ and simplify:

$$x = \frac{2/3}{2}\left(-1 \pm \sqrt{1 - \frac{4 \cdot 2/27}{(2/3)^3}}\right) = -\frac{1}{3}$$

Step 2: Pick out any new solution(s) gained. In this case, we have two copies of the same solution:

$$x_1 = x_2 = -\frac{1}{3}$$

Step 3: Write the final form:

$$x^3 - \frac{x}{3} - \frac{2}{27} = \left(x - \frac{2}{3}\right)\left(x + \frac{1}{3}\right)^2$$

## 12.3 Applications

**Depressed Cubic in Disguise**

Consider the rather exotic quantity

$$\left(7+\sqrt{50}\right)^{1/3}+\left(7-\sqrt{50}\right)^{1/3} ,$$

which might seem impossible to evaluate, namely because $7-\sqrt{50}$ is surely negative. Proceeding boldly, let us store the whole quantity in a variable $x$, and then calculate $x^3$:

$$\begin{aligned} x^3 &= \left(\left(7+\sqrt{50}\right)^{1/3}+\left(7-\sqrt{50}\right)^{1/3}\right)^3 \\ &= 7+\sqrt{50}+3\left(7+\sqrt{50}\right)^{2/3}\left(7-\sqrt{50}\right)^{1/3}+3\left(7+\sqrt{50}\right)^{1/3}\left(7-\sqrt{50}\right)^{2/3}+7-\sqrt{50} \\ &= 14+3\left(\left(7+\sqrt{50}\right)^{1/3}\left(7-\sqrt{50}\right)^{1/3}\right)\left(\left(7+\sqrt{50}\right)^{1/3}+\left(7-\sqrt{50}\right)^{1/3}\right) \\ &= 14+3\left(-1\right)^{1/3}x \end{aligned}$$

Evidently then, we find

$$x^3=-3x+14 ,$$

the equation of a depressed cubic. From here, it's probably easier to guess that $x=2$ rather than deploy the rest of the machinery, however. With the value for $x$ found, we write the answer:

$$\left(7+\sqrt{50}\right)^{1/3}+\left(7-\sqrt{50}\right)^{1/3}=2$$

# 13 Polynomial Division

A brute-force method for manipulating an expression entails long division of polynomials. The setup and procedure for polynomial division is identical to elementary methods for arithmetic. To illustrate polynomial division, consider a ratio such as

$$\frac{x^2 - 9x - 10}{x + 1},$$

and set up the corresponding division problem:

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & & & \\ \cline{2-4} x+1 & x^2 & -9x & -10 \end{array}$$

To proceed, divide the first term in the dividend by the first term in the divisor to get $x^2/x = x$. Place the result ($x$) in the quotient field. Then, distribute $x$ into the divisor and subtract the result from the dividend:

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & & x & \\ \cline{2-4} x+1 & x^2 & -9x & -10 \\ & x^2 & x & \\ \cline{2-4} & & -10x & -10 \end{array}$$

We may now regard $-10x - 10$ as the updated dividend and repeat the process. Dividing the respective leading terms, we find $-10x/x = -10$, and update our progress as follows:

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & & x & -10 \\ \cline{2-4} x+1 & x^2 & -9x & -10 \\ & x^2 & x & \\ \cline{2-4} & & -10x & -10 \\ & & -10x & -10 \\ \cline{3-4} & & & 0 \end{array}$$

With a new dividend of zero, the process halts, and we can read off the answer:

$$\frac{x^2 - 9x - 10}{x + 1} = x - 10$$

## 13.1 Remainders

Taking a more informative example, consider the ratio

$$\frac{\left(x^4 + x + 1\right)^2}{x^2 - 1} = \frac{x^8 + 2x^5 + 2x^4 + x^2 + 2x + 1}{x^2 - 1}.$$

Note that the numerator should always contain a higher-degree polynomial than the denominator. Setting up and starting the hard work, we have:

$$
\begin{array}{r|rrrrrrrr}
 & & x^6 & & +x^4 & +2x^3 & +3x^2 & +2x & +4 \\
\hline
x^2-1 & x^8 & & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & x^8 & -x^6 & & & & & & \\
\cline{2-9}
 & & x^6 & +2x^5 & +2x^4 & & +x^2 & +2x & +1 \\
 & & x^6 & & -x^4 & & & & \\
\cline{3-9}
 & & & 2x^5 & +3x^4 & & +x^2 & +2x & +1 \\
 & & & 2x^5 & & -2x^3 & & & \\
\cline{4-9}
 & & & & 3x^4 & +2x^3 & +x^2 & +2x & +1 \\
 & & & & 3x^4 & & -3x^2 & & \\
\cline{5-9}
 & & & & & 2x^3 & +4x^2 & +2x & +1 \\
 & & & & & 2x^3 & & -2x & \\
\cline{6-9}
 & & & & & & 4x^2 & +4x & +1 \\
 & & & & & & 4x^2 & & -4 \\
\cline{7-9}
 & & & & & & & 4x & +5
\end{array}
$$

The next step *would* be to try dividing $4x$ by $x^2$, however the result (and any following it) will contain factors of $x^{-1}$. Reading off the quotient while regarding $(4x+5)\,/\,\left(x^2-1\right)$ as the remainder, we assemble the result:

$$\frac{x^8+2x^5+2x^4+x^2+2x+1}{x^2-1} = x^6+x^4+2x^3+3x^2+2x+4+\frac{4x+5}{x^2-1}$$

## 13.2 Application to Factoring

Suppose we are interested in factoring a quantity $x^n-a$, where $n$ is any integer greater than zero. In general, the number of zeros, i.e. number of potential solutions to

$$x^n - a = 0\,,$$

is equal to $n$, the highest-degree term in the expression. At the very least though, we can be sure that $x = a^{1/n}$ is one valid solution.

Using polynomial division, a term $x-a^{1/n}$ can be factored out of $x^n-a$. Setting up the division problem, we write:

$$
\begin{array}{r|rr}
 & & \\
\hline
x-a^{1/n} & x^n & -a
\end{array}
$$

Without specifying $n$, it's not clear where the division process ought to terminate. Carrying out the division steps *anyway*, we find, after four steps:

$$\frac{x^n-a}{x-a^{1/n}} = x^{n-1}+a^{1/n}x^{n-2}+a^{2/n}x^{n-3}+a^{3/n}x^{n-4}+\frac{a^{4/n}x^{n-4}-a}{x-a^{1/n}}$$

Evidently, the maximum number of division steps should not exceed the degree number $n$, otherwise the exponent on $x$ becomes negative. Tidy up the equation by multiplying $x-a^{1/n}$ into each side:

$$x^n-a = \left(x-a^{1/n}\right)\left(x^{n-1}+a^{1/n}x^{n-2}+a^{2/n}x^{n-3}+a^{3/n}x^{n-4}\right)+\left(a^{4/n}x^{n-4}-a\right)$$

Next, suppose instead of strictly four steps, the division process carries out $j$ steps, where we now have

$$x^n-a = \left(x-a^{1/n}\right)\left(x^{n-1}+a^{1/n}x^{n-2}+a^{2/n}x^{n-3}+\cdots+a^{(j-1)/n}x^{n-j}\right)+\left(a^{j/n}x^{n-j}-a\right)\,.$$

If $j$ is tuned to equal $n$, the remainder term vanishes, and the above becomes

$$x^n-a = \left(x-a^{1/n}\right)\left(x^{n-1}+a^{1/n}x^{n-2}+a^{2/n}x^{n-3}+\cdots+a^{(n-2)/n}x+a^{(n-1)/n}\right)\,.$$

**Summation Notation**

In order to avoid writing a long polynomial battered with exponents, we can 'spot the pattern' in the sum, and use condensed notation as follows:

$$x^n - a = \left(x - a^{1/n}\right)\left(\sum_{k=1}^{n} a^{(k-1)/n} x^{n-k}\right)$$

This result achieves the task of factoring a known solution $x - a^{1/n}$ out of $x^n - a$. The price we pay is that all other solutions are embedded in a potentially long polynomial.

Example 1

Factor:

$$x^3 - 8$$

Step 1: Identify variables:

$$n = 3 \qquad a = 8$$

Step 2: Write the factored expression in summation notation:

$$x^3 - 8 = \left(x - 8^{1/3}\right)\left(\sum_{k=1}^{3} 8^{(k-1)/3} x^{3-k}\right)$$

Step 3: Simplify:

$$x^3 - 8 = (x-2)\left(x^2 + 2x + 4\right)$$

Example 2

Factor:

$$x^4 - 9$$

Step 1: Identify variables:

$$n = 4 \qquad a = 9$$

Step 2: Write the factored expression in summation notation:

$$x^4 - 9 = \left(x - 9^{1/4}\right)\left(\sum_{k=1}^{4} 9^{(k-1)/4} x^{4-k}\right)$$

Step 3: Simplify:

$$\begin{aligned} x^4 - 9 &= \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}x^2 + 3x + 3\sqrt{3}\right) \\ &= \left(x - \sqrt{3}\right)\left(x^2\left(x + \sqrt{3}\right) + 3\left(x + \sqrt{3}\right)\right) \\ &= \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)\left(x^2 + 3\right) \end{aligned}$$

# 14 Partial Fractions

While polynomial division is best-suited for breaking apart 'top-heavy' ratios, another technique is needed to grapple with 'bottom-heavy' ratios, called *partial fractions*. Starting with the case where the denominator has a degree-two polynomial in factored form, observe that such a ratio can be can be split into the sum of two terms, each containing a degree-one polynomial:

$$\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

The unknowns $A$, $B$ are easily determined in terms of $a$, $b$, $c$ $d$. By setting $x = 0$, and then $x = 1/c$, respectively, we gain two equations

$$bA + aB = -d \qquad\qquad c = A + B\,,$$

solved by

$$A = \frac{ac+d}{a-b} \qquad\qquad B = \frac{bc+d}{b-a}\,,$$

which could have been inferred by choosing values $x = a$, $x = b$. This method generalizes to higher-degree polynomial denominators, as shown for the degree-three case:

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

In the general case, if a polynomial $p(x)$ is already factored into linear and quadratic terms, then for each factor $x - a$, there exists a term

$$\frac{A}{x-a}\,,$$

where $A$ must be determined in context.

Example 1

Find the equivalent ratio as a sum of partial fractions:

$$\frac{2x+1}{(x-3)(x-4)}$$

Step 1: Rewrite the ratio as a sum:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Step 2: Solve for $A$ and $B$ to get:

$$A = -7 \qquad\qquad B = 9$$

Step 3: Assemble the result:

$$\frac{2x+1}{(x-3)(x-4)} = \frac{-7}{x-3} + \frac{9}{x-4}$$

Example 2

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{a^2 - x^2}$$

Step 1: Factor the denominator:

$$\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}$$

Step 3: Solve for $A$ and $B$ to get:

$$A = \frac{1}{2a} \qquad B = \frac{1}{2a}$$

Step 4: Assemble the result:

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left(\frac{1}{a-x} + \frac{1}{a+x}\right)$$

## 14.1 Repeated Roots

Of course, the partial fraction expansion is prone to error if we run into division by zero, i.e. the case $a = b$. To handle a ratio having two repeated roots in the denominator, we use a partial fraction expansion

$$\frac{1}{(x-a)^2(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{B}{x-b},$$

admitting a separate term for each instance of $(x-a)$. This pattern generalizes to three repeated roots, and so on:

$$\frac{1}{(x-a)^3(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \frac{B}{x-b}$$

## 14.2 Quadratic Factors

Factors of the form $x^2 + ax + b$ occurring in the denominator can be balanced by an $Ax + B$-term according to

$$\frac{1}{(x^2+ax+b)(x-c)} = \frac{Ax+B}{x^2+bx+c} + \frac{C}{x-c}.$$

If a factor like $\left(x^2+ax+b\right)^2$ occurs, extra terms are needed:

$$\frac{1}{(x^2+ax+b)^2(x-c)} = \frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{C}{x-c}$$

Example 3

Find the equivalent ratio as a sum of partial fractions:

$$\frac{1}{x^4-1}$$

Step 1: Factor the denominator:

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

Step 2: Rewrite the ratio as a sum:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Step 3: Multiply through by $(x-1)(x+1)\left(x^2+1\right)$:

$$1 = A(x+1)\left(x^2+1\right) + B(x-1)\left(x^2+1\right) + (Cx+D)(x-1)(x+1)$$

Step 4: Let $x = 1$, $x = -1$, $x = 0$, and $x = 2$ to isolate each coefficient:

$$A = \frac{1}{4} \qquad B = -\frac{1}{4} \qquad D = -\frac{1}{2} \qquad C = 0$$

Step 5: Assemble the result:

$$\frac{1}{x^4-1} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)$$

## 14.3 Mixed Cases

Certain situations call for polynomial division *and* partial fractions. For instance, in the ratio

$$\frac{x^3+4}{x^2+x} \, ,$$

the numerator contains a higher-degree polynomial than the denominator. Carrying out the division problem

$$x^2+x \overline{\big) \; x^3 \quad +4 \quad} \, ,$$

we end up with a quotient and a remainder as follows:

$$\frac{x^3+4}{x^2+x} = (x-1) + \frac{x+4}{x^2+x}$$

Next, take the remainder term in isolation and use partial fraction analysis to write

$$\frac{x+4}{x^2+x} = \frac{x+4}{x\,(x+1)} = \frac{A}{x} + \frac{B}{x+1} \, ,$$

where we easily get $A = 4$, $B = -3$. In summary then, we find:

$$\frac{x^3+4}{x^2+x} = x - 1 + \frac{4}{x} - \frac{3}{x+1}$$

# 15 Recursive Sequences

## 15.1 Applied Polynomial Division

Let us restate the chief result from polynomial division, namely

$$x^n - a = \left(x - a^{1/n}\right)\left(\sum_{k=1}^{n} a^{(k-1)/n} x^{n-k}\right) .$$

This time, make the substitution

$$-a^{1/n} = \frac{1}{x} \qquad \leftrightarrow \qquad a = \left(\frac{-1}{x}\right)^n ,$$

so the above becomes

$$\frac{x^n - (-1/x)^n}{x + 1/x} = \sum_{k=1}^{n} \left(\frac{-1}{x}\right)^{k-1} x^{n-k} = \sum_{k=1}^{n} (-1)^{k-1} x^{n+1-2k} .$$

Choosing $n = 1$, $n = 2$, $n = 3$, and so on, the results start a simple-enough pattern:

$$\frac{x^n - (-1/x)^n}{x + 1/x} = \begin{cases} n = 1 : \; x^0 \\ n = 2 : \; x^1 - x^{-1} \\ n = 3 : \; x^2 - x^0 + x^{-2} \\ n = 4 : \; x^3 - x^1 + x^{-1} - x^{-3} \\ n = 5 : \; x^4 - x^2 + x^0 - x^{-2} + x^{-4} \\ n = 6 : \; x^5 - x^3 + x^1 - x^{-1} + x^{-3} - x^{-5} \end{cases}$$

Labeling the $n$th result as $C_n$, we equivalently have

$$\frac{x^n - (-1/x)^n}{x + 1/x} = \begin{cases} C_1 = 1 \\ C_2 = x^1 - x^{-1} \\ C_3 = -C_1 + x^2 + x^{-2} \\ C_4 = -C_2 + x^3 - x^{-3} \\ C_5 = -C_3 + x^4 + x^{-4} \\ C_6 = -C_4 + x^5 - x^{-5} \end{cases} ,$$

where it's evident that all odd-indexed $C_n$ have a seed value of $C_1 = 1$, and all even-indexed $C_n$ seed from $C_2 = x - 1/x$. (Of course, subsequent odd $C_n$ depend on $x$ as well.)

### Large-n Behavior

Supposing we choose any even-valued $n$, the coefficient $C_n$ and its neighbors relate by

$$\begin{aligned} n \text{ even:} \qquad & C_n = -C_{n-2} + x^{n-1} - x^{-(n-1)} \\ n+1 \text{ odd:} \qquad & C_{n+1} = -C_{n-1} + x^n + x^{-n} , \end{aligned}$$

begging the ratio

$$R_e = \frac{C_{n+1}}{C_n} = \frac{-C_{n-1} + x^n + x^{-n}}{-C_{n-2} + x^{n-1} - x^{-(n-1)}} .$$

Similarly, we can start with an odd $n$ and write:

$$\begin{aligned} n \text{ odd:} \qquad & C_n = -C_{n-2} + x^{n-1} + x^{-(n-1)} \\ n+1 \text{ even:} \qquad & C_{n+1} = -C_{n-1} + x^n - x^{-n} , \end{aligned}$$

giving

$$R_o = \frac{C_{n+1}}{C_n} = \frac{-C_{n-1} + x^n - x^{-n}}{-C_{n-2} + x^{n-1} + x^{-(n-1)}} .$$

Next, let us examine the quantities $x^n$, $x^{-n}$ with $n$ growing very large. Regardless of whether $x$ is less than one or greater than one (but not equal to one), either of $x^n$, $x^{-n}$ will grow very large, whereas the other will grow very small. If we take the case with $x > 1$, then $x^{-n}$ and $x^{-(n-1)}$ become negligible, and we find

$$x > 1 : \qquad R_e \approx R_o \approx \frac{C_{n+1}}{C_n} \approx \frac{-C_{n-1} + x^n}{-C_{n-2} + x^{n-1}} = x \left( \frac{-C_{n-1} + x^n}{-x \cdot C_{n-2} + x^n} \right) ,$$

strongly suggesting that both even- and odd-index terms obey

$$x > 1 : \qquad C_{n+1} \approx x \cdot C_n$$

as $n$ grows large.

Taking $x < 1$ instead, the terms $x^n$, $x^{n-1}$ turn out much smaller than $x^{-n}$, $x^{-(n-1)}$, and are neglected. The respective ratios reduce to

$$R_e \approx \frac{-C_{n-1} + x^{-n}}{-C_{n-2} - x^{-(n-1)}} \qquad R_o \approx \frac{-C_{n-1} - x^{-n}}{-C_{n-2} + x^{-(n-1)}}$$

or

$$R_e \approx -\frac{1}{x} \left( \frac{-x \cdot C_{n-1} + x^{-(n-1)}}{C_{n-2} + x^{-(n-1)}} \right) \qquad R_o \approx -\frac{1}{x} \left( \frac{x \cdot C_{n-1} + x^{-(n-1)}}{-C_{n-2} + x^{-(n-1)}} \right) ,$$

suggesting, for large $n$,

$$x < 1 : \qquad C_{n+1} \approx \frac{-C_n}{x} .$$

## 15.2 Lucas Numbers

Our previous achievements have established that

$$C_n = \frac{x^n - (-1/x)^n}{x + 1/x} = \sum_{k=1}^{n} (-1)^{k-1} \, x^{n+1-2k} ,$$

where all $C_n$ are 'seeded' by

$$C_1 = 1 \qquad C_2 = x - \frac{1}{x} ,$$

and for large $n$ and fixed $x$, we know:

$$x > 1 : \qquad C_{n+1} \approx x \cdot C_n$$

$$x < 1 : \qquad C_{n+1} \approx \frac{-C_n}{x}$$

---

Having not specified $x$ yet, the freedom remains to make whatever mess of these equations we wish. The simplest foreseeable nontrivial case sets $C_2 = 1$, which immediately restrains $x$ to obeying

$$x - \frac{1}{x} = 1 ,$$

which is in fact a quadratic equation

$$x^2 - x - 1 = 0 .$$

Instead of jumping to solve the equation, note that by substituting $x \to -1/x$, the very same quadratic equation emerges. Thus, whatever the solutions are, calling them $x_1 = \phi$ and $x_2 = \psi$, they have the relationship

$$\phi \cdot \psi = -1 \qquad \phi + \psi = 1 .$$

This also means we may identify

$$x \to \phi \qquad\qquad \frac{-1}{x} \to \psi \, .$$

To indicate that the coefficients $C_n$ are no longer general, relabel $C_n \to F_n$ to get:

$$F_n = \left.\frac{x^n - (-1/x)^n}{x + 1/x}\right|_{x=\phi} = \frac{\phi^n - \psi^n}{\phi - \psi} = \begin{cases} F_1 = 1 \\ F_2 = \phi + \psi \\ F_3 = -F_1 + \phi^2 + \psi^2 \\ F_4 = -F_2 + \phi^3 + \psi^3 \\ F_5 = -F_3 + \phi^4 + \psi^4 \\ F_6 = -F_4 + \phi^5 + \psi^5 \end{cases} = -F_{j-2} + L_{n-1}$$

It's possible to simplify each of the $F_n$ by hand, but we shall resist this temptation, instead introducing the substitution $L_n = \phi^n + \psi^n$ in hopes of spotting a pattern:

$$\begin{aligned} L_1 &= \phi^1 + \psi^1 = 1 \\ L_2 &= \phi^2 + \psi^2 = (\phi + \psi)^2 - 2\phi\psi = 1^2 + 2 \\ L_3 &= \phi^3 + \psi^3 = (\phi + \psi)\left(\phi^2 - \phi\psi + \psi^2\right) = \phi^2 + 1 + \psi^2 = 1 + L_2 \\ L_4 &= \phi^4 + \psi^4 = \left(\phi^2 + \psi^2\right)^2 - 2\phi^2\psi^2 = L_2^2 - 2 \\ L_5 &= \phi^5 + \psi^5 = (\phi + \psi)\left(\phi^4 - \phi^3\psi + \phi^2\psi^2 - \phi\psi^3 + \psi^4\right) \\ &= \phi^4 + \phi^2 + 1 + \psi^2 + \psi^4 = L_3 + L_4 \\ L_6 &= \phi^6 + \psi^6 = \left(\phi^3 + \psi^3\right)^2 - 2\phi^3\psi^3 = L_3^2 + 2 \end{aligned}$$

Evidently, the behavior splits between odd and even bins

$$L_{j \text{ odd}} = L_{j-1} + L_{j-2} \qquad\qquad L_{j \text{ even}} = L_{j/2}^2 - 2 \cdot (-1)^{j/2} \ ,$$

where explicitly:

$$\begin{aligned} L_0 &= 2 \\ L_1 &= 1 \\ L_2 &= 1^2 + 2 = 3 \\ L_3 &= L_2 + L_1 = 4 \\ L_4 &= L_2^2 - 2 = 3^2 - 2 = 7 \\ L_5 &= L_4 + L_3 = 7 + 4 = 11 \\ L_6 &= L_3^2 + 2 = 4^2 + 2 = 18 \\ L_7 &= L_6 + L_5 = 18 + 11 = 29 \end{aligned}$$

The coefficients $L_n$ have a special name called the *Lucas numbers*

$$L_n = \{2, 1, 3, 4, 7, 11, 18, 29, \dots\} \ ,$$

generated by

$$L_n = \phi^n + \psi^n \, .$$

## 15.3 Fibonacci Numbers

Previously, we found

$$F_n = \left.\frac{x^n - (-1/x)^n}{x + 1/x}\right|_{x=\phi} = \frac{\phi^n - \psi^n}{\phi - \psi} = -F_{n-2} + L_{n-1} \ ,$$

with $L_n$ being the $n$th Lucas number. From this, we can immediately list the famed *Fibonacci numbers*

$$\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_3 &= -1 + L_2 = 2 \\
F_4 &= -1 + L_3 = 3 \\
F_5 &= -2 + L_4 = 5 \\
F_6 &= -3 + L_5 = 8 \\
F_7 &= -5 + L_6 = 13 \\
F_8 &= -8 + L_7 = 21 \ ,
\end{aligned}$$

or

$$\{F\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\} \ .$$

**Extension to Negatives**

We can repeat the calculation by substituting $x \to 1/x$, leading to

$$\tilde{F}_n = \left.\frac{x^n - (-1/x)^n}{x + 1/x}\right|_{x=1/\phi} = \frac{(-\psi)^n - (-\phi)^n}{\phi - \psi}$$

with:

$$\begin{aligned}
\tilde{F}_1 &= 1 = F_1 \\
\tilde{F}_2 &= -\frac{(\phi+\psi)\cancel{(\phi-\psi)}}{\cancel{(\phi-\psi)}} = -1 = -F_2 \\
\tilde{F}_3 &= \frac{\cancel{(\phi-\psi)}\left(\phi^2 + \phi\psi + \psi^2\right)}{\cancel{(\phi-\psi)}} = -1 + L_2 = F_3 \\
\tilde{F}_4 &= \frac{\left((-\psi)^2 + (-\phi)^2\right)\cancel{(\phi-\psi)((-\psi)+(-\phi))}^{-1}}{\cancel{(\phi-\psi)}} = -\left(L_1^2 + 2\right) = -F_4
\end{aligned}$$

Evidently, the even-indexed terms flip sign, where the odd-indexed terms remain the same. The extended Fibonacci numbers are thus:

$$\{F\} = \{\dots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, \dots\}$$

## 15.4 Solving for x

Recall that the numbers $\phi$, $\psi$ obeying

$$\phi \cdot \psi = -1 \qquad\qquad \phi + \psi = 1$$

are solutions of the equation

$$x^2 - x - 1 = 0 \ ,$$

generated by the choice

$$1 = C_2 = x + \frac{1}{x} \ .$$

With this construction, we have discovered the set of Lucas numbers, along with the Fibonacci numbers

$$L_n = \phi^n + \psi^n \qquad\qquad F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \ ,$$

without ever needing the numerical values of $\phi$, $\psi$.

Let us finally attain these values numerically. Solving $x^2 - x - 1 = 0$ using the quadratic formula, we get:

$$x_1 = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618034\ldots$$
$$x_2 = \psi = -\frac{1}{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.618034\ldots$$

These are responsible for the overall 'growth' in the Lucas and Fibonacci numbers:

$$L_n = \frac{\left(1+\sqrt{5}\right)^n + \left(1-\sqrt{5}\right)^n}{2^n}$$

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

## 15.5 Modifying x

Since the Lucas-Fibonacci number tour started at the special case $C_2 = 1 = x - 1/x$, it's natural to try other initial values for $C_2$, giving rise to a myriad of Fibonacci-like number sequences. Stepping through a few initial $C_2$-values, we find:

$$C_2 = 1: \qquad x^2 - 1 - x = 0 \qquad \rightarrow \qquad x_1 = \frac{1 \pm \sqrt{5}}{2}$$
$$C_2 = 2: \qquad x^2 - 1 - 2x = 0 \qquad \rightarrow \qquad x_2 = 1 \pm \sqrt{2}$$
$$C_2 = 3: \qquad x^2 - 1 - 3x = 0 \qquad \rightarrow \qquad x_3 = \frac{3 \pm \sqrt{13}}{2}$$
$$C_2 = 4: \qquad x^2 - 1 - 4x = 0 \qquad \rightarrow \qquad x_4 = 2 \pm \sqrt{5}$$
$$C_2 = m: \qquad x^2 - 1 - mx = 0 \qquad \rightarrow \qquad x_m = \frac{m \pm \sqrt{m^2+4}}{2}$$

Each of the above cases yield two solutions corresponding to the same sequence generated by:

$$\frac{x^n - (-1/x)^n}{x + 1/x} = \begin{cases} C_1 = 1 \\ C_2 = x^1 - x^{-1} \\ C_3 = -C_1 + x^2 + x^{-2} \\ C_4 = -C_2 + x^3 - x^{-3} \\ C_5 = -C_3 + x^4 + x^{-4} \\ C_6 = -C_4 + x^5 - x^{-5} \end{cases}$$

Working these out carefully, we find a family of Fibonnacci-like sequences:

$$\{F\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$
$$\{F_2\} = \{1, 2, 5, 12, 29, 70, 169, 408, \ldots\}$$
$$\{F_3\} = \{1, 3, 10, 33, 109, 360, 1189, 3927, \ldots\}$$
$$\{F_4\} = \{1, 4, 17, 72, 305, 1292, 5473, 23184, \ldots\}$$

## 15.6 Recursion Relations

So far, we have generated sequences always starting with $C_1 = 1$, and with $C_2$ any integer 1, 2, 3, and so on. The case $C_2 = 1$ leads to the Lucas numbers $\{L\}$, along with the Fibonacci numbers $\{F\}$, which are related by

$$F_n = -F_{n-2} + L_{n-1}\,.$$

Pursuing the arbitrary-integer case

$$C_2 = x + \frac{1}{x} = p \; ,$$

much of the analysis remains the same, however we pick up an extra factor of $p$ in a modified version of the Lucas numbers:

$$\tilde{L}_{j \text{ odd}} = p \cdot \tilde{L}_{j-1} + \tilde{L}_{j-2} \qquad\qquad \tilde{L}_{j \text{ even}} = \tilde{L}^2_{j/2} - 2 \cdot (-1)^{j/2}$$

The equation relating neighboring coefficients becomes

$$C_n = -C_{n-2} + \tilde{L}_{n-1} \; .$$

To proceed, let us write two more instances of the equation above, however we will advance the index $n \to n+1$ for each new instance

$$\begin{aligned} C_n &= -C_{n-2} + \tilde{L}_{n-1} \\ C_{n+1} &= -C_{n-1} + \tilde{L}_n \\ C_{n+2} &= -C_n + \tilde{L}_{n+1} \; , \end{aligned}$$

and then multiply the first equation by a factor of $-1$, and the second by a factor of $-p$

$$\begin{aligned} -C_n &= C_{n-2} - \tilde{L}_{n-1} \\ -pC_{n+1} &= pC_{n-1} - p\tilde{L}_n \\ C_{n+2} &= -C_n + \tilde{L}_{n+1} \; , \end{aligned}$$

and sum all three equations:

$$C_{n+2} - pC_{n+1} - C_n = -C_n + pC_{n-1} + C_{n-2} + \tilde{L}_{n+1} - p\tilde{L}_n - \tilde{L}_{n-1}$$

Regrouping terms, this is

$$(C_{n+2} - pC_{n+1} - C_n) + (C_n - pC_{n-1} - C_{n-2}) = \left(\tilde{L}_{n+1} - p\tilde{L}_n - \tilde{L}_{n-1}\right) \; ,$$

which can only be true if

$$C_n = pC_{n-1} + C_{n-2} \; ,$$

known as a *recursion relation.* In the special case $p = 1$, we recover

$$F_n = F_{n-1} + F_{n-2} \; ,$$

the main characteristic of the Fibonacci sequence.

# 16 Geometric Series

Consider the curious ratio

$$\frac{x^n - 1}{x - 1} \,,$$

which can be studied by taking the $a = 1$-case of the equation

$$x^n - a = \left(x - a^{1/n}\right)\left(\sum_{k=1}^{n} a^{(k-1)/n} x^{n-k}\right)$$

derived by polynomial division. Choosing a few values for $n$, we may write

$$\frac{x^2 - 1}{x - 1} = 1 + x$$
$$\frac{x^3 - 1}{x - 1} = 1 + x + x^2$$
$$\frac{x^4 - 1}{x - 1} = 1 + x + x^2 + x^3 \,,$$

where for a general $n$, the above becomes

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1} = \sum_{k=1}^{n} x^{k-1} \,.$$

Before proceeding, let us break apart the left side of the equation to move all $n$-dependence to the right:

$$\frac{1}{1 - x} = \sum_{k=1}^{n} x^{k-1} + \frac{x^n}{1 - x}$$

Note that the number of zeros in $(x^n - 1)\,/\,(x - 1)$ is equal to $n$, the degree of the numerator. Meanwhile, as $n$ increases, the number of polynomial terms on the right side of the equations grows steadily and predictably. If we take $n$ to be *infinitely* large, the number of zeros approaches infinity, and so does the number of terms in the polynomial on the right. This is a rather boring scenario if $x^n$ itself approaches infinity, however if $x$ satisfies $-1 < x < 1$, then $x^n$ approaches *zero*. With $n$ set to infinity, the above becomes

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \qquad\qquad -1 < x < 1 \,,$$

called the *geometric series*. The geometric series applies to the domain $-1 < x < 1$, called the *basin of convergence*.

Example 1

A basketball is dropped from 10 feet and bounces up 6 feet. On each bounce, the ball recovers 3/5 of its previous height. What is the total distance traveled by the ball, supposing it bounces forever?

Step 1: Add up the total distance accumulated during each movement downward:

$$D_1 = 10 + \frac{3}{5} \cdot 10 + \left(\frac{3}{5}\right)^2 \cdot 10 + \left(\frac{3}{5}\right)^3 + \cdots$$
$$D_1 = 10 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)$$

Step 2: Add up the total distance accumulated during each movement upward:

$$D_2 = 6 + \frac{3}{5} \cdot 6 + \left(\frac{3}{5}\right)^2 \cdot 6 + \left(\frac{3}{5}\right)^3 + \cdots$$
$$D_2 = 6 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)$$

Step 3: Compare each infinite sequence to the geometric series, and note that:

$$1+\frac{3}{5}+\left(\frac{3}{5}\right)^2+\cdots=\frac{1}{1-3/5}=\frac{5}{2}$$

Step 4: Assemble the total distance moved in feet:

$$D_1+D_2=10\cdot\frac{5}{2}+6\cdot\frac{5}{2}=40$$

## 16.1 Squaring the Series

It's possible to derive an infinite series expansion for $1/(1-x)^2$ by squaring the geometric series. Carrying this out carefully, we find

$$\begin{aligned}\left(1+x+x^2+x^3+\cdots\right)^2 = 1 &+ x+x^2+x^3+\cdots\\ &+x+x^2+x^3+x^4+\cdots\\ &+x^2+x^3+x^4+x^5\cdots\ ,\end{aligned}$$

simplifying to

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots\ .$$

In the same spirit, the geometric series can be raised to higher powers. Listing the first few, we have:

$$\begin{aligned}\frac{1}{(1-x)^3}&=1+3x+6x^2+10x^3+15x^4+\cdots\\ \frac{1}{(1-x)^4}&=1+4x+10x^2+20x^3+35x^4+\cdots\end{aligned}$$

## 16.2 Change of Variables

### Negative Argument

Starting with a standard geometric series

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots\ ,$$

we can let $x\rightarrow -x$ to discover an alternating sequence

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots\ .$$

This has similar consequence in the 'squared' version, where

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots$$

becomes

$$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+\cdots\ .$$

The same goes for higher powers:

$$\begin{aligned}\frac{1}{(1+x)^3}&=1-3x+6x^2-10x^3+15x^4-\cdots\\ \frac{1}{(1+x)^4}&=1-4x+10x^2-20x^3+35x^4+\cdots\end{aligned}$$

### Squared Argument

Starting with the geometric series and letting $x \to x^2$, we get a result having only even terms

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

and similarly,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots .$$

As a sanity check, we can verify that $1/(1-x) + 1/(1+x)$ produces the proper series, which indeed checks out:

$$\begin{aligned}
\frac{1}{1-x} + \frac{1}{1+x} &= \left(1 + x + x^2 + x^3 + \cdots\right) + \left(1 - x + x^2 - x^3 - \cdots\right) \\
\frac{2}{1-x^2} &= 1 + 1 + x - x + x^2 + x^2 + x^3 - x^3 + \cdots \\
\frac{1}{1-x^2} &= 1 + x^2 + x^4 + x^6 + \cdots
\end{aligned}$$

### Relation to Pascal's Triangle

If we take $z = 1 - x$, having basin of convergence $0 < z < 2$, the geometric series reads

$$\frac{1}{1-x} = \frac{1}{z} = 1 + (1-z) + (1-z)^2 + (1-z)^3 + \cdots .$$

Now, observe that each power of $(1-z)$ occurs on the right side, which means the series contains the same information as the full 'negative version' of the Pascal triangle. Plucking off (literally removing) the right-most coefficient from each row of Pascal's triangle, and assigning climbing powers of $z$, we have a sequence

$$1 - z + z^2 - z^3 + z^4 - \cdots .$$

With the modified triangle, remove the 'new' right-most coefficient from each row, and line up these terms to get

$$1 - 2z + 3z^2 - 4z^3 + \cdots .$$

This pattern can be repeated indefinitely, giving rise to the next sequence

$$1 - 3z + 6z^2 - 10z^3 + \cdots .$$

and so on. Note however that these sequences are not news. Indeed, each is a geometric series, respectively equal to

$$\frac{1}{1+z} \qquad \frac{1}{(1+z)^2} \qquad \frac{1}{(1+z)^3} ,$$

which establish a clear pattern. Evidently then, we find:

$$\frac{1}{z} = \frac{1}{1+z} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)^3} + \cdots$$

If we change variables once more, namely $y = 1/(1+z)$, the original geometric series pops back out:

$$\frac{1}{z} = \frac{1}{1-y} = 1 + y^2 + y^3 + \cdots$$

## 16.3 Zeno's Paradox

An ancient 'paradox' originating in Greece began with Zeno of Elia, as recalled by Aristotle:

*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*

This sounds fine, but then the ancient Greeks take the argument off the rails:

*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.*

In other words, according to Zeno, a moving object can never reach its destination, as it must go half-way *first*, but to reach the half-way point, it has to reach the quarter-way point, and so on. The object will thus never reach its destination, and even worse, it's not clear where the object gets stuck, or if the motion ever started at all.

#### Spatial Sum

While Zeno's paradox has continued to keep certain philosophers busy over the centuries, it is of little concern to anyone aware of geometric series. Consider a racetrack that has a length of precisely one unit, and at the start of this track we place a frog or other jumping device. Following a program of jumps inspired by Zeno, suppose the frog jumps toward the end of the track one half-unit, and then one quarter-unit, and the one eight-unit, and so on. Adding up the length of each jump, the frog moves total distance

$$D = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots ,$$

simplifying to $D = 1$. Evidently, the frog reaches the goal, telling us that jumping 'halfway there' at every step is a good-enough program of motion for reaching a destination, at least mathematically.

Of course, in order to do this, we have assumed a kind of 'ideal frog' that can jump an infinite number of times within a finite duration. While valid in a mathematical sense, the *physical* implausibility of such a frog, which does not matter, still helps to perpetuate Zeno's paradox to this day: if the frog couldn't exist, the math is wrong, and the paradox lives on (or some such nonsense).

#### Temporal Sum

For the unconvinced, we must talk about *real* motion, such as on a racetrack, where the leaping frog is swapped out for something more practical. Suppose instead we observe a rocket moving with constant velocity $V$ along a straight track of length $L$. Without even thinking, we know the relation $V = LT$ must hold, where $T$ is the time required to move along the whole track.

Using geometric series, we may *still* slice up the motion in a Zeno-inspired fashion. Let $t_1$ be the time needed to traverse half of the racetrack. Similarly, let $t_2$ be the time needed to cover one quarter of the track, etc. As a typical 'rate' problem, we have

$$\frac{L}{2} = V \cdot t_1 \qquad \frac{L}{4} = V \cdot t_2 \qquad \frac{L}{8} = V \cdot t_3 ,$$

and so on, for each interval.

Proceed by solving each equation for $t_i$ and take their infinite sum to reconstruct the total time

$$T = t_1 + t_2 + t_3 + \cdots = \frac{L}{V}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) ,$$

after factoring the right side. Now, in order for $V = LT$ to hold, the parenthesized quantity needs to vanish. Perhaps not surprisingly, this is the same series that cropped up previously, and sums to exactly 1. That is, we find

$$T = \frac{L}{V}(1) = \frac{L}{V} ,$$

and we verify that there is no paradox. (In fact, one *could* start with the temporal sum and work backwards to 'discover' the geometric series.)

## 16.4 Applications

### Repeating Decimals

The geometric series can be used to make sense of decimal numbers, particularly numbers with repeating decimals. Consider any decimal number $N$ with any number of repeating digits:

$$N = 0.abcd \ldots abcd \ldots .$$

If the number of repeating digits is $Q$, an equivalent representation of $N$ is given by

$$\begin{aligned} N &= \frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots + \frac{a/10}{10^Q} + \frac{b/100}{10^Q} + \frac{c/1000}{10^Q} + \cdots \\ N &= \left( \frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots \right) \left( 1 + \frac{1}{10^Q} + \frac{1}{10^{2Q}} + \frac{1}{10^{3Q}} + \cdots \right) , \end{aligned}$$

where the sequence on the right is none other than the geometric series with $x = 1/\left(10^Q\right)$. Simplifying, we have

$$N = \left( \frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots \right) \left( \frac{1}{1 - 10^{-Q}} \right) ,$$

where the sequence on the left is *non-repeating*, i.e., terminates after $Q$ terms. If the last unique digit is $q$, we write

$$N = (0.abcd \ldots q) \left( \frac{1}{1 - 10^{-Q}} \right) .$$

Multiplying the numerator and denominator by $10^Q$, we can shed the decimal notation and treat $abcd \ldots q$ as a large integer such that

$$N = \frac{abcd \ldots q}{10^Q - 1} .$$

Evidently, any repeating decimal can be written in the fractional form above. It's also clear why non-repeating decimals cannot be represented as a fraction, as an infinite number of digits $abcd \ldots$ would be needed, and $10^Q$ also becomes infinite.

---

In the special case that $Q = 1$, meaning the decimal has format

$$N = 0.aaaa \ldots ,$$

we have

$$N = a \times \left( \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots \right) ,$$

where the parenthesized quantity is a geometric series with $x = 1/10$, thus

$$N = a \times \left( \frac{1}{1 - 1/10} - 1 \right) = \frac{a}{9} ,$$

in accordance with the general formula above. As a corollary, we can immediately note that

$$\begin{aligned} 1/9 &= 0.1111 \ldots \\ 3/9 &= 0.3333 \ldots \\ 6/9 &= 0.6666 \ldots \\ 8/9 &= 0.8888 \ldots , \end{aligned}$$

and in particular:

$$\frac{9}{9} = 1 = 0.9999 \ldots$$

### Infinite Sequence Analysis

The geometric series helps to evaluate non-obvious sums. For instance, the sum

$$A = \sum_{k=0}^{\infty} \frac{k}{2^k}$$

may appear impenetrable due to the extra $k$ in the numerator, but we can reason around this. Since the first term in the series is identically zero, we have

$$A = 0 + \sum_{k=1}^{\infty} \frac{k}{2^k} \,,$$

and letting $n = k - 1$, the above becomes

$$A = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+1}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n}{2^n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n} \,.$$

On the right, the first sum is simply $A$ again. The second sum should stand out as a standard geometric series, as

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \,.$$

Finally then, we find

$$A = \frac{A}{2} + 1 \,,$$

or

$$A = 2 = \sum_{k=0}^{\infty} \frac{k}{2^k} \,.$$

The same trick can be used to evaluate a harder sum, such as

$$B = \sum_{k=0}^{\infty} \frac{k^2}{2^k} \,.$$

Observing that the leading term is zero, and using the same substitution $n = k - 1$, we get

$$B = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n^2+2n+1}{2^n} = \frac{B}{2} + A + 1 \,,$$

giving

$$B = 6 = \sum_{k=0}^{\infty} \frac{k^2}{2^k} \,.$$

## 16.5 Alternate Derivations

The geometric series can be derived in several ways. With a formal approach finished, some shortcuts can help recover the series in a pinch, or on a rainy afternoon.

### The Division Shortcut

If all you remember is $1/\left(1-x\right)$, the rest can be derived from polynomial division to get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x} \,.$$

**The G-Shortcut**

If you can remember the 'series' part only, let the series equal any variable such as $G$, then multiply through by $x$ to write $xG$:

$$G = 1 + x + x^2 + x^3 + \cdots + x^n$$
$$xG = x + x^2 + x^3 + x^4 + \cdots + x^{n+1}$$

Next, take the difference $G - xG$

$$G - xG = 1 + x - x + x^2 - x^2 + x^3 - x^3 + \cdots - x^{n+1} \ ,$$

and observe that most terms cancel. Factor $(1 - x)$ from the left side to get

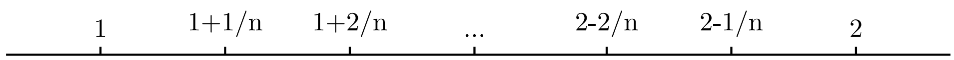
$$G\,(1 - x) = 1 - x^{n+1} \ ,$$

and finally:

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

**Number Line Derivation**

Divide the number line from $x = 1$ to $x = 2$ into $n$ bins of the same width as shown:




Or, written slightly differently:

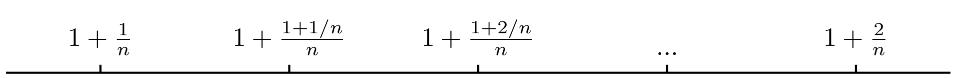

To be clear, each entry is equivalent in each number line, for instance

$$1 + \frac{2}{n} = 2 - \frac{n-2}{n}$$

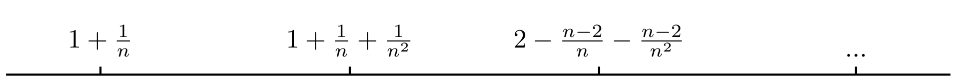
checks out, as so for all matching terms.

To make this less abstract, imagine a Zeno-Inspired 'jumping game' where we begin at the number $x = 1$, and then plan a jump to $x = 2$. However, the game dictates that we may only jump $1/n$th of the way there. Starting at $x = 1$, we thus land at $x = 1 + 1/n$.

This game may repeat: start at $x = 1 + 1/n$, and plan a jump to the next tick mark, $x = 1 + 2/n$, landing $1/n$th of the way there. Divide the interval between the two points into $n$ equal bins to illustrate this:




Simplifying as done above, this number line is equivalent to:




Evidently, we have landed at $x = 1 + 1/n + 1/n^2$.

Running the game again, start at $x = 1 + 1/n + 1/n^2$ plan a jump to the next tick mark located at

$$1 + \frac{1}{n} + \frac{2}{n^2} = 2 - \frac{n-2}{n} - \frac{n-2}{n^2} \ .$$

Of course, we only land $1/n$th of the way there. After drawing the appropriate number line, we end up zoomed in between two points as shown:


Of course, the game would have us divide *that* interval into $n$ bins, and continue on and on. Reading the emerging pattern, we skip the details and write a general restriction for $x$ after an arbitrary number $q$ jumps:

$$1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\cdots+\frac{1}{n^q}<x<2-\frac{n-2}{n}-\frac{n-2}{n^2}-\frac{n-2}{n^3}-\cdots-\frac{n-2}{n^q}\ .$$

Denote the sequence

$$H=1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\cdots+\frac{1}{n^q}$$

such that the inequality reads

$$H<x<2-(n-2)\left(H-1\right)\ .$$

Since $x$ is seated comfortably between the two sides of the inequality, let us momentarily omit $x$ and write

$$H<2-(n-2)\left(H-1\right)\ ,$$

readily simplifying to

$$H<\frac{1}{1-1/n}\ .$$

Restoring all original variables, the above reads

$$1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\cdots+\frac{1}{n^q}<x<\frac{1}{1-1/n}\ .$$

If the game is played such that $q\rightarrow\infty$, we have $x$ being squeezed into a smaller and smaller interval, but $x$ never exceeds $1/\left(1-1/n\right)$. In this regime, the left side of the inequality eventually becomes equal to the right side (both of which equal $x$), so we must have

$$\frac{1}{1-1/n}=1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\cdots\ .$$

Introduce a final variable $z=1/n$ to pull the above into familiar form.

# 17 Factorial Operator

An interesting generalization of multiplication leads to the *factorial* operator, appearing as an exclamation point ( ! ) directly after an integer. The factorial operator transforms an integer into a product of the base number and all lesser integers, as in

$$4! = 4 \times 3 \times 2 \times 1 \; .$$

Factorials aren't easily added or subtracted, but division and multiplication is straightforward:

$$\frac{6!}{3!} \times 4! = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} \times 4 \times 3 \times 2 \times 1 = 4 \times 6!$$

For a general integer $n$, we can play similar games:

$$\begin{aligned} n! &= n\,(n-1)! \\ n! &= n\,(n-1)\,(n-2)! \\ \frac{n!}{(n-3)!} &= n\,(n-1)\,(n-2) \end{aligned}$$

Carrying the above pattern to its limit, we would have

$$\frac{n!}{(n-n)!} = n\,(n-1)\,(n-2)\cdots(1) = n! \; ,$$

implying the special identity for zero-factorial:

$$0! = 1$$

Example 1

Insert any combination of parentheses and operators to make the following statement true:

$$1\;1\;1\;1 = 5$$

$$(1+1+1)! - 1 = 5$$

Example 2

Insert any combination of parentheses and operators to make the following statement true:

$$2\;0\;1\;6 = 1$$

$$\frac{(2+0+1)!}{6} = 1$$

## 17.1 Factorials in Scientific Notation

To convert a factorial number $X!$ to scientific notation, apply the base-ten logarithm operator and use the addition identity to write

$$\log_{10}(X!) = \log_{10}(X) + \log_{10}(X-1) + \log_{10}(X-2) + \cdots + \log_{10}(1) \; .$$

Using summation notation, the above condenses to

$$\log_{10}(X!) = \sum_{k=0}^{X-1} \log_{10}(X-k) = Y \; ,$$

where $Y$ resolves to the sum $C+N$, with $N$ being an integer and $C$ being the fractional component. Thus, we have

$$X! = 10^{(\log_{10}(X!))} = 10^{Y} = 10^{C} \times 10^{N} \; ,$$

or using the so-called 'integer' operator:

$$X! = 10^{Y-\text{int}(Y)} \times 10^{\text{int}(Y)}$$

Example 3

Convert the number $X = 1000!$ to scientific notation.

Step 1: Using a computer or calculator, compute:

$$\sum_{k=0}^{1000-1} \log_{10}(1000-k) = 2567.604644222133$$

Step 2: Identify $Y \approx 2567.6046442221336$, so then

$$C \approx 0.6046442221336 \qquad N = 2567$$

Step 3: Assemble the result using $X = 10^C \times 10^N$:

$$1000! \approx 10^{0.6046442221336} \times 10^{2567} \approx 4.0238726008 \times 10^{2567}$$

## 17.2 Applications

**Infinite Sequence Analysis**

The factorial operator used in conjunction with summation notation can be used to analyze infinite sequences. For an example, let us calculate the infinite sum

$$A = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots .$$

Begin by spotting the pattern in the terms given, where evidently the $n$th term is

$$a_n = \frac{n}{(n+1)!} .$$

Then, using summation notation, we write

$$A = \sum_{k=1}^{\infty} a_n = \sum_{k=1}^{\infty} \frac{n}{(n+1)!} .$$

To proceed, introduce a variable $q = n+1$, and simplify like mad:

$$A = \sum_{q=2}^{\infty} \frac{q-1}{q!} = \sum_{q=2}^{\infty} \frac{q}{q!} - \sum_{q=2}^{\infty} \frac{1}{q!}$$

$$A = \sum_{q=2}^{\infty} \frac{1}{(q-1)!} - \sum_{q=2}^{\infty} \frac{1}{q!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=2}^{\infty} \frac{1}{n!}$$

$$A = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} - \sum_{n=2}^{\infty} \frac{1}{n!}$$

$$A = 1$$

Finally then,

$$1 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots .$$

# 18 Binomial Theorem

## 18.1 Binomial Coefficients

Recall that the expansions of $(a+b)^n$ can be manually attained for any integer $n$ using the distributive property. With enough patience, we eventually generate Pascal's triangle to keep track of the coefficients. That is,

$$\begin{aligned}(a+b)^0 &= 1\\ (a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

leads to:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The individual entries of Pascal's triangle are called *binomial coefficients*, and are contained in the neat formula

$$C_n^m = \frac{n!}{m!\,(n-m)!}\ ,$$

where $n$ is the row counting downward from the top, starting from $n=0$. The number $m$ denotes a given entry counting from the left, also starting from $m=0$. For example, to generate the list 1, 3, 3, 1, we set $n=3$ and write:

$$C_3^0 = \frac{3!}{0!\,(3-0)!} = 1 \qquad C_3^1 = \frac{3!}{1!\,(3-1)!} = 3$$

$$C_3^2 = \frac{3!}{2!\,(3-2)!} = 3 \qquad C_3^3 = \frac{3!}{3!\,(3-3)!} = 1$$

Using binomial coefficients, a general binomial expansion is easily written in summation notation:

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

---

To derive an interesting identity, let $a=b$ in the above to get:

$$2^n = \sum_{m=0}^{n} C_n^m$$

We can also set $a=-b$ such that $a/b=-1$ to gain another identity for $n>0$:

$$0 = \sum_{m=0}^{n} C_n^m\,(-1)^m$$

**Negative Exponents**

Binomial coefficients also come into play when handling expansions of $1/\left(a+b\right)^n$. Recall from our study of geometric series that the equations

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 + \cdots \\
\frac{1}{\left(1+x\right)^2} &= 1 - 2x + 3x^2 - 4x^3 + \cdots \\
\frac{1}{\left(1+x\right)^3} &= 1 - 3x + 6x^2 - 10x^3 + \cdots
\end{aligned}$$

contain infinite sequences on the right. To make sense of these, write a version of Pascal's triangle generated by $\left(a-b\right)^n$ to get:

$$\begin{array}{ccccccccc}
 & & & & +1 & & & & \\
 & & & +1 & & -1 & & & \\
 & & +1 & & -2 & & +1 & & \\
 & +1 & & -3 & & +3 & & -1 & \\
+1 & & -4 & & +6 & & -4 & & +1
\end{array}$$

Starting at the $n$th row down on the left edge, and then reading 'south-east', the above coefficients $\tilde{C}_n^m$ match the diagonal rows of the triangle. For a given diagonal $n$, the sequence runs through all integer $m$ from zero to infinity. That is,

$$\frac{1}{\left(1+x\right)^n} = 1 + \tilde{C}_n^1 x + \tilde{C}_{n+1}^2 x^2 + \tilde{C}_{n+2}^3 x^3 + \cdots = \sum_{m=0}^{\infty} \tilde{C}_{n+m-1}^m x^m$$

Note that for even values of $m$, the coefficients $\tilde{C}_n^m$ are positive, and thereby equal to $C_n^m$, whereas for for negative $m$, the coefficients are simply $-C_n^m$. To capture this, we write

$$\frac{1}{\left(1+x\right)^n} = \sum_{m=0}^{\infty} \left(-1\right)^m C_{n+m-1}^m x^m \,,$$

where the coefficient $C_{n+m}^m$ resolves to

$$C_{n+m-1}^m = \frac{\left(m+n-1\right)!}{m!\,\left(n-1\right)!}\,.$$

For example, to generate the coefficients for $1/\left(1+x\right)^4$, we set $n=4$ and start grinding for all positive integer $m$:

$$C_3^0 = \frac{\left(0+4-1\right)!}{0!\,\left(4-1\right)!} = 1 \qquad C_4^1 = \frac{\left(1+4-1\right)!}{1!\,\left(4-1\right)!} = 4$$

$$C_5^2 = \frac{\left(2+4-1\right)!}{2!\,\left(4-1\right)!} = 10 \qquad C_6^3 = \frac{\left(3+4-1\right)!}{3!\,\left(4-1\right)!} = 20$$

Of course, the whole apparatus generalizes for expansions of $1/\left(a+b\right)^n$. Generalzing the above result, we get:

$$\frac{1}{\left(a+b\right)^n} = \sum_{m=0}^{\infty} \left(-1\right)^m C_{n+m-1}^m \frac{a^m}{b^{n+m}}$$

For another order-four example, we set $n=4$ in the above to generate the expansion of $1/\left(a+b\right)^4$:

$$\begin{aligned}
\frac{1}{\left(a+b\right)^4} &= C_3^0 \frac{a^0}{b^4} - C_4^1 \frac{a^1}{b^5} + C_5^2 \frac{a^2}{b^6} - C_6^3 \frac{a^3}{b^7} + \cdots \\
&= \frac{1}{b^4} - \frac{4a}{b^5} + \frac{10a^2}{b^6} - \frac{20a^3}{b^7} + \cdots
\end{aligned}$$

## 18.2 Binomial Theorem

The summary of our results, namely

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m \qquad C_n^m = \frac{n!}{m!\,(n-m)!}$$

and

$$\frac{1}{(a+b)^n} = \sum_{m=0}^{\infty} (-1)^m C_{n+m-1}^m \frac{a^m}{b^{n+m}} \qquad C_{n+m-1}^m = \frac{(m+n-1)!}{m!\,(n-1)!}\,,$$

constitute the *binomial theorem.*

**Abusing the Theorem**

As derived, the binomial theorem should only work for integer $n$. If we write out the coefficients $C_n^m$ in a certain way though, namely

$$C_n^m = \frac{n!}{m!\,(n-m)!} = \frac{n\,(n-1)\,(n-2)\cdots(n-(m-1))}{m!}\,,$$

we can start to address *try* non-integer $n$.

Looking at

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

however, the upper limit on the summation is clearly nonsense, as $n$ is not an integer. The best move here, originally performed by Newton, is to the let the sum run to *infinity.* This only makes sense if the individual terms in the sum tend to zero for large $m$.

To try this out, let us attempt to calculate $\sqrt{3}$, which should follow from the $n = 1/2$-case of

$$\begin{aligned}\sqrt{3} = (4-1)^n = 4^n\left(1-\left(\frac{1}{4}\right)^n\right) &= 4^n \sum_{m=0}^{\infty} C_n^m 1^{n-m}\left(\frac{-1}{4}\right)^m \\ &= 4^n\left(C_n^0\left(\frac{-1}{4}\right)^0 + C_n^1\left(\frac{-1}{4}\right)^1 + C_n^2\left(\frac{-1}{4}\right)^2 + C_n^3\left(\frac{-1}{4}\right)^3 + \cdots\right)\,,\end{aligned}$$

simplifying to

$$\begin{aligned}\frac{\sqrt{3}}{2} &= 1 + \frac{1/2}{1!}\left(\frac{-1}{4}\right)^1 + \frac{(1/2)\,(1/2-1)}{2!}\left(\frac{-1}{4}\right)^2 + \frac{(1/2)\,(1/2-1)\,(1/2-2)}{3!}\left(\frac{-1}{4}\right)^3 + \cdots \\ &= 1 + \frac{1}{2}\left(\frac{-1}{4}\right)^1 - \frac{1}{8}\left(\frac{-1}{4}\right)^2 + \frac{1}{16}\left(\frac{-1}{4}\right)^3 - \frac{5}{128}\left(\frac{-1}{4}\right)^4 + \frac{7}{256}\left(\frac{-1}{4}\right)^5 - \cdots\,,\end{aligned}$$

leading, quite correctly, to:

$$\sqrt{3} = 1.7320508075\ldots$$

## 18.3 The ‘Choose’ Notation

An alternative notation for the binomial coefficients is the ‘choose’ notation, given by

$$C_n^m = \frac{n!}{m!\,(n-m)!} = \binom{n}{m}\,,$$

which reads ‘$n$ choose $m$’.

**Summation Identity**

Using the so-called ‘choose’ notation, it turns out that

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1},$$

proven by brute force:

$$\begin{aligned}\binom{n}{m} + \binom{n}{m+1} &= \frac{n!}{m!\,(n-m)!} + \frac{n!}{(m+1)!\,(n-(m+1))!} \\ &= \frac{n!\,(m+1)}{(m+1)!\,(n-m)!} + \frac{n!\,(n-m)}{(m+1)!\,(n-m)!} \\ &= \frac{(n+1)!}{(m+1)!\,(n-m)!} \\ &= \binom{n+1}{m+1}\end{aligned}$$

On Pascal’s triangle, the summation identity proves that the sum of a pair of coefficients equals the one ‘below’.

**Reflection Identity**

The ‘axial symmetry’ in Pascal’s triangle is contained in the identity

$$\binom{n}{m} = \binom{n}{n-m},$$

proven as follows:

$$\binom{n}{n-m} = \frac{n!}{(n-m)!\,(n-(n-m))!} = \frac{n!}{m!\,(n-m)!} = \binom{n}{m}$$

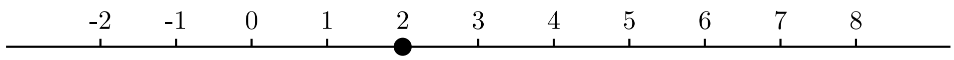
# 19 Concepts of Motion

In the most general sense, the idea of *motion* concerns with change in the position of an object with respect to time. To quantify motion, it is required to set up a local coordinate system that is considered at rest with respect to the objects measured against it. For most purposes, the Earth is considered 'at rest', lending to a myriad of coordinate systems - maps, highway distance markers, latitude/longitude lines, etc., - against which motion can be judged.

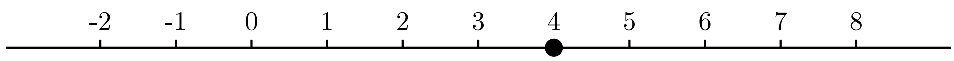
## 19.1 One-Dimensional Motion

The simplest kind of motion takes place in one dimension. This may manifest as a train on a straight track, a meteor coasting in open space, a ball dropped straight from a building, etc. In such a case, we can take all motion to be along the $x$-direction, ignoring other dimensions $y$ and $z$. (Or, choose one sensible variable and ignore the other two.)

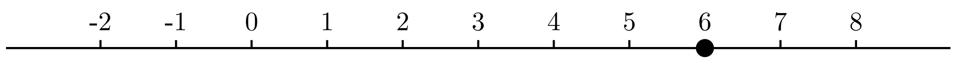
To visualize one-dimensional motion, one may introduce a number line (a large-enough ruler) and decide where where to place $x = 0$. Supposing we're given the position of any object at a given time, such as a train being $x = 2$ kilometers from a station at $t = 1$ hours, we may plot:




Next, suppose the train is $x = 4$ kilometers from the same station at $t = 2$ hours:



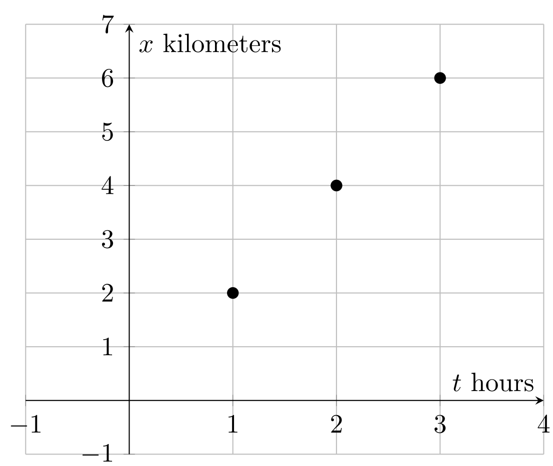

... And then at $t = 3$ hours, suppose the train is $x = 6$ kilometers from the station:




For as long as the train, or any object, maintains its straight-line motion, its position can be represented on a number line at any instant in time.

## 19.2 Time as a Dimension

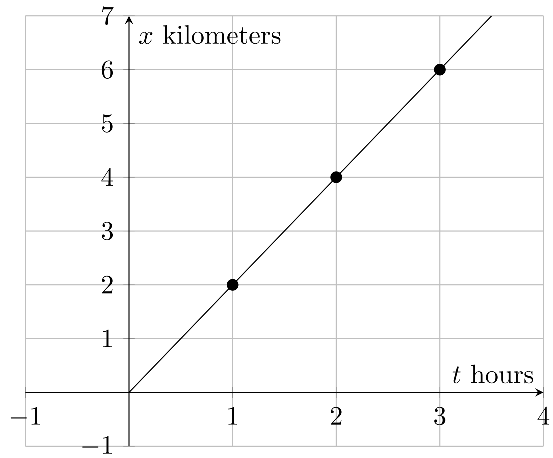
A number line plot representing one-dimensional motion of an object must be updated at each instant in time, which eventually becomes cumbersome. It would be better to include the 'time' variable ($t$) on the same graph as the 'space' variable ($x$), giving rise to a *two-dimensional* plot. Using the train-and-station example above, the same three number lines correspond to three points in the following:




In the above, we may regard the 'time' axis as running horizontally, whereas distance (from the station) extends vertically.

## 19.3 Uniform Linear Motion

Continuing the example of a train moving on a straight track, assume now that the train only moves at *uniform* speed. In this regime, we are allowed to connect the given points with a straight line as shown:

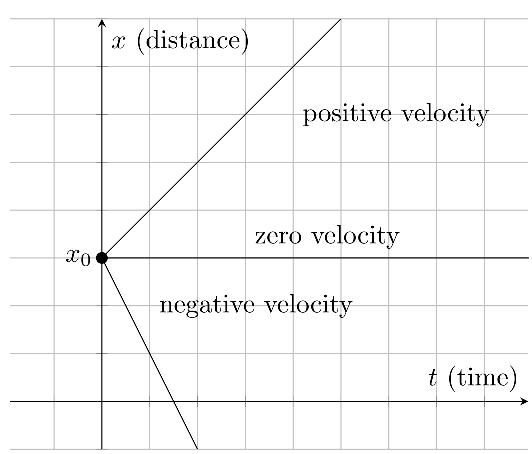



In the graph above, the line connecting the known points illustrates the train's entire motion over several hours. The line extends through the origin ($t = 0, x = 0$), meaning the train leaves the station ($x = 0$) at zero time ($t = 0$). In fact, the straight line obeys *all* of the qualities previously covered in straight line analysis. Instead of the form $y = mx + b$ though, we write a modified version that reflects our change of variables:

$$x = x_0 + vt$$

That is, the position $x$ of a linearly-moving object is given by its initial position $x_0$ plus the quantity $vt$. The variable $x_0$ plays the role of the $y$-intercept. The variable $v$ is called the *velocity*, and is equal to the slope of the line in the $xt$-plot.

In terms of velocity, we can visualize any linear motion starting from a given point $x = x_0$. For positive velocities, the position $x$ steadily increases, whereas for negative velocities, the position steaily decreases. Only the special case $v = 0$ corresponds to a stationary object as shown:




**Example**

On a canoe trip, you paddle upstream for one mile, at which point your hat falls into the river. Ten minutes later, you turn around and catch up to the hat at the starting point of the trip. Calculate the speed of the river.

Denote the speed of the river as $\tilde{v}$, and the local paddle speed of the canoe as $v$. This means the canoe's net speed is $v - \tilde{v}$ upstream, and $v + \tilde{v}$ downstream. Let $D$ equal the total distance traveled by the canoe before turning around, and let $T$ equal the time required for the canoe to return from $D$ to the starting point. So far, we may write two equations:

$$\begin{aligned} 1\,mi + (v - \tilde{v})\,10\,min &= D \\ (v + \tilde{v})\,T &= D \end{aligned}$$

Note that the time required for the hat to float downstream to its starting point is $T$ plus ten minutes, so we may also write

$$1\,mi = \tilde{v}\,(T + 10\,min) \ .$$

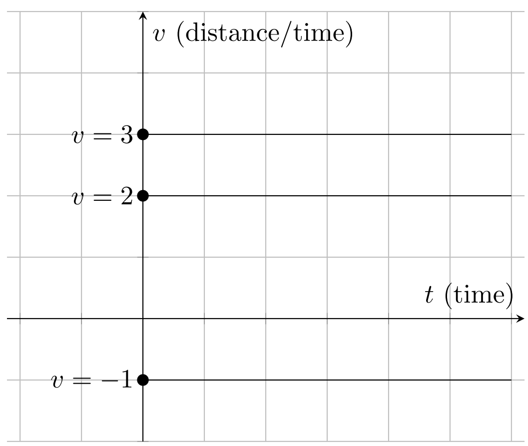
Eliminating $D$ between the first two equations and using the third equation to replace $\tilde{v}T$, we find $T = 10min$, or

$$\tilde{v} = \frac{1\,mi}{20\,min} \ .$$

This result can actually be seen intuitively with no equations. Imagine an observer in a boat that stays with the hat. According to this observer, you swim away for ten minutes, and then swim back in ten minutes. Covering a distance of one mile, we reason $\tilde{v}$ to be one mile per twenty minutes.

## 19.4 Distance as Area

Let us conceive of a new graph that maintains time ($t$) on the horizontal axis, but plots velocity ($v$) on the vertical axis. For a few constant-velocity values, these appear as horizontal lines:




Now comes a crucial observation. *The distance traveled by an object is equal to the* ***area*** *under the velocity plot.* The velocity plot does not indicate the initial position. That is, $x_0$ is completely hidden, however, we can easily calculate displacements from $x_0$, or

$$x - x_0 = \text{(area under velocity plot)}$$

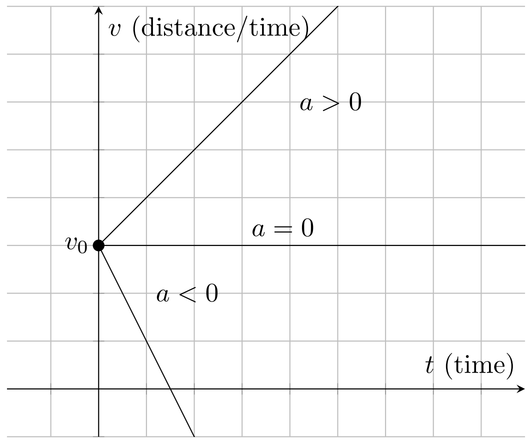
For uniform motion, the area translates to that of a rectangle having height $v$ and width $t$, or

$$x - x_0 = \text{(base)} \cdot \text{(height)} = t \cdot v \ ,$$

simplifying to the familiar $x = x_0 + vt$. Interpreting distance as area, it may turn out that the quantity (base) $\cdot$ (height) comes out negative, which corresponds to backward motion.

## 19.5 Uniform Acceleration

Next, suppose the velocity is itself allowed to increase or decrease in time, a phenomenon called *acceleration*, often denoted $a$. The case of *uniform acceleration* occurs when the velocity changes at a constant rate. The following plot sketches several cases of uniform acceleration:




In the velocity-versus-time graph, uniform acceleration causes $v$ to appear as a straight line, obeying

$$v = v_0 + at \, .$$

The 'intercept' variable $v_0$ is interpreted as the initial velocity. The slope of the graph is the acceleration $a$. One beautiful aspect of this setup is that the change in position is *still* equal to the area under the velocity curve. In this case, we are adding the area of a rectangle to that of a triangle, both having base $t$. That is:

$$x - x_0 = (\text{rectange area}) + (\text{triangle area})$$
$$x - x_0 = (t \cdot v_0) + \left( \frac{1}{2} \cdot t \cdot (v - v_0) \right) \, ,$$

where solving for $x$, we get a position equation that is *quadratic* in time

$$x = x_0 + v_0 t + \frac{a}{2} t^2 \, ,$$

known as the chief equation of *kinematics*.

To get a sense of what a position plot looks like, complete the square in the time variable to write

$$x = \left( x_0 - \frac{v_0^2}{2a} \right) + \frac{a}{2} \left( t + \frac{v_0}{a} \right)^2 \, .$$

The vertex occurring at $(t_{\text{vert}}, x_{\text{vert}})$ is calculated by setting $t = -v_0/a$, giving

$$(t_{\text{vert}}, x_{\text{vert}}) = \left( \frac{-v_0}{a}, x_0 - \frac{v_0^2}{2a} \right) \, .$$

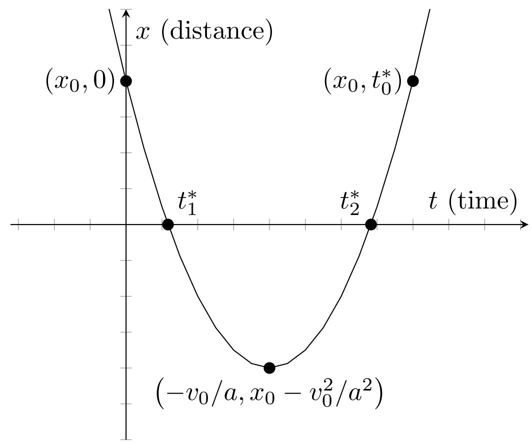
There exists a condition for which the position returns to $x = x_0$, given by

$$t_0^* = \frac{-2v_0}{a} \, .$$

Note that if $v_0/a$ resolves to a positive number, the above condition is not met for positive time values. We may also determine the $t$-intercepts, corresponding to the point(s) satisfying $x = 0$:

$$t_{1,2}^* = \frac{v_0}{a} \left( 1 \pm \sqrt{1 - \frac{2x_0 a}{v_0^2}} \right)$$

The summary of our findings is contained in the following graph, choosing $x_0 > 0$, $v_0 < 0$, $a > 0$:




## 19.6 Kinematic Identities

For an object undergoing uniform acceleration, we already have two equations

$$v = v_0 + at \qquad\qquad x = x_0 + v_0 t + \frac{a}{2}t^2$$

to describe the motion. A simple *kinematic identity* emerges if we solve for $v_0$ to get $v_0 = v - at$, and then insert this factor into the $x$-equation:

$$\begin{aligned} x &= x_0 + (v - at)\,t + \frac{a}{2}t^2 \\ x &= x_0 + vt - \frac{a}{2}t^2 \end{aligned}$$

While equivalent to the original $x$-equation, it obscures the initial velocity $v_0$ in trade for the updated velocity $v$.

To gain another identity, let us add and subtract a factor of $-v_0 t/2$ from the $x$-equation, and then reshuffle the terms as shown

$$\begin{aligned} x &= x_0 + v_0 t + \frac{a}{2}t^2 \\ &= x_0 + v_0 t + \frac{v_0 t}{2} - \frac{v_0 t}{2} + \frac{a}{2}t^2 \\ &= x_0 + \frac{v_0 t}{2} + \frac{1}{2}(v_0 + at)\,t \\ x &= x_0 + \frac{1}{2}(v_0 + v)\,t\,, \end{aligned}$$

telling us that the change in position depends on the *average velocity*

$$\bar{v} = \frac{v_0 + v}{2}$$

multiplied by the time variable:

$$x = x_0 + \bar{v}t$$

Starting with the identity $x = x_0 + \bar{v}t$, we may replace $t$ using $v = v_0 + at$ to derive an equation that elimnates the time variable altogether:

$$\begin{aligned} x - x_0 &= \bar{v}t \\ x - x_0 &= \frac{1}{2}(v_0 + v)\left(\frac{v - v_0}{a}\right) \\ 2a\,(x - x_0) &= (v_0 + v)\,(v - v_0) \\ 2a\,(x - x_0) &= v^2 - v_0^2 \end{aligned}$$

Solving for $v^2$, we land at

$$v^2 = v_0^2 + 2a\,(x - x_0)\ .$$

**Summary**

To summarize, we now have five equations describing one-dimensional motion under constant acceleration, each purposefully containing all but one of the variables in play:

| Identity: | Missing Variable: |
|---|---|
| $v = v_0 + at$ | $x - x_0$ |
| $x = x_0 + v_0 t + at^2/2$ | $v$ |
| $x = x_0 + (v_0 + v)\,t/2$ | $a$ |
| $x = x_0 + vt - at^2/2$ | $v_0$ |
| $v^2 = v_0^2 + 2a\,(x - x_0)$ | $t$ |

## 19.7 Free-fall Acceleration

Near Earth's surface, we know from experience that objects tend to fall downward due to gravity. It turns out that the gravitational effect on a free-falling object resolves to a *constant* acceleration, denoted $g$, having magnitude

$$g = 9.8\,\frac{m}{s^2}\ .$$

If we orient $+x$ as the 'upward' direction, the acceleration due to gravity is $a = -g$.

**Example: Two Balls and a Building**

A ball is thrown straight up at speed $v_0$ from the edge of the roof of a building of height $h$. A second ball is dropped from the roof $t_0$ seconds later. Ignoring air drag, determine the relationship between $v_0$, $h$, and $t_0$ such that the two balls hit the ground simultaneously.

Let $y_1$ represent the trajectory of the first ball, and $y_2$ represent the second ball. According to the information given, we may write

$$y_1 = h + v_0 t - \frac{g}{2}t^2 \qquad\qquad y_2 = h - \frac{g}{2}\,(t - t_0)^2\ ,$$

where $h$ is the height of the building, and $v_0$ is the initial velocity of the first ball. Next, let each ball reach the ground $y_1 = y_2 = 0$ at time $t = t^*$ to write

$$0 = h + v_0 t^* - \frac{g}{2}\,(t^*)^2 \qquad\qquad 0 = h - \frac{g}{2}\,(t^* - t_0)^2\ ,$$

and the job now is to eliminate $t^*$ between both equations. Solving for $t^*$ in each, we have

$$t^* = \frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2h}{g}} \qquad\qquad t^* = t_0 \pm \sqrt{\frac{2h}{g}}\ ,$$

combining to give

$$\frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2h}{g}} = t_0 \pm \sqrt{\frac{2h}{g}}\ .$$

To ease up the notation, let $a = v_0/g$ and $b = 2h/g$ to write

$$a \pm \sqrt{a^2 + b} = t_0 \pm \sqrt{b}\ .$$

Solving for $b$, we easily find

$$b = \frac{\left(a^2 - (a - t_0)^2\right)^2}{4\,(a - t_0)^2} = \frac{\left(2at_0 - t_0^2\right)^2}{4\,(a - t_0)^2} \;,$$

or, after eliminating $a$, $b$ and simplifying:

$$h = \frac{g}{2}\left(\frac{gt_0^2/2 - v_0 t_0}{gt_0 - v_0}\right)^2$$

Since the denominator in the above can never be zero, we must have

$$v_0 < gt_0 = v_{max} \;.$$

That is, if $v_0$ is too large, the second ball is always ahead of the first ball, and hits the ground first. On the other hand, the numerator tells us

$$v_0 > \frac{gt_0}{2} = v_{min} \;,$$

otherwise the first ball will already have passed the roof when the second ball is released, and they cannot hit the ground at the same time.

**Example: Wishing Well Problem**

A stone is dropped from rest into a well at sea level. The splash is heard 2.059 seconds after release. What is the depth of the well? Choices are:

$$20.77\,m \qquad 19.60\,m \qquad 23,564\,m \qquad 18.43\,m \qquad 39.20\,m$$

Take the speed of sound as 330 meters per second. Use no calculator.

From the information given, we have

$$2.059\;s = \frac{D}{330\,m/s} + \sqrt{\frac{2D}{g}} \;,$$

which ends up being quadratic in $D$:

$$(2.059\;s)^2 - 2D\frac{2.059\;s}{330\,m/s} + \frac{D^2}{(330\,m/s)^2} = \frac{2D}{9.8\,m/s^2}$$

Approximating each numerical term (thereby omitting the $D^2$-term due to the large denominator), the above can be rewritten and simplified:

$$4.24 - \frac{4.12\,D}{330\,m} + 0 \approx \frac{D}{4.9\,m} \qquad \rightarrow \qquad D\left(\frac{1}{4.9} + \frac{4.12}{330}\right) \approx 4.24\,m$$

$$D\left(1 + \frac{20.2}{330}\right) \approx 20.78\,m \qquad \rightarrow \qquad D\left(1 + \frac{2}{33}\right) \approx 20.78\,m$$

$$D \approx \frac{20.78\,m}{1.06} \approx 19.60\,m$$

## 19.8 Uniform Jerk

Kicking things up a notch, we may inquire about non-constant acceleration, a phenomenon called *jerk*, denoted $j$. For a uniform jerk $j_0$, the acceleration begins at an initial value $a_0$ and subsequently obeys

$$a = a_0 + j_0 t \;.$$

From here the analysis is almost completely analogous to the uniform-acceleration case, with all variables 'shifted' as follows:

| Equation: | Missing Variable: |
|---|---|
| $a = a_0 + j_0 t$ | $v - v_0$ |
| $v = v_0 + a_0 t + j_0 t^2/2$ | $a$ |
| $v = v_0 + (a_0 + a)\, t/2$ | $j_0$ |
| $v = v_0 + at - j_0 t^2/2$ | $a_0$ |
| $a^2 = a_0^2 + 2j_0\,(v - v_0)$ | $t$ |

Conspicuously absent from the above are any equations for $x$. At the very least, we know that

$$x = x_0 + v_0 t + \frac{a}{2} t^2 + J \ ,$$

where the 'jerk term' $J$ must have something to do with $j_0$ and $t$, but what is the precise relationship? To motivate the result, let us rewrite the above in the format

$$x = x_0\left(t^0\right) + v_0\left(\frac{t^1}{1}\right) + a_0\left(\frac{t^2}{2}\right) + j_0\left(\frac{t^m}{n}\right) \ ,$$

where any unknown information is cast into $m$ and $n$. For ease, let us relabel each term such that

$$x = q_1 + q_2 + q_3 + q_4 \ ,$$

where

$$q_1 = x_0\left(t^0\right) \qquad q_2 = v_0\,(t)$$

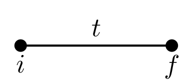
$$q_3 = a_0\left(\frac{t^2}{2}\right) \qquad q_4 = j_0\left(\frac{t^m}{n}\right) \ .$$

**Geometric Analysis**

We shall gain some traction by interpreting each $q$-term *geometrically*. First, we may view $q_1$ as the product of $x_0$ and a dimensionless point, which is called $t^0$ for the sake of starting a pattern, illustrated as shown:

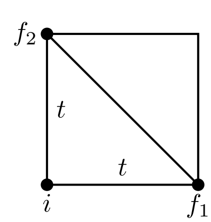
$$q_1 = x_0 \times t^0 = x_0 \times \quad \bullet$$

Next, $q_2$ is the product of $v_0$ and $t$, represented as $v_0$ multiplied by a line of length $t$:

$q_2 = v_0 \times t = x_0 \times$ 


You can imagine 'jumping' from point $i$ to point $f$ in order to step across the 'line of time'.

Now things get interesting. The $q_3$-term is the product of $a_0$ and *half* of the area of a square of side $t$:

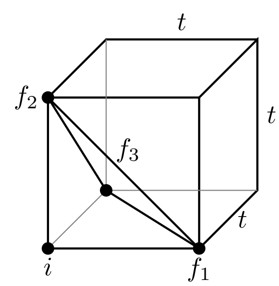
$q_3 = a_0 \times t^2/2 = a_0 \times$ 


Either edge of the triangle can be 'jumped to' starting from point $i$, and and landing at either of $f_1$ or $f_2$. The enclosed shape is a triangle having half of the area of the square, or $t^2/2$.

Finally comes the $q_4$-term. By now, we see see that the geometric interpretation of $q_4$ should extend the square by one dimension, thereby involving a cube, so surely

$$q_4 = j_0 \left( \frac{t^3}{n} \right) .$$

To determine $n$, let us draw a cube of side $t$, and extend the $i$, $f_1$, $f_2$ construction to three dimensions:




Now, we're not looking for an area, but instead the volume bounded by $i$, $f_1$, $f_2$, $f_3$ as shown. This turns out to be exactly one sixth of the volume of the cube, so the geometric interpretation implies $n = 6$. Putting the result together, we finally have a candidate equation for uniform jerk:

$$x = x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 .$$

### Shifted-Time Analysis

To establish this result more carefully, let the time value increase by some amount $h$. In turn, the position should update from $x$ to a new point $x^*$. Carrying this out, we find

$$x^* = x_0 + v_0 (t + h) + \frac{a_0}{2} (t + h)^2 + \frac{j_0}{6} (t + h)^3 ,$$

expanding to

$$x^* = x_0 + v_0 (t + h) + \frac{a_0}{2} \left(t^2 + 2th + h^2\right) + \frac{j_0}{6} \left(t^3 + 3th^2 + 3t^2 h + h^3\right) .$$

Next, combine like terms in powers of $h$ to get

$$x^* = \left( x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 \right) + h \left( v_0 + a_0 t + \frac{j_0}{2} t^2 \right) + \frac{h^2}{2} (a_0 + j_0 t) + \frac{h^3 j_0}{6} ,$$

simplifying, almost miraculously, to

$$x^* = x + vh + \frac{a}{2} h^2 + \frac{j_0}{6} h^3 .$$

At a glance, it seems like nothing new was said, for if we simply lock $t = 0$ and let $h$ play the role of the time variable, we simply get $x = x_0 + v_0 t + a_0 t^2/2 + j_0 t^3/3$. We could thus set up a second time shift $h \to h + w$, and arrive at yet another similar relation, and repeat ad infinitum. Note though, that the same simplification does not occur if the coefficient under $j_0$ was any number other than six.

### Uniform Snap

Of course, even the jerk can have a rate of change, a quantity usually called *snap*, or more traditionally, *jounce*. Denoting a uniform snap as $k_0$, the appropriate kinematic equation, at least up to the unknown factor $A$, looks like

$$x = x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 + \frac{k_0}{A} t^4 ,$$

with a new $t^4$-term tacked onto the end.

Proceeding exactly as above, shift the time variable such that $t \to t + h$, and collect like terms in powers of $h$. The order-four term expands as

$$(\text{snap term}) = \frac{k_0}{A}\left(t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4\right) ,$$

therefore terms in groups of $h^0$, $h^1$, $h^2/2$, and $h^3/6$, respectively, come out to

$$x = x_0 + v_0 t + \frac{a}{2}t^2 + \frac{j_0}{6}t^3 + \frac{k_0}{A}t^4$$

$$v = v_0 + a_0 t + \frac{j_0}{2}t^2 + k_0\frac{4}{A}t^3$$

$$a = a_0 + j_0 t + k_0\frac{12}{A}t^2$$

$$j = j_0 + k_0\frac{24}{A}t \; .$$

The constant $A$ is be determined by any of the above equations, most transparently the last equation. Since we are dealing with uniform snap, it follows that the jerk obeys a linear relationship in time

$$j = j_0 + k_0 t \; ,$$

telling us $A = 24$.

# 20 Two-Dimensional Motion

The ideas of displacement, velocity, acceleration, etc., can be applied to *two-dimensional motion.* When visualizing two dimensions, we (typically) align the horizontal direction with the $x$-axis, and the vertical direction with the $y$ axis. Motion in two dimensions is analyzed using the same equations that apply to the one dimensional case, however most variables gain an $x$- or $y$-subscript. For instance, the constant-acceleration case is generally described by

$$x = x_0 + v_{x_0} t + \frac{a_x}{2} t^2 \qquad y = y_0 + v_{y_0} t + \frac{a_y}{2} t^2$$

$$v_x = v_{x_0} + a_x t \qquad v_y = v_{y_0} + a_y t \ ,$$

and so on.

The total speed of the object, denoted $v_0$, is not simply the sum $v_{x_0}$ of $v_{y_0}$. Instead, the velocity components relate to the speed by the Pythagorean theorem:

$$v = \sqrt{v_{x_0}^2 + v_{y_0}^2}$$

### Free-Fall Case

Typically, we are interested in the kinematics of an object near Earth's surface, in which case the $x$-acceleration is precisely zero, and the $y$-acceleration is $-g$, the local gravitation constant. In this case, the above equations simplify to

$$x = x_0 + v_{x_0} t \qquad y = y_0 + v_{y_0} t - \frac{g}{2} t^2$$

$$v_x = v_{x_0} \qquad v_y = v_{y_0} - gt \ .$$

That is, the $x$-velocity remains constant, and only the $y$-component is subject to acceleration.

## 20.1 Eliminating Time

From one-dimensional motion analysis, we are able to plot, analyze, or ponder $x$ and $y$ on separate graphs with time represented by the horizontal axis, but it would be handy to deal with $x$ and $y$ together, perhaps to generate a $xy$-plot of the motion. To achieve this, solve the $x$-equation for $t$, and replace all instances of $t$ within the $y$-equation:

$$y = y_0 + v_{y_0} \left( \frac{x - x_0}{v_{x_0}} \right) - \frac{g}{2} \left( \frac{x - x_0}{v_{x_0}} \right)^2$$

The quantity $(x - x_0) / v_{x_0}$ plays the role of $t$ in the above, so let us develop this result by completing the square:

$$y = \left( y_0 + \frac{v_{y_0}^2}{2g} \right) - \frac{g}{2} \left( \frac{x - x_0}{v_{x_0}} - \frac{v_{y_0}}{g} \right)^2$$

### Vertex

Now, we can discern that the vertex occurs when the second argument is zero. If this occurs at some special value $x_{vert}$, we find

$$x_{vert} = x_0 + \frac{v_{x_0} v_{y_0}}{g} \ ,$$

implying

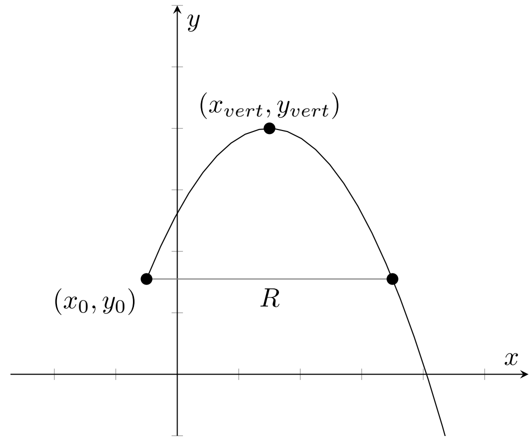
$$y_{vert} = y_0 + \frac{v_{y_0}^2}{2g} \ ,$$

so the equation can be written

$$y = y_{vert} - \frac{g}{2 v_{x_0^2}} \left( x - x_{vert} \right)^2 \ .$$

## 20.2 Two-Dimensional Plot

We can now plot the motion of an object starting from the point $(x_0, y_0)$ at time $t = 0$, subject to uniform gravitational free-fall acceleration:




### Range

In the graph above, we have assumed $v_{x_0} > 0$ and $v_{y_0} > 0$, meaning the object was tossed diagonally upward. The maximum height reached corresponds to $y = y_{vert}$, which depends only on $v_{y_0}$ and $g$. Eventually, the object returns to its original height, and the displacement along $x$ at which this occurs is call the *range*, denoted $R$. Working this out, we find

$$y_{vert} - y_0 = \frac{v_{y_0}^2}{2g} = \frac{g}{2v_{x_0}^2}\left(R - x_{vert}\right)^2 \qquad \to \qquad R = x_{vert} + \frac{v_{x_0} v_{y_0}}{g} \ ,$$

or

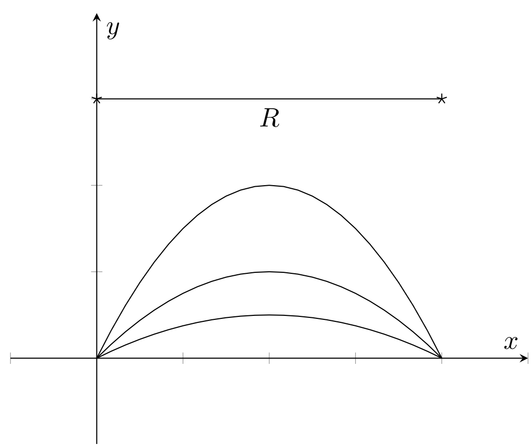
$$R = x_0 + \frac{2v_{x_0} v_{y_0}}{g} \ .$$

### Range Equivalence

Looking back the the 'horizontal range' equation

$$R = x_0 + \frac{2v_{x_0} v_{y_0}}{g} \ ,$$

we can quickly observe that mutually swapping $v_{x_0} \leftrightarrow v_{y_0}$ leaves the range unchanged. In fact, the range remains the same so long as the combination $v_{x_0} v_{y_0}$ remains constant. Of course, the resulting trajectories appear different as sketched below:



**Flight Time**

Starting at $t = 0$ with $x = x_0$, $y = y_0$, the flight time of an object can be calculated from either the $x$-equation or the $y$-equation. Choosing the easier case, we know

$$t = \frac{x - x_0}{v_{x_0}} \, ,$$

where substituting $R/2$ and $R$, respectively, gives the flight time to the vertex and to the 'range' point:

$$t_{vert} = \frac{x_{vert} - x_0}{v_{x_0}} = \frac{v_{y_0}}{g}$$

$$t_R = \frac{R - x_0}{v_{x_0}} = \frac{2v_{y_0}}{g}$$

For completeness, it's worth having an equation that delivers $t$ for a given value of $y$. To do so, begin with $y = y_0 + v_{y_0} t - gt^2/2$, and complete the square to get

$$y = y_{vert} - \frac{g}{2}\left(t - \frac{v_{y_0}}{g}\right)^2 \, .$$

Continuing to solve for $t$, observe there are two solutions to $t$ for any given $y$:

$$t = t_{vert} \pm \sqrt{\frac{2}{g}\left(y_{vert} - y\right)}$$

## 20.3 Envelope of Trajectories

For free-fall motion, the initial velocities $v_{x_0}$, $v_{y_0}$ determine the overall shape of the parabolic trajectory over time. If we have $v_{x_0} = 0$, the initial motion is consistently straight upward, whereas if $v_{y_0} = 0$, the motion begins horizontally, but for a vanishing instant before gravity shows an effect. For all cases with $v_{x_0}$, $v_{y_0}$ nonzero, we get a parabolic trajectory as previously studied. The so-called *envelope of trajectories* concerns all possible paths of motion that can occur form a given starting point, supposing the initial speed is fixed. That is, we seek all pairs $v_{x_0} > 0$, $v_{y_0} > 0$ that satisfy

$$\text{constant} = v_0 = \sqrt{v_{x_0}^2 + v_{y_0}^2} \, .$$

To proceed, begin with

$$y = y_0 + v_{y_0}\left(\frac{x - x_0}{v_{x_0}}\right) - \frac{g}{2}\left(\frac{x - x_0}{v_{x_0}}\right)^2 \, ,$$

and then note that

$$\frac{1}{v_{x_0}^2} = \frac{1}{v_0^2}\left(1 + \frac{v_{y_0}^2}{v_{x_0}^2}\right)$$

to get

$$y = y_0 + \frac{v_{y_0}}{v_{x_0}}\left(x - x_0\right) - \frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\left(1 + \frac{v_{y_0}^2}{v_{x_0}^2}\right) \, ,$$

which is in fact a quadratic equation in the variable $v_{y_0}/v_{x_0}$, particularly

$$y = \left(y_0 - \frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right) + \left(x - x_0\right)\frac{v_{y_0}}{v_{x_0}} - \left(\frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right)\frac{v_{y_0}^2}{v_{x_0}^2} \, ,$$

or in more manageable form:

$$
\begin{aligned}
0 &= A\left(\frac{v_{y_0}}{v_{z_0}}\right)^2 + B\left(\frac{v_{y_0}}{v_{z_0}}\right) + C \\
A &= -\left(\frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right) \\
B &= \left(x - x_0\right) \\
C &= \left(y_0 - \frac{g}{2}\left(x - x_0\right)^2 \frac{1}{v_0^2}\right) - y = -y + y_0 + A
\end{aligned}
$$

Supposing we needed to solve for the variable $v_{y_0}/v_{x_0}$, surely we would use the quadratic formula

$$
\frac{v_{y_0}}{v_{x_0}} = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \, ,
$$

but the crucial quantity is $B^2 - 4AC$. As per usual, this must resolve to a positive number, or zero at the very least. The special case $B^2 = 4AC$ defines an 'edge' that divides valid solutions from invalid solutions. This is in fact the envelope of trajectories. Setting $B^2 = 4AC$ and simplifying, we find

$$
\begin{aligned}
\left(x - x_0\right)^2 &= -4A\left(y - y_0\right) + 4A^2 \\
\frac{-2\cancel{A}v_0^2}{g} &= -4\cancel{A}\left(y - y_0\right) + 4A \cdot \cancel{A} \\
y - y_0 &= \frac{v_0^2}{2g} + A \, ,
\end{aligned}
$$

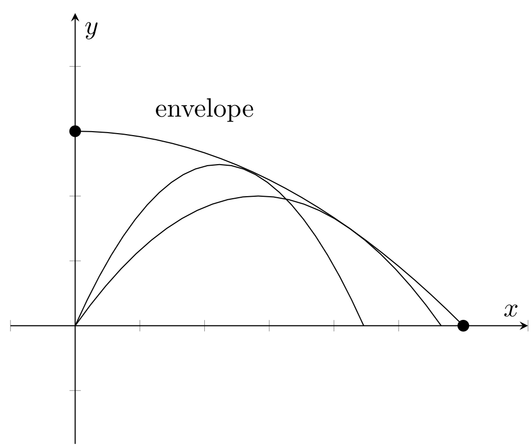
and finally,

$$
y - y_0 = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}\left(x - x_0\right)^2
$$

We see the envelope of trajectories is itself another quadratic equation, intercepting the $y$- and $x$-axes respectively at

$$
y_{int} = y_0 + \frac{v_0^2}{2g} \qquad\qquad x_{int} = x_0 + \frac{v_0^2}{g} \, .
$$

For simplicity, let us set $x_0 = y_0 = 0$ to generate the following graph, showing the envelope of trajectories enclosing several typical paths of motion:




**Maximal Range**

For given values of $v_{x_0}$ and $v_{y_0}$, we know the horizontal range of a free-falling object is

$$
R = x_0 + \frac{2v_{x_0}v_{y_0}}{g} \, .
$$

Meanwhile, for a given speed $v_0$, the greatest possible range corresponds to the extreme edge of the envelope of trajectories,

$$x_{int} = x_0 + \frac{v_0^2}{g} \ .$$

Equating these, we come up with a restriction on $v_0$, $v_{x_0}$, $v_{y_0}$ that corresponds to the maximal range:

$$2v_{x_0}v_{y_0} = v_0^2$$

To proceed, recall that $v_0^2 = v_{x_0}^2 + v_{y_0}^2$, and factor the result:

$$\begin{aligned} 0 &= v_{x_0}^2 + v_{y_0}^2 - v_{x_0}v_{y_0} \\ 0 &= (v_{x_0} + v_{y_0})\,(v_{x_0} - v_{y_0}) \end{aligned}$$

Evidently, the two 'solutions' for $v_{x_0}$ are $\pm v_{y_0}$, and the negative result may be discarded. Finally, we find that

$$v_{x_0} = v_{y_0} = \frac{v_0}{\sqrt{2}}$$

corresponds to the maximal-range trajectory.

# 21 Taylor Polynomial

## 21.1 Return to 1D Motion

Our study of one-dimensional motion went atypically deep, eventually turning up a kinematic equation involving position, velocity, acceleration, jerk, and snap, each related to escalating orders of time:

$$x = x_0 + v_0 t + \frac{a_0}{2}t^2 + \frac{j_0}{6}t^3 + \frac{k_0}{24}t^4$$

The initial values $x_0$, $v_0$, $a_0$, $j_0$ each correspond to $t = 0$, where otherwise $x$, $v$, $a$, $j$ each vary with time.

Recall that, in order to justify the coefficients under $j_0$ and $k_0$, we introduced a shift in the time such that $t \to t + h$, which gave rise to the remarkable equation

$$x^* = x + vh + \frac{a}{2}h^2 + \frac{j}{6}h^3 + \frac{k_0}{24}h^4 \, .$$

The updated position $x^*$ corresponds to the position at the shifted time $t + h$. Each variable on the right, namely $x$, $v$, $a$, and $j$, occur as their *real-time* values at time $t$. (This is quite different than each having the 0-subscript, which bases all motion from $t = 0$.) For completeness, these evaluate to:

$$\begin{aligned} v &= v_0 + a_0 t + \frac{j_0}{2}t^2 + \frac{k_0}{6}t^3 \\ a &= a_0 + j_0 t + \frac{k_0}{2}t^2 \\ j &= j_0 + k_0 t \end{aligned}$$

## 21.2 Change of Base-Point

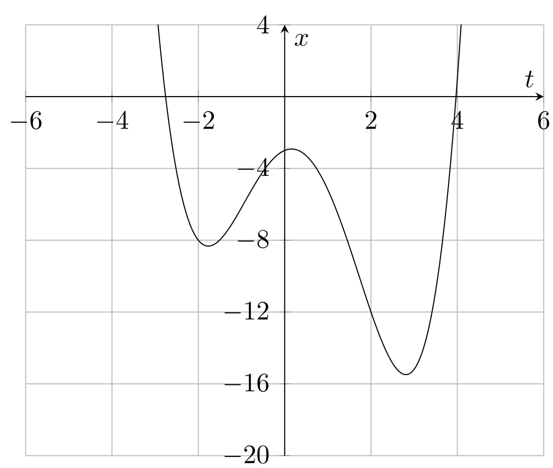
To illustrate what the $x^*$-equation does in practice, consider the following trajectory with coefficients

$$x_0 = -3, \qquad v_0 = 1, \qquad \frac{a_0}{2} = -3, \qquad \frac{j_0}{6} = \frac{-1}{2}, \qquad \frac{k_0}{24} = \frac{5}{16} \, ,$$

such that

$$x = -3 + t - 3t^2 - \frac{1}{2}t^3 + \frac{5}{16}t^4 \, ,$$

plotted below:



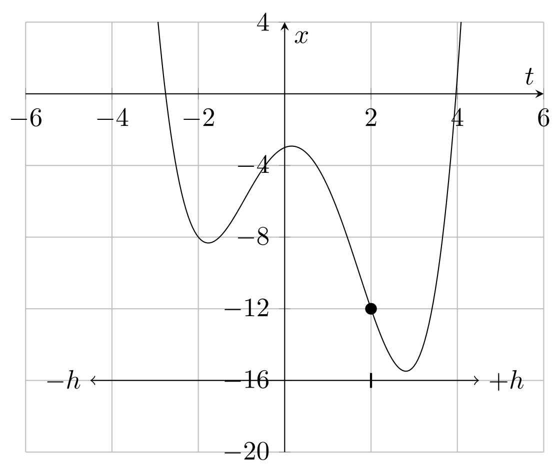
Choosing any time-value such as $t = 2$, we next compute $x$, $v$, $a$, and $j$ at this value:

$$
\begin{aligned}
x_{t=2} &= \left( x_0 + v_0 t + \frac{a_0}{2} t^2 + \frac{j_0}{6} t^3 + \frac{k_0}{24} t^4 \right) \Bigg|_{t=2} = -12 \\
v_{t=2} &= \left( v_0 + a_0 t + \frac{j_0}{2} t^2 + \frac{k_0}{6} t^3 \right) \Bigg|_{t=2} = -7 \\
a_{t=2} &= \left( a_0 + j_0 t + \frac{k_0}{2} t^2 \right) \Bigg|_{t=2} = 3 \\
j_{t=2} &= \left( j_0 + k_0 t \right) \Bigg|_{t=2} = 12 \\
k_0 &= \frac{15}{2}
\end{aligned}
$$

Thus, the $x^*$-equation evaluated at $t = 2$ resolves to

$$x^*_{t=2} = -12 - 7h + \frac{3}{2} h^2 + 2h^3 + \frac{5}{16} h^4 \, .$$

Plotting this solution, we get a curve identical to the original one:




It takes little to imagine that, instead of 'centering on' $t = 2$, we could have based the new curve on any *base-point* $t_0$. The corresponding $x^*$-equation will have a different set of coefficients fitting the form

$$x^*_{t_0} = x_{t_0} + v_{t_0} h + \frac{a_{t_0}}{2} h^2 + \frac{j_{t_0}}{6} h^3 + \frac{k_{t_0}}{24} h^4 \, .$$

## 21.3 Notation and Bookkeeping

If the pattern was not obvious yet, we may replace each denominator by its equivalent representation as a factorial:

$$x^*_{t_0} = x_{t_0} + v_{t_0} h + \frac{a_{t_0}}{2!} h^2 + \frac{j_{t_0}}{3!} h^3 + \frac{k_{t_0}}{4!} h^4 \, .$$

Next, recall that the effective time variable $h$ is referenced from the base value $t_0$ via

$$h = t - t_0 \, ,$$

thus $h$ can be eliminated:

$$x^*_{t_0} = x_{t_0} + v_{t_0} \left( t - t_0 \right) + \frac{a_{t_0}}{2!} \left( t - t_0 \right)^2 + \frac{j_{t_0}}{3!} \left( t - t_0 \right)^3 + \frac{k_{t_0}}{4!} \left( t - t_0 \right)^4 \, .$$

We may further conceive that the snap $k_0$ itself can vary with time, and so may *its* rate of change, and so on ad infinitum. Supposing there are $n$ *total* terms on the right, the above generalizes to

$$x^*_{t_0} = x_{t_0} + v_{t_0}\left(t - t_0\right) + \frac{a_{t_0}}{2!}\left(t - t_0\right)^2 + \cdots + \frac{k_n}{n!}\left(t - t_0\right)^n \ ,$$

where the final coefficient $k_n$ is a constant in time.

## 21.4 Change of Domain

While all of the results so far have been framed as the variable $x$ changing with respect to time, the whole apparatus, by complete analogy, applies to a curve $y$ that depends on $x$. By changing from the $t$-domain to the $x$-domain, the quantity $t - t_0$ becomes $x - x_0$, and $x^*_{t_0}$ becomes $y^0_{x_0}$. Furthermore, the base-value velocity $v_{t_0}$ becomes the base-value slope $m_{x_0}$. Since acceleration, jerk, snap, etc., must be renamed, let us introduce the notation $m^{(q)}$ such that:

$$\begin{aligned} v_{t_0} &\rightarrow m^{(1)}_{x_0} \\ a_{t_0} &\rightarrow m^{(2)}_{x_0} \\ j_{t_0} &\rightarrow m^{(3)}_{x_0} \\ k_{t_0} &\rightarrow m^{(4)}_{x_0} \end{aligned}$$

Specifically, each of the $m^{(q)}_{x_0}$ at a given base-point $x_0$ come from their initial values $m^{(q)}_0$ such that

$$\begin{aligned} y_{x_0} &= \left(y_0 + m^{(1)}_0 x + \frac{m^{(2)}_0}{2!}x^2 + \frac{m^{(3)}_0}{3!}x^3 + \frac{m^{(4)}_0}{4!}x^4 + \cdots\right)\Bigg|_{x=x_0} \\ m^{(1)}_{x_0} &= \left(m^{(1)}_0 + m^{(2)}_0 x + \frac{m^{(3)}_0}{2!}x^2 + \frac{m^{(4)}_0}{3!}x^3 + \frac{m^{(5)}_0}{4!}x^4 + \cdots\right)\Bigg|_{x=x_0} \\ m^{(2)}_{x_0} &= \left(m^{(2)}_0 + m^{(3)}_0 x + \frac{m^{(4)}_0}{2!}x^2 + \frac{m^{(5)}_0}{3!}x^3 + \frac{m^{(6)}_0}{4!}x^4 + \cdots\right)\Bigg|_{x=x_0} \\ m^{(3)}_{x_0} &= \left(m^{(3)}_0 + m^{(4)}_0 x + \frac{m^{(5)}_0}{2!}x^2 + \frac{m^{(6)}_0}{3!}x^3 + \frac{m^{(7)}_0}{4!}x^4 + \cdots\right)\Bigg|_{x=x_0} , \end{aligned}$$

and so on.

---

Pulling our achievements all together, we have

$$y^*_{x_0} = y_{x_0} + m^{(1)}_{x_0}\left(x - x_0\right) + \frac{m^{(2)}_{x_0}}{2!}\left(x - x_0\right)^2 + \cdots + \frac{m^{(n)}_{x_0}}{n!}\left(x - x_0\right)^n \ ,$$

or in summation notation,

$$y^*_{x_0} = y_{x_0} + \sum_{q=1}^{n} \frac{m^{(q)}_{x_0}}{q!}\left(x - x_0\right)^q \ ,$$

a result called a *Taylor polynomial*. If the number of terms is infinite, the polynomial becomes a *Taylor series*. The special case $x_0 = 0$ of a Taylor series is known as a *Maclaurin series*.

## 21.5 Applications

### Experimental Rocket

Suppose an experimental rocket that moves with constant jerk $J$ is observed on a linear track. If the rocket is measured at position $X$ at time $T$ with velocity $V$ and acceleration $A$, determine the path of motion for all times before and after $T$. What must be the 'initial' position, velocity, and acceleration at $t = 0$?

As a Taylor polynomial, the entire path of motion is immediately given by

$$x_T^* = X + V(t-T) + \frac{A}{2}(t-T)^2 + \frac{J}{6}(t-T)^3 \ .$$

Meanwhile, the same path of motion, in terms of the initial launch parameters, is given by

$$x = x_0 + v_0 t + \frac{a_0}{2}t^2 + \frac{j_0}{6}t^3 \ .$$

Exploiting the Taylor polynomial, note that both $x_T^*$ and $x$ are related by a change of base-point, namely from $t = T$ to $t = 0$. Then, the initial and final launch parameters are related by

$$\begin{aligned}
X &= \left( x_0 + v_0 t + \frac{a_0}{2}t^2 + \frac{j_0}{6}t^3 \right)\bigg|_T \\
V &= \left( v_0 + a_0 t + \frac{j_0}{2}t^2 \right)\bigg|_T \\
A &= (a_0 + j_0 t)\bigg|_T \\
J &= j_0 \ .
\end{aligned}$$

After simplifying a little, we are left with a system of three linear equations with three unknowns

$$\begin{aligned}
0 &= -X + x_0 + v_0 T + \frac{a_0}{2}T^2 + \frac{J}{6}T^3 \\
0 &= -V + v_0 + a_0 T + \frac{J}{2}T^2 \\
0 &= -A + a_0 + JT \ ,
\end{aligned}$$

easily solved by hand:

$$\begin{aligned}
x_0 &= X - VT + \frac{A}{2}T^2 - \frac{J}{6}T^3 \\
v_0 &= V - AT + \frac{J}{2}T^2 \\
a_0 &= A - JT
\end{aligned}$$

Perhaps not surprisingly, the $x_0$-equation is what we get by setting $t = 0$ in the $x_T^*$-equation, a nice reality check. Alongside this are the results for $v$ and $a$ at $t = 0$.

**Path Between Points (Optional)**

Consider two fixed points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in the Cartesian plane. The simplest way to connect the pair of points is to a draw straight line between $p_1$ and $p_2$, and this is surely the shortest path. However, we may be interested in paths that are not the shortest, but perhaps 'smoother' ones. This only makes sense if we have information about the slope of the path as it intersects $p_1$ and $p_2$, so let us consider each slope $m_{x_1}^{(1)}$, $m_{x_2}^{(1)}$ fixed. To add extra precision, take $m_{x_1}^{(2)}$, $m_{x_2}^{(2)}$ fixed as well, analogous to specifying acceleration in the time domain. Let all other higher terms $m_{x_{1,2}}^{(n>2)}$ equal zero.

To find a path that 'smoothly' connects $p_1$ to $p_2$, postulate the solution as a polynomial with unknown coefficients:

$$y = y_0 + m_0^{(1)}x + \frac{m_0^{(2)}}{2}x^2 + \frac{m_0^{(3)}}{6}x^3 + \frac{m_0^{(4)}}{24}x^4 + \frac{m_0^{(5)}}{120}x^5$$

Then, using the information given, we have:

$$\begin{aligned}
y_{x_1} &= \left( y_0 + m_0^{(1)} x + \frac{m_0^{(2)}}{2} x^2 + \frac{m_0^{(3)}}{6} x^3 + \frac{m_0^{(4)}}{24} x^4 + \frac{m_0^{(5)}}{120} x^5 \right) \Bigg|_{x=x_1} \\
m_{x_1}^{(1)} &= \left( m_0^{(1)} + m_0^{(2)} x + \frac{m_0^{(3)}}{2} x^2 + \frac{m_0^{(4)}}{6} x^3 + \frac{m_0^{(5)}}{24} x^4 \right) \Bigg|_{x=x_1} \\
m_{x_1}^{(2)} &= \left( m_0^{(2)} + m_0^{(3)} x + \frac{m_0^{(4)}}{2} x^2 + \frac{m_0^{(5)}}{6} x^3 \right) \Bigg|_{x=x_1}
\end{aligned}$$

$$\begin{aligned}
y_{x_2} &= \left( y_0 + m_0^{(1)} x + \frac{m_0^{(2)}}{2} x^2 + \frac{m_0^{(3)}}{6} x^3 + \frac{m_0^{(4)}}{24} x^4 + \frac{m_0^{(5)}}{120} x^5 \right) \Bigg|_{x=x_2} \\
m_{x_2}^{(1)} &= \left( m_0^{(1)} + m_0^{(2)} x + \frac{m_0^{(3)}}{2} x^2 + \frac{m_0^{(4)}}{6} x^3 + \frac{m_0^{(5)}}{24} x^4 \right) \Bigg|_{x=x_2} \\
m_{x_2}^{(2)} &= \left( m_0^{(2)} + m_0^{(3)} x + \frac{m_0^{(4)}}{2} x^2 + \frac{m_0^{(5)}}{6} x^3 \right) \Bigg|_{x=x_2}
\end{aligned}$$

That is, we get a system of six equations and six unknowns, which has a definite solution. Proceeding by hand would be daunting, so let us write the above as an augmented matrix:

$$\begin{bmatrix}
1 & x_1 & x_1^2/2 & x_1^3/6 & x_1^4/24 & x_1^5/120 & y_{x_1} \\
1 & x_2 & x_2^2/2 & x_2^3/6 & x_2^4/24 & x_2^5/120 & y_{x_2} \\
0 & 1 & x_1 & x_1^2/2 & x_1^3/6 & x_1^4/24 & m_{x_1}^{(1)} \\
0 & 1 & x_2 & x_2^2/2 & x_2^3/6 & x_2^4/24 & m_{x_2}^{(1)} \\
0 & 0 & 1 & x_1 & x_1^2/2 & x_1^3/6 & m_{x_1}^{(2)} \\
0 & 0 & 1 & x_2 & x_2^2/2 & x_2^3/6 & m_{x_2}^{(2)}
\end{bmatrix}$$

This is as far we we can drive the analysis without choosing specific numbers for $x_1$, $x_2$, along with each term in the right-most column of the augmented matrix.

---

To continue, let us choose a few example numbers such that

$$\begin{aligned}
p_1 &= (-2,-2) & m_{x_1}^{(1)} &= 1 & m_{x_1}^{(2)} &= -\frac{1}{2} \\
p_1 &= \left(1, \frac{5}{4}\right) & m_{x_1}^{(1)} &= \frac{1}{2} & m_{x_1}^{(2)} &= \frac{1}{2} \,,
\end{aligned}$$

and the above matrix becomes

$$\begin{bmatrix}
1 & (-2) & (-2)^2/2 & (-2)^3/6 & (-2)^4/24 & (-2)^5/120 & -2 \\
1 & (1) & (1)^2/2 & (1)^3/6 & (1)^4/24 & (1)^5/120 & 5/4 \\
0 & 1 & (-2) & (-2)^2/2 & (-2)^3/6 & (-2)^4/24 & 1 \\
0 & 1 & (1) & (1)^2/2 & (1)^3/6 & (1)^4/24 & 1/2 \\
0 & 0 & 1 & (-2) & (-2)^2/2 & (-2)^3/6 & -1/2 \\
0 & 0 & 1 & (1) & (1)^2/2 & (1)^3/6 & 1/2
\end{bmatrix},$$

which is suitable to be solved by Gaussian elimination. Letting a machine do the work, the row-reduced echelon form of the matrix is

$$\begin{bmatrix}
43/81 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 94/81 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 \cdot 149/324 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -6 \cdot 23/162 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 24 \cdot 19/162 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 120 \cdot 7/162 & 1
\end{bmatrix},$$

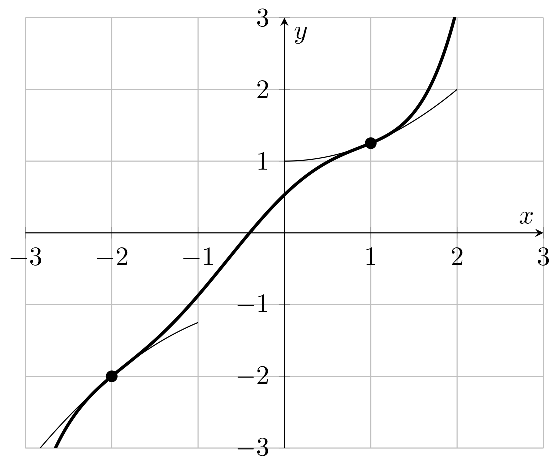
meaning:

$$y_0 = \frac{43}{81} \qquad m_0^{(1)} = \frac{94}{81} \qquad m_0^{(2)} = \frac{-2 \cdot 149}{324}$$
$$m_0^{(3)} = \frac{-6 \cdot 23}{162} \qquad m_0^{(4)} = \frac{24 \cdot 19}{162} \qquad m_0^{(5)} = \frac{120 \cdot 7}{162}$$

Finally then, we have the solution,

$$y = \frac{43}{81} + \frac{94}{81}x - \frac{149}{324}x^2 - \frac{23}{162}x^3 + \frac{19}{162}x^4 + \frac{7}{162}x^5 \, .$$

Plotting this solution while including the 'target' points, along with some indication of the slope at each, we generate a beautiful graph:



# 22 Euler's Constant

## 22.1 Derivation

Using the binomial theorem, a curious result emerges while expanding $(a+b)^n$ with $a = 1$ and $b = 1/n$. Inserting these values into

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m \qquad\qquad C_n^m = \frac{n!}{m!\,(n-m)!} \ ,$$

the above becomes

$$\left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m \frac{1}{n^m} = \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}\frac{1}{n^m} \ .$$

As a side exercise, let us examine the factorial terms involving the integer $n$. By definition, we know

$$n! = n\,(n-1)\,(n-2)\,(n-3)\cdots(1) \ ,$$

or

$$\frac{n!}{(n-m)!} = n\,(n-1)\,(n-2)\,(n-3)\cdots(n-m+1) \ .$$

On the right, the product of $n$-terms expands to a very messy polynomial. Denoting the coefficients $P_m$, the above becomes

$$\frac{n!}{(n-m)!} = P_m n^m + P_{m-1} n^{m-1} + P_{m-2} n^{m-2} + \cdots P_1 n = \sum_{k=1}^{m} P_k n^k \ .$$

Of course, the end coefficient is always $P_m = 1$, so we pluck this term from the infinite series to write

$$\frac{n!}{(n-m)!} = n^m + \sum_{k=1}^{m-1} P_k n^k = n^m + q \ ,$$

where the summation is now stored in a variable $q$. So far then, we have

$$\left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} \frac{1}{m!}\frac{n^m+q}{n^m} \ .$$

Next, let us take the integer $n$ to be extremely large, perhaps infinitely large. Recall that the quantity $q$ contains a mess of $n$-terms and coefficients, however all of these are of power $m-1$ or less. Thus, for large-enough $n$, the raw $n^m$-term dominates over $q$, so $q$ is *effectively* forgotten. That is,

$$\frac{n^m+q}{n^m} \approx \frac{n^m}{n^m} = 1 \qquad\qquad n, m \gg q \gg 1 \ ,$$

so the above becomes, in the very large-$n$ regime,

$$\left(1+\frac{1}{n}\right)^n \approx \sum_{m=0}^{n} \frac{1}{m!} \ .$$

On the left side, we see inside the parentheses that $1/n$ becomes very small for very large $n$, so $1 + 1/n$ is just barely greater than one. Raising this quantity to the power $n$, the result grows again, but we cannot predict by how much from pure inspection. Trying a few $n$-values, we find:

$$\begin{aligned}
n &= 1 & \left(1+\frac{1}{1}\right)^1 &= 2.0\\
n &= 2 & \left(1+\frac{1}{2}\right)^2 &= 2.25\\
n &= 3 & \left(1+\frac{1}{3}\right)^3 &\approx 2.370\\
n &= 10 & \left(1+\frac{1}{10}\right)^{10} &\approx 2.5937\\
n &= 100 & \left(1+\frac{1}{100}\right)^{100} &\approx 2.7048\\
n &= 1000000 & \left(1+\frac{1}{1000000}\right)^{1000000} &\approx 2.71828
\end{aligned}$$

Evidently, the result never goes above 3, but rather settles on

$$e = 2.7182818284590\ldots\ ,$$

an irrational number called *Euler's constant*, denoted $e$. Getting back to the problem on hand, let us kick $n$ all the way to infinity, finally giving:

$$e = \sum_{m=0}^{\infty} \frac{1}{m!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

## 22.2 Natural Exponent

Starting with the approximate sum in the derivation above, we can change the summation variable via $n \to n/x$ to produce

$$\left(1+\frac{x}{n}\right)^{n/x} \approx \sum_{m=0}^{n/x} \frac{1}{m!}\ ,$$

where $n$ is still considered to be a very large integer. Raising each side to the power $x$, we have

$$\left(1+\frac{x}{n}\right)^{n} \approx \left(\sum_{m=0}^{n/x} \frac{1}{m!}\right)^x\ .$$

Meanwhile, the previous derivation can be repeated with $b = x/n$, so we pick up an extra $x^m$ the whole way through:

$$\left(1+\frac{x}{n}\right)^{n} \approx \sum_{m=0}^{n} \frac{x^m}{m!}$$

Eliminating $(1 + x/n)^n$ between both results, and also letting $n$ run to infinity, we find

$$\left(\sum_{m=0}^{\infty} \frac{1}{m!}\right)^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}\ ,$$

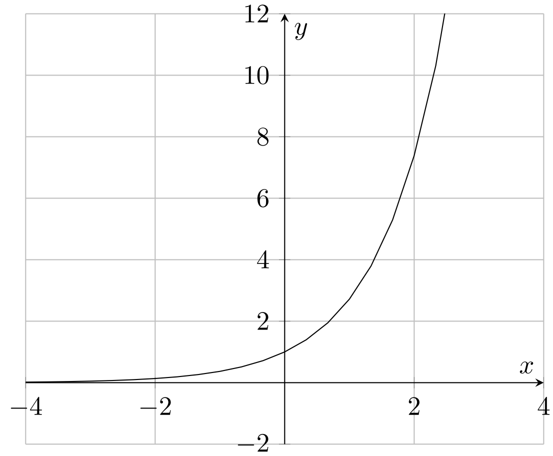
translating to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\ .$$

Of course, we may replace $x$ with the product $ax$, where $a$ is called a *scale factor*, to write

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots\ .$$

Since $x$ was never restricted, it's value can be *any* number, i.e. $-\infty < x < \infty$. A plot of $y = e^x$ is shown below:




## 22.3 Natural Logarithm

Recall that the logarithm operator helps pick apart exponentiation, where if $y = n^x$, then

$$\log_n(y) = \log_n(n^x) = x \ .$$

If the base number $n$ equals the natural exponential $e$, the above becomes

$$\log_e(y) = \log_e(e^x) = x \ .$$

The combination $\log_e$ is called the *natural logarithm*, abbreviated ln.

### Calculating Natural Logarithms

A curious interpretation of the natural logarithm begins with the equation

$$y = \frac{n^x - 1}{x} \ ,$$

readily implying, for some base number $q$, that

$$\log_q(1 + xy) = x \log_q(n) \ ,$$

equivalent to

$$\log_q\left((1 + xy)^{1/x}\right) = \log_q(n) \ .$$

Next, let us substitute $1/x = k$, giving

$$\log_q\left(\left(1 + \frac{y}{k}\right)^k\right) = \log_q(n) \ .$$

On the left, the argument inside side the logarithm looks tantalizingly close to the definition of Euler's constant so long as we let $k$ become arbitrarily large. In this large-$k$ regime, let us rename $y \rightarrow \tilde{y}$, so we have

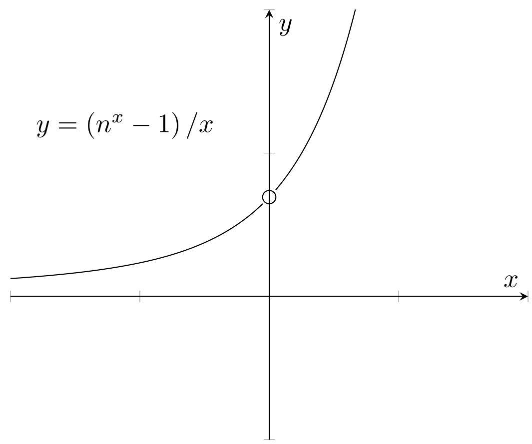
$$\log_q\left(e^{\tilde{y}}\right) = \log_q(n) \ ,$$

where if we finally set $q = e$, the natural logarithm 'undoes' the natural exponential on the left, leaving

$$\tilde{y} = \ln(n) \ .$$

---

Looking again at the original $y$-equation, the curve 'behaves' nicely everywhere except $x = 0$, leading to division by zero, which must be abruptly skipped when plotted:




Now we cash in on the $\tilde{y}$-calculation, which was found by letting $x = 1/k$ become arbitrarily small. Since an arbitrarily small number is essentially zero, it must follow that $\tilde{y}$ fills in the missing point in the plot above. This provides a way to approximate the value of $\tilde{y} = \ln n$ by sampling the curve to the immediate left and/or right of $x = 0$. In the special case $n = e$, it follows that $\tilde{y}_e = \ln e = 1$.

## 22.4 Connection to Taylor Polynomial

Writing the Taylor polynomial

$$y^*_{x_0} = \sum_{q=0}^{n} \frac{m^{(q)}_{x_0}}{q!} (x - x_0)^q$$

alongside the equation for Euler's 'natural' exponential

$$e^x = \sum_{q=0}^{\infty} \frac{x^q}{q!} \; ,$$

we immediately notice that the natural exponential is a Taylor series (infinite terms), based at the point $x_0 = 0$, with all $m^{(q)}_{x_0} = 1$. This also means we can change the base-point of the exponential quite easily:

$$e^{x - x_0} = \sum_{q=0}^{\infty} \frac{(x - x_0)^q}{q!}$$

## 22.5 Hyperbolic Curves

Starting with $e^{ax}$, and then setting $a = -1$, a curious 'split' occurs in the infinite series expansion. Note that for $a = -1$, all even-powered $x$ terms remain positive, whereas all odd-powered terms become negative:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \; .$$

Separating the positive terms from the negative terms, we have

$$e^{-x} = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \; ,$$

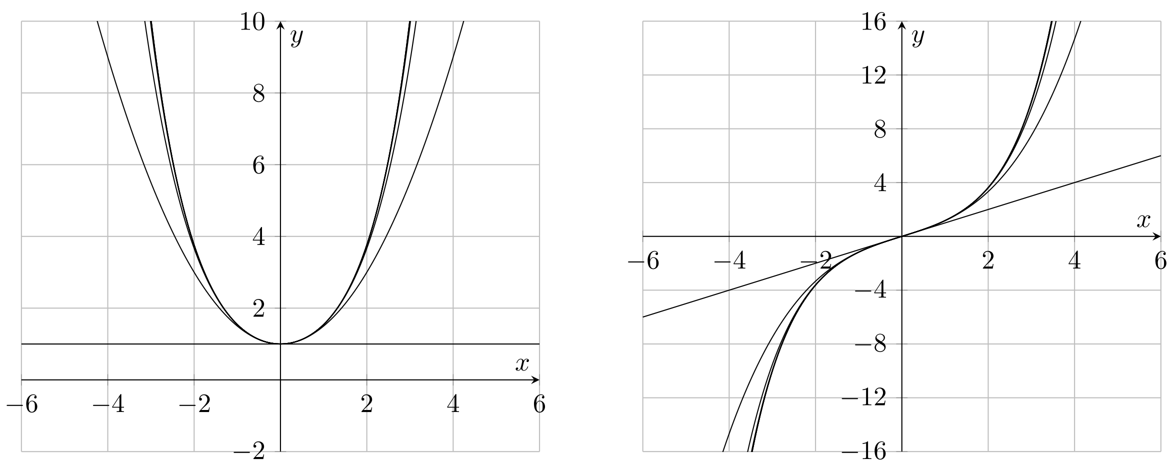
or in summation notation,

$$e^{-x} = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} - \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} \,.$$

Taking the 'even' terms and the 'odd' terms of terms in isolation, let us store a truncated version of each sum in its own variable, namely

$$y_{even} \approx \sum_{m=0}^{n} \frac{x^{2m}}{(2m)!} \qquad\qquad y_{odd} \approx \sum_{m=0}^{n} \frac{x^{2m+1}}{(2m+1)!} \,,$$

where each sum terminates at $n$. By plotting $y_{even}$ for $n = 1, 2, 3, 4, 5$, we see a 'bowl' shape emerge and squeeze inward, but not all the way. After $n = 5$, the curves become indistinguishable in the graph below (left):




Playing the same game for the odd terms, we get another plot (right) that eventually settles into a definite shape. Of course, we would need *all* powers of $n$ in order to get the true plots by the present method. This can be avoided by solving for $y_{even}$ and $y_{odd}$ explicitly. If we write $e^x$ and $e^{-x}$ side-by-side, we have

$$e^{x} = y_{even} + y_{odd} \qquad\qquad e^{-x} = y_{even} - y_{odd} \,,$$

a system with two equations and two variables. The sum and difference of these equations, respectively, tell us

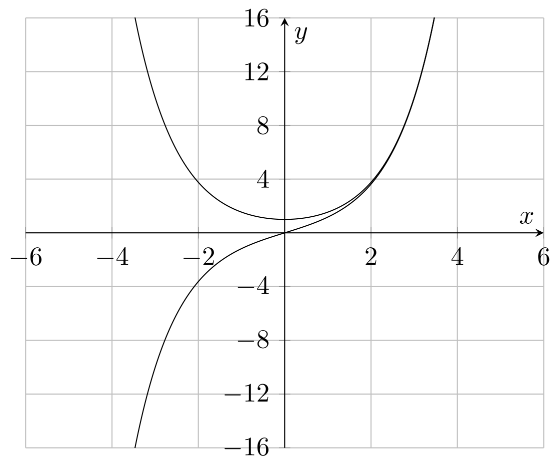
$$y_{even} = \frac{e^x + e^{-x}}{2} \qquad\qquad y_{odd} = \frac{e^x - e^{-x}}{2} \,.$$

The $y_{even}$-quantity has a special name called a *hyperbolic cosine*, whereas $y_{odd}$ is called a *hyperbolic sine*. Putting everything together, we have discovered

$$\text{Hyperbolic Cosine} = \frac{e^x + e^{-x}}{2} = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$$

$$\text{Hyperbolic Sine} = \frac{e^x - e^{-x}}{2} = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} \,.$$

On the same plot, these appear as:


16
12
8
4
y
x
−6
−4
−2
2
4
6
−4
−8
−12
−16

# 23 Area Under Polynomial Curves

From our study of one-dimensional motion, note that the constant-acceleration kinematics equation

$$v = v_0 + at$$

gives rise to the position

$$x = x_0 + v_0 t + \frac{a}{2} t^2 \ .$$

Most importantly, recall that the $x$-equation was derived from the crucial observation that the displacement $x - x_0$ is equal to the *area* under the velocity plot.

In developing the Taylor polynomial, we found that a polynomial equation

$$x = x_0 + m_0^{(1)} t + \frac{m_0^{(2)}}{2!} t^2 + \frac{m_0^{(3)}}{3!} t^3 + \frac{m_0^{(4)}}{4!} t^4 + \cdots$$

is equivalent to a sum

$$x^*_{t_0} = x_{t_0} + \sum_{q=1}^{n} \frac{m_{t_0}^{(q)}}{q!} (t - t_0)^q \ ,$$

where all coefficients are related by

$$\begin{aligned} x_{t_0} &= \left( x_0 + m_0^{(1)} t + \frac{m_0^{(2)}}{2!} t^2 + \frac{m_0^{(3)}}{3!} t^3 + \frac{m_0^{(4)}}{4!} t^4 + \cdots \right) \Bigg|_{t=t_0} \\ v_{t_0} = m_{t_0}^{(1)} &= \left( m_0^{(1)} + m_0^{(2)} t + \frac{m_0^{(3)}}{2!} t^2 + \frac{m_0^{(4)}}{3!} t^3 + \frac{m_0^{(5)}}{4!} t^4 + \cdots \right) \Bigg|_{t=t_0} \\ a_{t_0} = m_{t_0}^{(2)} &= \left( m_0^{(2)} + m_0^{(3)} t + \frac{m_0^{(4)}}{2!} t^2 + \frac{m_0^{(5)}}{3!} t^3 + \frac{m_0^{(6)}}{4!} t^4 + \cdots \right) \Bigg|_{t=t_0} \ , \end{aligned}$$

and so on.

Now, if we are still to believe that the area under the velocity plot is equal to the displacement, it should follow that the curve

$$v = m_0^{(1)} + m_0^{(2)} t + \frac{m_0^{(3)}}{2!} t^2 + \frac{m_0^{(4)}}{3!} t^3 + \frac{m_0^{(5)}}{4!} t^4 + \cdots$$

encloses an area

$$x - x_0 = m_0^{(1)} t + \frac{m_0^{(2)}}{2!} t^2 + \frac{m_0^{(3)}}{3!} t^3 + \frac{m_0^{(4)}}{4!} t^4 + \cdots \ .$$

By the same token, the area under the acceleration plot $a$ encloses an area $v - v_0$, and so on. This relationship must continue through jerk, snap, and all the way up.

---

By a change of notation, the whole apparatus applies to the $xy$-plane, where the curve

$$y_{curve} = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + \cdots$$

encloses the area

$$(A - A_0)_{curve} = c_1 x + \frac{c_2}{2} x^2 + \frac{c_3}{3} x^3 + \frac{c_4}{4} x^4 + \frac{c_5}{5} x^5 + \cdots \ .$$

The coefficients have been recast by letting

$$c_q = \frac{m_0^{(q)}}{(q-1)!} \ .$$

Any number of coefficients are allowed to be zero. For instance, if we need to calculate the area under the curve $y = x^4$, the result immediately stands out as $A - A_0 = x^5/5$.

## 23.1 Area Under Natural Exponential Curve

So far, we have affirmed that a curve

$$y = m_0^{(0)} + m_0^{(1)}x + \frac{m_0^{(2)}}{2!}x^2 + \frac{m_0^{(3)}}{3!}x^3 + \frac{m_0^{(4)}}{4!}x^4 + \cdots$$

encloses an area equal to

$$A - A_0 = m_0^{(0)}x + \frac{m_0^{(1)}}{2!}x^2 + \frac{m_0^{(2)}}{3!}x^3 + \frac{m_0^{(3)}}{4!}x^4 + \cdots .$$

Since the natural exponent has all $m_{x_0}^{(q)} = 1$, we then find the curve

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

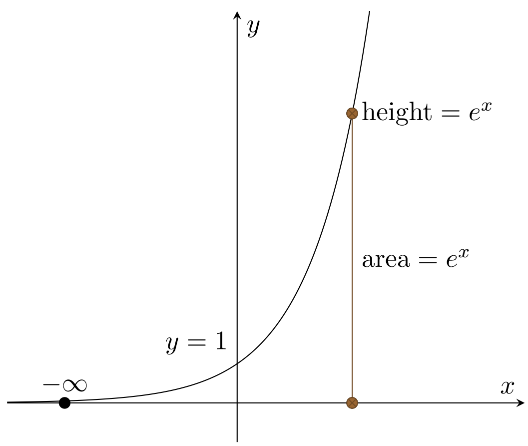
encloses an area equal to

$$A - A_0 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x - 1 .$$

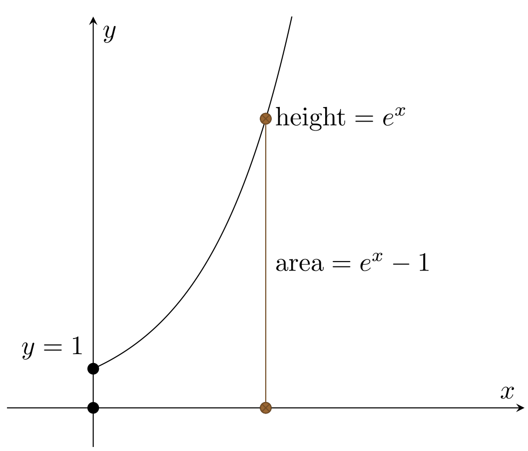
These results are identical up to the value of $A_0$, which is so far unrestricted, simply shifting the entire plot up or down the $y$-axis. We may introduce a new constant $C$ with $A_0 = 1 - C$ such that

$$A = e^x + C ,$$

indicating, remarkably, that *the curve $y = e^x$ encloses an area equal to $A = e^x$ plus an arbitrary constant.* Choosing this constant to be zero, and starting the curve at $x = -\infty$, we produce the precise circumstance for which $e^x$ encloses its own area (note the plot where $x < 0$ is not to scale.):




Or, we may begin the plot at $x = 0$, in which case the enclosed area is $e^x - 1$:



## 23.2 Area Under Hyperbolic Curves

Taking a second look at either one of the hyperbolic curves, for instance

$$y_{even} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots ,$$

we take note that the area enclosed under $y_{even}$ is

$$(A - A_0)_{even} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots ,$$

which is none other than $y_{odd}$, up to an arbitrary constant. Of course, setting $A_0 = 0$ makes $A_{even}$ identical to $y_{odd}$. Similarly, we may start with

$$y_{odd} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots ,$$

and notice the enclosed area is

$$(A - A_0)_{odd} = \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots ,$$

resulting in $y_{even}$, up to an arbitrary constant. Choosing $A_0 = 1$, we find $A_{odd}$ identical to $y_{even}$, establishing the cyclic relationship:

$$\begin{aligned} A_{even} &= y_{odd} \\ A_{odd} &= y_{even} \end{aligned}$$

# 24 Periodic Curves

As we continue to dissect curves of different kinds - linear, quadratic, higher-order polynomials, etc., we may eventually ponder which kinds of plots can repeat themselves in a regular way, a property called *periodicity*. This means that, along the *entire* curve, if a particular value $y$ occurs at any given $x_0$, then $y$ occurs again at $x_0 + L$, where $L$ is a fixed length.

To grapple with periodicity, begin with the the Taylor series of a curve based at $x = 0$, namely

$$y = m_0^{(0)} + m_0^{(1)} x + \frac{m_0^{(2)}}{2} x^2 + \frac{m_0^{(3)}}{6} x^3 + \frac{m_0^{(4)}}{24} x^4 + \cdots ,$$

and introduce the shift $x \to x + x_0$, giving an equation for the updated position $y^*$:

$$y^* = m_0^{(0)} + m_0^{(1)} (x + x_0) + \frac{m_0^{(2)}}{2} (x + x_0)^2 + \frac{m_0^{(3)}}{6} (x + x_0)^3 + \frac{m_0^{(4)}}{24} (x + x_0)^4 + \cdots .$$

Expand each polynomial, and collect like terms in powers of $x_0$. (This exercise should be familiar.)

$$\begin{aligned} y^* = & \left( m_0^{(0)} + m_0^{(1)} x + \frac{m_0^{(2)}}{2} x^2 + \frac{m_0^{(3)}}{6} x^3 + \cdots \right) \\ & + x_0 \left( m_0^{(1)} + m_0^{(2)} x + \frac{m_0^{(3)}}{2} x^2 + \frac{m_0^{(4)}}{6} x^4 + \cdots \right) \\ & + \frac{x_0^2}{2} \left( m_0^{(2)} + m_0^{(3)} x + \frac{m_0^{(4)}}{2} x^2 + \frac{m_0^{(5)}}{6} x^3 + \cdots \right) \\ & + \frac{x_0^3}{6} \left( m_0^{(3)} + m_0^{(4)} x + \frac{m_0^{(5)}}{2} x^2 + \frac{m_0^{(6)}}{6} x^3 + \cdots \right) \\ & \cdots \end{aligned}$$

In the above, the first group of terms on the right is simply $y$, the non-updated position.

## 24.1 Periodicity Condition

Now we enforce the periodicity condition, which is to set, $y^* = y$ when $x = L$, eliminating the $y$-terms:

$$\begin{aligned} 0 = & \, x_0 \left( m_0^{(1)} + m_0^{(2)} L + \frac{m_0^{(3)}}{2} L^2 + \frac{m_0^{(4)}}{6} L^3 + \cdots \right) \\ & + \frac{x_0^2}{2} \left( m_0^{(2)} + m_0^{(3)} L + \frac{m_0^{(4)}}{2} L^2 + \frac{m_0^{(5)}}{6} L^3 + \cdots \right) \\ & + \frac{x_0^3}{6} \left( m_0^{(3)} + m_0^{(4)} L + \frac{m_0^{(5)}}{2} L^2 + \frac{m_0^{(6)}}{6} L^3 + \cdots \right) \\ & \cdots \end{aligned}$$

For any $x_0$, we must have each parenthesized quantity independently resolving to zero:

$$\begin{aligned} 0 &= m_0^{(1)} + m_0^{(2)} L + \frac{m_0^{(3)}}{2} L^2 + \frac{m_0^{(4)}}{6} L^3 + \cdots \\ 0 &= m_0^{(2)} + m_0^{(3)} L + \frac{m_0^{(4)}}{2} L^2 + \frac{m_0^{(5)}}{6} L^3 + \cdots \\ 0 &= m_0^{(3)} + m_0^{(4)} L + \frac{m_0^{(5)}}{2} L^2 + \frac{m_0^{(6)}}{6} L^3 + \cdots \\ & \cdots \end{aligned}$$

Next, sum all equations together, and group like terms in $L$:

$$\begin{aligned} 0 &= \left(m_0^{(1)} + m_0^{(2)} + m_0^{(3)} + \cdots\right) + L\left(m_0^{(2)} + m_0^{(3)} + m_0^{(4)} + \cdots\right) \\ &+ \frac{L^2}{2}\left(m_0^{(3)} + m_0^{(4)} + m_0^{(5)} + \cdots\right) + \cdots \end{aligned}$$

Evidently, for the curve to exhibit periodicity, the following condition must be satisfied:

$$0 = m_0^{(1)} + m_0^{(2)} + m_0^{(3)} + \cdots$$

While the periodicity condition restricts the relationship between each $m_0^{(j)}$, it doesn't tell us *how* to calculate each term, nor does it contain any information on $L$. The most obvious pattern to try has each $\left|m_0^{(j)}\right| = 1$ with alternating signs, and to let the sum run infinitely, producing

$$y = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots ,$$

however this is simply the equation for $e^{-x}$, a clearly non-periodic curve. In such cases like this, $L$ is interpreted as infinite.

## 24.2 Sinusoidal Curves

Let us try a new attempt, now separating all even terms from all odd terms in powers of $x$, leading to a pair of periodicity conditions, one for each:

$$\begin{aligned} 0 &= m_0^{(1)} + m_0^{(3)} + m_0^{(5)} + \cdots \\ 0 &= m_0^{(2)} + m_0^{(4)} + m_0^{(6)} + \cdots \end{aligned}$$

As a checkpoint, note that setting all $m_0^{(j)} = 1$ results in the hyperbolic curves explored above, up to the constant $m_0^{(0)}$, which must be manually set to one. This term simply shifts the plot up or down the $y$-axis, anyway - it does not depend on $x$.

Letting the signs alternate on each $m_0^{(3)}$-term, we produce a pair of infinite polynomials, $y_{odd} = S_x$, and $y_{even} = C_x$, respectively:

$$\begin{aligned} S_x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ C_x &= m_0^{(0)} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

We're now juggling with two curves instead of one, analogous to the hyperbolic case. By setting $m_0^{(0)} = 1$, we can relate the $S_x$- and $C_x$-curves by enclosed area. Going term-by-term, we find, for the area under the $S$-curve,

$$(A - A_0)_S = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots = 1 - C_x ,$$

where setting $A_0 = -1$, we get the clean result

$$A_S = -C_x .$$

Similarly, the area under the $C_x$-curve leads to

$$(A - A_0)_C = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = S ,$$

where setting $A_0$ we get the corresponding equation

$$A_C = S_x .$$

The cyclic relation between $S_x$, $C_x$, $A_S$, and $A_C$ now involves a minus sign, differing from the hyperbolic case. It now turns out that we have to take the are under $S$ or $C$ *four* times in order to return to where we started.

**Unity Condition**

To proceed, lets us calculate $S_x^2$ and $C_x^2$ in hopes of spotting a pattern.

$$\begin{aligned} S_x^2 &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)^2 \\ &= S_x x - S_x \frac{x^3}{3!} + S_x \frac{x^5}{5!} - S_x \frac{x^7}{7!} + \cdots \\ &= x^2 - \left(\frac{2}{3!}\right) x^4 + \left(\frac{2}{5!} + \frac{1}{3!3!}\right) x^6 - \left(\frac{2}{7!} + \frac{2}{3!5!}\right) x^8 + \cdots \\ &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots \end{aligned}$$

$$\begin{aligned} C_x^2 &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2 \\ &= C_x - C_x \frac{x^2}{2!} + C_x \frac{x^4}{4!} - C_x \frac{x^6}{6!} \cdots \\ &= 1 - \left(\frac{2}{2!}\right) x^2 + \left(\frac{2}{4!} + \frac{1}{2!2!}\right) x^4 - \left(\frac{2}{6!} + \frac{2}{4!2!}\right) x^6 + \left(\frac{2}{8!} + \frac{2}{6!2!} + \frac{1}{4!4!}\right) x^8 - \cdots \\ &= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \frac{x^8}{315} - \cdots \end{aligned}$$

Putting each result side-by-side, we have found

$$\begin{aligned} S_x^2 &= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots \\ C_x^2 &= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \frac{x^8}{315} - \cdots , \end{aligned}$$

where the trailing terms on the right are turning out the same. It should follow that the sum of $S_x^2$ and $C_x^2$ cancels these terms perfectly, so we find

$$S_x^2 + C_x^2 = 1 ,$$

which we shall call the *unity condition*.

**Range**

By inspection of the infinite polynomials for $S$ and $C$, we have, near $x = 0$,

$$S_{x=0} = 0 \qquad\qquad C_{x=0} = 1 .$$

Slowly increasing $x$ causes $S$ to start increasing, and causes $C$ to start decreasing. Eventually, as $x$ gets away from zero, the $S$-curve increases to some maximum value $S_{max}$, corresponding to $x^*$, before decreasing again for $x > x^*$. Keeping in mind that the $S$-curve is tracking the area under the $C$-curve, it must follow that $x^*$ corresponds to a minimum value in $C$, namely $C_{x^*} = 0$. That is, after the point $x^*$, the $C$-curve dips below the $x$-axis and starts contributing *negative* area, causing the $S$-curve to bow down. So far then, we know

$$S_{x^*} = \text{maximum} \qquad\qquad C_{x^*} = 0 .$$

Since we already have the unity condition on hand, we immediately know $S_{x^*} = 1$. Moreover, since the $S$-curve consists entirely of odd-powered polynomials, we know $S_{-x^*} = -S_{x^*} = -1$. Similarly, the minimum of the $C$-curve is also $-1$. In summary, we find

$$-1 \le S_x \le 1 \qquad\qquad -1 \le C_x \le 1 .$$

### Period

Now we determine the periodicity of a sinusoidal curve. If we write

$$S_{x=0} = 0 \qquad\qquad C_{x=0} = 1\,,$$

and then shift $x \to x + L$ such that, we must land at the same place:

$$S_{x=L} = 0 \qquad\qquad C_{x=L} = 1$$

Meanwhile, we will soon prove, but for now simply use the fact that $S_x^2 + C_x^2 = 1$ has a geometric interpretation, namely a right triangle with a hypotenuse of 1. As $x$ sweeps across one period $L$, the $S$- and $C$- curves occupy the full range of their values, tracing out a *circle* in the plane. The full period $L$ is therefore equal to the full distance around the unit circle, precisely $2\pi$, where

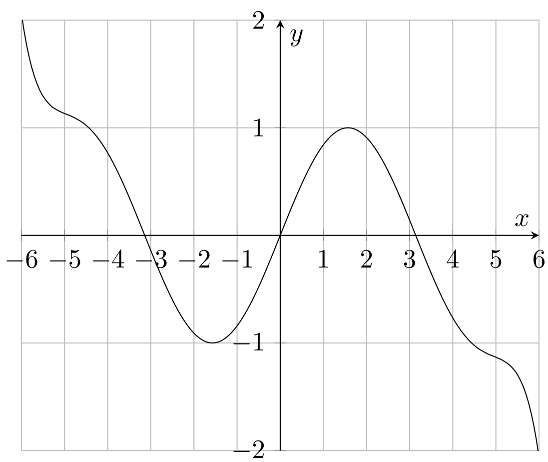
$$\pi = 3.14159265358979\ldots\,,$$

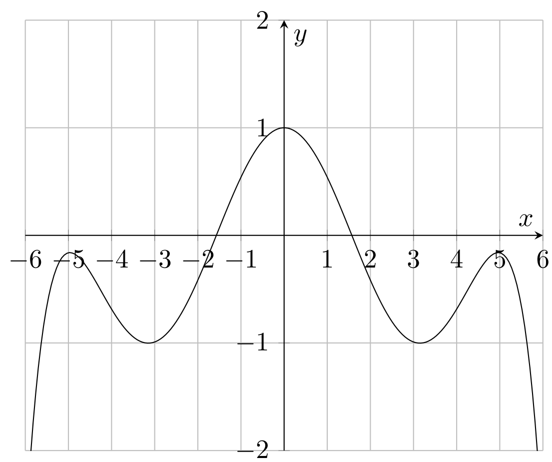
or

$$L = 2\pi\,.$$

### Partial Sum Plots

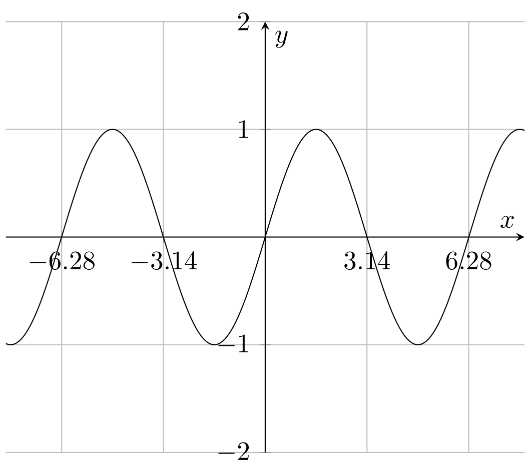
Plotting the first five terms of the $S$- and $C$-curves, respectively, we can see the 'expected' behavior occurring neat $x = 0$, but the plots go wild outside of this domain:

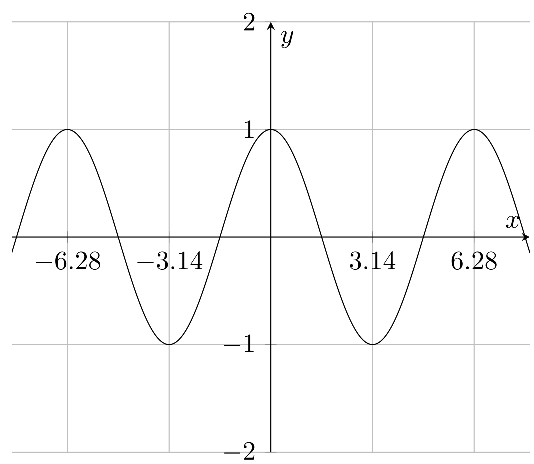



### Sine, Cosine

By including 'all' terms in each sum, we get a nice set of $S$- and $C$-curves curves, respectively:



The $S$- and $C$-curves are formally called the *sine*, and *cosine*, respectively:

$$\text{Sine Curve} = S_x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\text{Cosine Curve} = C_x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

**Tangent**

Looking again at the polynomial expressions for the sine and cosine, it should make sense that the ratio $T_x = S_x/C_x$ should also be periodic, for instance

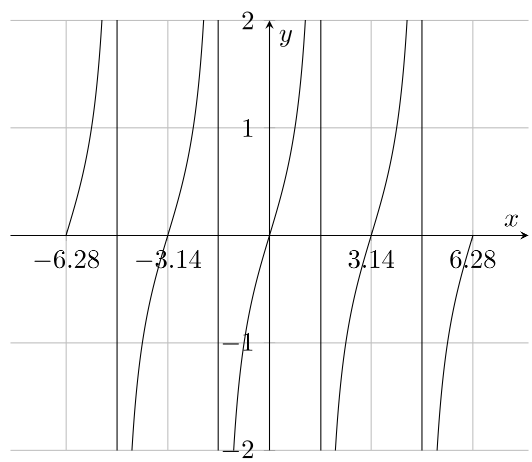
$$\frac{S_x}{C_x} = \frac{x - x^3/3! + x^5/5! - \cdots}{1 - x^2/2! + x^4/4! - \cdots}.$$

By brute polynomial division, the right side can be simplified to yield another infinite polynomial. On the left, the ratio $S_x/C_x$ is called the *tangent* of $x$, given by

$$T_x = \frac{S_x}{C_x} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

The sine, cosine, and tangent fall into a class called *trigonometric* curves.

Because the cosine curve passes through $y = 0$ at $\pm\pi/2$, $\pm 3\pi/2$, etc., the tangent plot is not clearly defined these points as shown.



# 25 Functions

By now, we've collected a number of mathematical curves that can be plotted in the Cartesian plane. To name a new, we have:

| | |
|---|---|
| line | $y = mx + b$ |
| parabola | $y = ax^2 + bx + c$ |
| cubic | $y = ax^3 + bx^2 + cx + d$ |
| positive radical | $y = \sqrt{x}$ |
| general polynomial | $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ |
| general exponent | $y = a^x$ |
| natural exponent | $y = e^x$ |
| hyperbolic cosine | $y = \left(e^x + e^{-x}\right)/2$ |
| hyperbolic sine | $y = \left(e^x - e^{-x}\right)/2$ |
| periodic cosine | $y = 1 - x^2/2! + x^4/4! - \cdots$ |
| periodic sine | $y = x - x^3/3! + x^5/5! - \cdots$ |
| periodic tangent | $y = x - x^3/3 + 2x^5/15 + \cdots$ |

While these curves each appear different when plotted, the crucial feature they share is: *a given input value $x$ is mapped to one single output value $y$.* Curves that obey this property fall into a class called *functions*. When a curve $y$ qualifies as a function, we label it

$$y = f(x) \ ,$$

where $f(x)$ literally translates to 'function of $x$'.

Any curve that 'doubles back' on itself does *not* qualify as a function, and fails the so-called *vertical line test*. This to say if a function $f$ produces multiple $y$-outputs for a given $x$-input, then $f$ is not a function. For instance, the equation of a circle of radius one centered at the origin reads

$$x^2 + y^2 = 1 \ .$$

Solving for $y$ gives *two* possible results

$$y = \pm\sqrt{1 - x^2} \ ,$$

and thus $y$ is not a function. However, if we simply break the result into two parts, each can be a function:

$$f_1(x) = \sqrt{1 - x^2} \qquad\qquad f_2(x) = -\sqrt{1 - x^2}$$

## 25.1 Domain and Range

### Domain

The set of all $x$-values for which a function $f(x)$ produces meaningful output is called the *domain* of $f(x)$.

Many functions, such as $f(x) = x^2$, produce a result for any choice of $x$, thus the domain is the set of all real numbers. Other functions go haywire for certain inputs, such as $f(x) = \sqrt{x}$, so the domain must be limited to the set of non-negative real numbers.

As a set, the domain shall be denoted $\{x\}$. For instance, a domain containing all real numbers is

$$\{x\} = (x \ : \ \infty < x < \infty) \ ,$$

whereas a domain that includes only negative numbers could be written

$$\{x\} = (x \ : \ x < 0) \ .$$

### Range

For a function $f(x)$, the *range* of the function is the set of all output values over a given domain. For instance, the function $f(x) = x^2$ excludes negative outputs, so its range is the set of positive real numbers

$$\{y\} = (y \,:\, y \geq 0) \ .$$

## 25.2 Classifying Functions

### One-To-One

A function is *one-to-one*, also known as *injective*, when every element in $\{x\}$ maps to a unique element in $\{y\}$. For example, general lines $f(x) = mx + b$ are one-to-one, but the parabola $f(x) = x^2$ is not one-to-one over the set of all real numbers. However, the parabola is one-to-one if we take the domain as $\{x\} = (x \,:\, 0 \leq x)$ or $\{x\} = (x \,:\, 0 \geq x)$ separately.

### Surjective

A function is *surjective* or (*onto*) if every single point $y$ in $\{y\}$ can be reached for some input $x$ in $\{x\}$ passed through the function.

Most curves we'll encounter are not surjective. Consider however the function $f(x) = 2x$ as applied to the domain of natural numbers

$$\{x\} = \mathbb{N} = \{1, 2, 3, \dots\} \ .$$

Applying $f(x)$ to each member of the domain, we find the range to be

$$\{y\} = \{2, 4, 6, \dots\} \ ,$$

which lists the even non-negative integers. Note though, that $\{y\}$ contains *all* even non-negative integers, so each member in $\{y\}$ is 'pointed to' by some $x$. In this context, $f(x) = 2x$ is a surjective function.

### Bijective

A function that is both one-to-one and surjective is called *bijective*. That is, a function is *bijective* if every member in $\{x\}$ uniquely leads to a member in $\{y\}$, and vice versa. For example, the function $f(x) = x^2$ over the domain of positive real numbers is both one-to-one and surjective, and is thus bijective.

## 25.3 Even and Odd Functions

Any function that obeys

$$f(-x) = f(x)$$

is called an *even function*. Any functions that obeys

$$f(-x) = -f(x)$$

is called an *odd function*. Said another way, even functions are *symmetric* about the line $x = 0$, and odd functions are *anti-symmetric* about the line $x = 0$.

Many functions are neither even nor odd overall, but every function can be split into even and odd components:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

In terms of $f(x)$ and $f(-x)$, the even and odd components of a function are

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

For example, recall that the natural exponent function $f(x)$ breaks nicely into the hyperbolic cosine (even) and the hyperbolic sine (odd):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$e^x = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$
$$e^x = \cosh(x) + \sinh(x) \ ,$$

where:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

## 25.4 Periodic Functions

A functions that repeats itself according to

$$f(x + nL) = f(x)$$

is called *periodic*, where $L$ is the period of the function, and $n$ is an integer.

#### Trigonometric Functions

We have painstakingly established two periodic functions already, namely the sine,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and the cosine,

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \ ,$$

each having period $L = 2\pi$. The ratio $\sin(x)/\cos(x)$, known as the tangent function, has the same period:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \ .$$

Note too that the sine function is entirely odd, whereas the cosine function is entirely even.

## 25.5 Inverse Functions

A function $y = f(x)$ that is bijective (one-to-one) admits an *inverse function* to $f$, denoted $f^{-1}$, such that

$$f^{-1}(f(x)) = x \ .$$

It immediately follows that the inverse of $f^{-1}$ is simply $f$ again, for if we apply $f$ to both sides, we have

$$f\left(f^{-1}(f(x))\right) = f(x) \ ,$$

or more plainly,

$$f\left(f^{-1}(y)\right) = y \ .$$

#### Symmetry

With the notion of inverse on hand, consider a pair of curves such that

$$y_1 = f(x) \qquad y_2 = f^{-1}(x) \ .$$

Now, choose two distinct points $x_1$ and $x_2$ in the domain of $y_1$, $y_2$, respectively. Evaluate $f(x_1)$ to come up with a value for $y_1$, and then insert that into the argument in $y_2$:

$$y_2 = f^{-1}(f(x_1)) = x_1$$

Similarly, evaluate $f^{-1}(x_2)$ to have a value for $y_2$, and insert that into the argument in $y_1$:

$$y_1 = f\left(f^{-1}(x_2)\right) = x_2$$

To summarize, we now have two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the Cartesian plane, subject to the conditions $y_2 = x_1$ and $y_1 = x_2$. Now, let us calculate the slope of the line formed between $P_1$ and $P_2$, namely

$$m = \frac{y_2 - y_1}{x_2 - x_1} ,$$

which quickly reduces to

$$m = \frac{y2 - y1}{y1 - y2} = -1 .$$

That is, the slope of the line connecting the points is precisely $-1$. This can only mean that the line that 'reflects' $P_1$ onto $P_2$ (and vice versa) has slope 1, which is the line $y = x$. This proves an interesting fact about a function and its inverse: $f(x)$ and $f^{-1}(x)$ are symmetric about the line $f(x) = x$.

If a function $f(x)$ is not one-to-one on its entire domain, it often occurs that $f(x)$ is one-to-one in a sub-domain. In this case, the inverse is well-defined in the sub-domain.

## 25.6 Case Study: Nested Radicals

Suppose we need to solve the rather exotic equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} = 2 ,$$

where the pattern in the square roots continues forever. This equation is surprisingly easy to grapple with, for if we let

$$F = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} ,$$

and then square both sides, we find

$$x + F = 4 .$$

Since $F = 2$ by construction, we immediately see that $x = 4 - 2 = 2$.

Next, write the more general equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}} = a$$

that replaces the 2 with an arbitrary constant $a$. Squaring both sides of the above while recycling the same $F$ used previously, we have

$$x + F = a^2 .$$

This equation has a trivial solution, namely $F = 0$, $a = 0$ gives $x = 0$. Stating the obvious, this means

$$\sqrt{0 + \sqrt{0 + \sqrt{0 + \sqrt{0 + \cdots}}}} = 0 .$$

Something curious occurs by setting $F = 1$, $a = 1$, giving $x + 1 = 1$, solved by $x = 0$. Writing this out though, we have stumbled upon

$$\sqrt{0 + \sqrt{0 + \sqrt{0 + \sqrt{0 + \cdots}}}} = 1 ,$$

which *can't* be right. To make sense of this, return to the definition of $F$ and square both sides to write

$$F^2 = x + F \ ,$$

and use the quadratic formula to isolate $F$:

$$F = \frac{1}{2} \pm \frac{\sqrt{1+4x}}{2}$$

By setting $x = 0$, it's immediately evident that there are two solutions to $F$, namely

$$F_1 = \frac{1}{2} + \frac{1}{2} = 1 \qquad\qquad F_0 = \frac{1}{2} - \frac{1}{2} = 0 \ .$$

Being multi-valued at $x = 0$, $F$ cannot qualify as a function, hence our avoidance of the $F(x)$-notation.

A different way to proceed is to write $x$ as a function of $F$, which is perfectly valid:

$$x(F) = F^2 - F = \left(F - \frac{1}{2}\right)^2 - \frac{1}{4}$$

---

## 25.7 Continuity and Smoothness

## 25.8 Asymptotes

### Vertical Asymptotes

### Horizontal Asymptotes

---

# 26 Parametric Equations

While the tool set for dealing with functions $y = f(x)$ is powerful, not all curves fit the paradigm of functions, specifically curves that 'double back' on themselves, failing the vertical line test. A more general means of handling curves (and by extension, surfaces) is called *parametric equations*. Instead of having one equation $y = f(x)$ to trace a curve, the job shall be split into two (or more) parametric equations that have of the form

$$x = x(t) \qquad\qquad y = y(t) \ ,$$

where $t$ is a new variable called a *parameter*. If it is algebraically possible to isolate the parameter $t$ in each equation, the familiar form $y = y(x)$ can be recovered. Parametric equations are in fact not a new acquaintance. In studying two-dimensional kinematics, we wrote equations of motion for the uniform-acceleration case,

$$x = x_0 + v_{x_0} t + \frac{a_x}{2} t^2 \qquad\qquad y = y_0 + v_{y_0} t + \frac{a_y}{2} t^2 \ ,$$

in where $t$ is the parameter.

---

Taking an example, consider the pair of equations

$$x(t) = t^2 + t \qquad\qquad y(t) = 2t - 1 \ .$$

Just as was the case for functions, we generate plots of parametric equations by producing ordered-pair solutions. That means to choose a given $t_0$, and then the corresponding $x(t_0)$, $y(t_0)$ denote a point in the plot.

If we insist on the 'old' way, the parameter $t$ can be eliminated to produce

$$y = -2 \pm \sqrt{1 + 4x} \ ,$$

which is actually *two* functions. On the other hand, we can solve in the other direction to write

$$x = \frac{1}{4}\left(y^2 + 4y + 3\right) \ ,$$

which isn't much easier to grapple with. To the credit of the ladder equation though, writing $x$ as a function of $y$ *does* qualify as a function by virtue of passing the 'horizontal line test'.