Integral Calculus
MANUSCRIPT

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Chapter 1

Integral Calculus

1 Area Under a Curve

The workhorse equation of differential calculus is undoubtedly the definition of the derivative of a function $f(x)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \,,$$

which gives the instantaneous slope of the function at x_0.

An extension of the derivative comes in the form of Taylor's theorem, which attempts to approximate the function $f(x)$ near a given point x_0:

$$f(x) \approx f(x_0) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}(x_0)(x - x_0)^q$$

Of course, Taylor's theorem embeds the first derivative as its first-order case.

As it turns out, the derivative formulas from differential calculus can be used backwards, so so speak, which is the basis of *integral calculus*.

Motivation

Consider a point x_0 in the domain of a 'well-behaved' function $f(x)$, and also consider a point x_1 that is arbitrarily close to x_0. By the derivative formula, we can surely write

$$\lim_{x_1 \to x_0} f(x_1) - f(x_0) = f'(x_0) \lim_{x_1 \to x_0} (x_1 - x_0) \,.$$

Also, consider another point x_2 that is arbitrarily close to x_1, which means

$$\lim_{x_2 \to x_1} f(x_2) - f(x_1) = f'(x_1) \lim_{x_2 \to x_1} (x_2 - x_1) \,,$$

and just to start a pattern, consider yet another point x_3 obeying

$$\lim_{x_3 \to x_2} f(x_3) - f(x_2) = f'(x_2) \lim_{x_3 \to x_2} (x_3 - x_2) \,.$$

For definiteness, let's have the x-variables relate by

$$x_0 < x_1 < x_2 < x_3 < \cdots < x_n \ ,$$

assuming the pattern keeps going. It also helps to label the interval between each:

$$\Delta x_j = x_{j+1} - x_j$$

With a list of n total equations, one sensible step is to take their sum. Tallying the right-hand sides first, we have

$$\begin{aligned}\text{RHS} &= f'(x_0)\,\Delta x_0 + f'(x_1)\,\Delta x_1 \\ &\quad + f'(x_2)\,\Delta x_2 + \cdots \ ,\end{aligned}$$

which can be written concisely as a sum

$$\text{RHS} = \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j$$

that goes out to n total terms.

As for the left-hand side, we have

$$\begin{aligned}\text{LHS} &= \lim_{x_1 \to x_0} f(x_1) - f(x_0) \\ &\quad + \lim_{x_2 \to x_1} f(x_2) - f(x_1) \\ &\quad + \lim_{x_3 \to x_2} f(x_3) - f(x_2) \\ &\quad + \cdots + \lim_{x_n \to x_{n-1}} f(x_n) - f(x_{n-1}) \\ &= f(x_n) - f(x_0)\end{aligned}$$

Notice how any given $f(x_j)$ present in the above has a negative counterpart, thus most terms in the above cancel out in pairs. This obliterates any notion of 'limit' on the left, as only the difference $f(x_n) - f(x_0)$ remains.

Putting the left side and the right side together, we seem to have discovered

$$f(x_n) - f(x_0) \approx \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j \ . \qquad (1.1)$$

On the left is simply the difference of a function at two points in its domain. The right, however, seems to be the total *area* of n rectangles, with the jth rectangle having height $f'(x_j)$ and width Δx_j.

As a whole, Equation (1.1) suggests a way to approximate the area under the curve $f'(x)$ between the endpoints x_0, x_n. The tricky part, in general, is finding whatever function $f(x)$ corresponds to the slope $f'(x)$, i.e. the notorious *antiderivative*.

1.1 Riemann Sums

At face value, Equation (1.1) can be implemented 'as-is' to approximate the area under $f'(x)$. To clean up the notation, make the substitution $f'(x) = g(x)$, and write the above as

$$f(x_n) - f(x_0) \approx S = \sum_{j=0}^{n-1} g\left(x_j^*\right)\Delta x \ ,$$

where the argument sent to $g(x)$ is denoted x_j^*. Furthermore, the subscript on Δx_j has been dropped with the understanding that each Δx_j is one and the same length given by

$$\Delta x = \frac{x_n - x_0}{n} \ ,$$

implying

$$x_j = x_0 + j\Delta x \ .$$

Left, Right, Midpoint Sum

The reason x_j^* gets special attention is there are no natural restrictions on where x_j^* occurs within the interval Δx_j. Right off the bat, there are three obvious options

$$x_j^* = \begin{cases} x_j & \text{Left sum} \\ x_{j+1} & \text{Right sum} \\ (x_j + x_{j+1})/2 & \text{Midpoint sum} \end{cases} \ ,$$

which sample from $g(x)$ differently. Explicitly, these mean:

$$\begin{aligned}\frac{S_{\text{Left}}}{\Delta x} &= g(x_0) + g(x_0 + \Delta x) \\ &\quad + g(x_0 + 2\Delta x) + \cdots + g(x_n - \Delta x) \\ \frac{S_{\text{Right}}}{\Delta x} &= g(x_0 + \Delta x) + g(x_0 + 2\Delta x) \\ &\quad + g(x_0 + 3\Delta x) + \cdots + g(x_n) \\ \frac{S_{\text{Mid}}}{\Delta x} &= g\left(x_0 + \frac{\Delta x}{2}\right) + g\left(x_0 + \frac{3\Delta x}{2}\right) \\ &\quad + g\left(x_0 + \frac{5\Delta x}{2}\right) + \cdots + g\left(x_n - \frac{\Delta x}{2}\right)\end{aligned}$$

Problem 1

Using the midpoint sum rule with $n = 10$ bins, approximate the area under the function

$$g(x) = 5x + 2$$

in the domain

$$-2 \le x \le 3 \ .$$

Answer: Let $x_0 = -2$, let $x_n = 3$, and $n = 10$ so that

$$\Delta x = \frac{x_n - x_0}{n} = \frac{3 - (-2)}{10} = \frac{1}{2} \ .$$

At step j in the sum we further have

$$x_j = x_0 + j\Delta x = -2 + \frac{j}{2} \, .$$

To prepare for the midpoint sum, note that

$$x_j^* = \frac{x_j + x_{j+1}}{2} = \frac{-7}{4} + \frac{j}{2} \, .$$

The midpoint sum S_M is given by

$$S_M = \sum_{j=0}^{n-1} g\left(x_j^*\right)\Delta x = \sum_{j=0}^{9}\left(5\left(\frac{-7}{4} + \frac{j}{2}\right) + 2\right)\frac{1}{2} \, ,$$

which simplifies nicely:

$$\begin{aligned} S_M &= \frac{1}{2}\sum_{j=0}^{9}\left(2 - \frac{35}{4}\right) + \frac{5}{4}\sum_{j=0}^{9} j \\ &= \frac{1}{2}(10)\left(2 - \frac{35}{4}\right) + \frac{5}{4}(45) \\ &= \frac{45}{2} = 22.5 \end{aligned}$$

Using the midpoint rule, 22.5 happens to be the exact solution to the stated problem, regardless of how many bins we choose. This brings out a special relationship between the midpoint rule and straight lines: the approximation is perfect.

<u>Problem 2</u>

Using the right sum rule with any number n bins, approximate the area under the function

$$g(x) = 4x - x^2$$

in the domain

$$0 \leq x \leq 4 \, .$$

Answer: Let $x_0 = 0$, let $x_n = 4$, and $n = 10$ so that

$$\Delta x = \frac{x_n - x_0}{n} = \frac{4-0}{n} = \frac{4}{n} \, .$$

At step j in the sum we further have

$$x_j = x_0 + j\Delta x = \frac{4j}{n} \, .$$

To prepare for the right sum rule, note that

$$x_{j+1} = \frac{4j}{n} + \frac{4}{n} \, .$$

Then, the right sum rule is

$$\begin{aligned} S_R &= \sum_{j=0}^{n-1} f'\left(x_{j+1}\right)\Delta x \\ &= \sum_{j=0}^{n-1}\left(4\left(\frac{4j}{n} + \frac{4}{n}\right) - \left(\frac{4j}{n} + \frac{4}{n}\right)^2\right)\frac{4}{n} \end{aligned}$$

Let $k = j + 1$ and simplify the right side to get

$$S_R = \frac{4^3}{n^3}\left(n\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2\right) \, .$$

By analyzing the remaining sums, it's straightforward to show that

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \, , \end{aligned}$$

and the sum simplifies to

$$S_R = \frac{32}{3}\left(1 - \frac{1}{n^2}\right)$$

This result contains a factor of n, allowing the exactness of S_R to be tuned. Note that $1/n^2$ vanishes for sufficiently large n, telling us the exact area under the curve is $32/3$.

Trapezoid Rule

An improvement over rectangle-based methods is the average the left- and right rules, which effectively turns rectangles into trapezoids, giving (you guessed it) the *trapezoid rule*:

$$\begin{aligned} S_{\text{Trap}} &= \frac{1}{2}\left(S_{\text{Left}} + S_{\text{Right}}\right) \\ &= \frac{1}{2}\sum_{j=0}^{n-1}\left(g\left(x_j\right) + g\left(x_{j+1}\right)\right)\Delta x \\ &= \frac{g\left(x_0\right) + g\left(x_n\right)}{2}\Delta x + \sum_{j=1}^{n-1} g\left(x_j\right)\Delta x \end{aligned}$$

Without summation notation, the above reads

$$\begin{aligned} \frac{S_{\text{Trap}}}{\Delta x} &= \frac{g\left(x_0\right)}{2} + g\left(x_0 + \Delta x\right) \\ &\quad + g\left(x_0 + 2\Delta x\right) + \cdots + \frac{g\left(x_n\right)}{2} \, . \end{aligned}$$

2 The Integral

There is a regime where all versions of the Riemann sum converge to the same answer, and that is when we impose the limit $\Delta x \to 0$ and simultaneously $n \to \infty$. In this limit, the entire picture gets squeezed together, and the area under a curve is approximated by an infinite number of vertical lines. In other words, the Riemann sum becomes an exact solution to the area under the curve $f'(x)$:

$$f(x_n) - f(x_0) = \lim_{n\to\infty} \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j\ .$$

Integral Notation

To update the above with cleaner notation, the summation is replaced by the 'integral', literally a giant 'S', via

$$\lim_{\Delta x\to 0} \sum \Delta x \to \int dx\ ,$$

which also replaces Δx with dx. The limits on the sum turn into *integration limits*, one 'lower' limit and one 'upper' limit:

$$\lim_{\Delta x\to 0} \sum_{j=0}^{n-1} \Delta x \to \int_{x_0}^{x_n} dx$$

All j-subscripts have also been dropped, as x is now understood to be a continuous variable inside the integral.

2.1 Fundamental Theorem

Using integral notation, the above is written

$$f(x_n) - f(x_0) = \int_{x_0}^{x_n} f'(x)\,dx\ .$$

While this is workable, it's customary to drop the n-subscript from $f(x_n)$, and this term becomes $f(x)$. To prevent a naming conflict on the right, swap the integration variable from x to t:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt\ . \tag{1.2}$$

This result is called the *fundamental theorem of calculus*, which is the full inversion of the definition of the derivative.

Role of the Antiderivative

A less tautological way to write Equation (1.2) is

$$f(x) - f(x_0) = \int_{x_0}^{x} g(t)\,dt\ ,$$

where $f'(t)$ is renamed to some given or otherwise evident function $g(t)$. The left-side function $f(x)$ is considered unknown.

In order to 'solve' the integral, $g(t)$ must be expressed as the derivative of something else, which means to find the antiderivative of $g(t)$. The 'something else' in this case has already been named, particularly $f(t)$:

$$f(x) - f(x_0) = \int_{x_0}^{x} \frac{d}{dt}(f(t))\,dt$$

The ability to evaluate an integral usually comes down to the ability to find the antiderivative of the function being integrated. This can be quite the chore, if not impossible.

With the proper antiderivative in place, the derivative and the integral on the right mutually annihilate, leaving $f(t)$ alone evaluated at the integration limits, i.e., the quantity $f(x) - f(x_0)$. One way to think of this is to cancel the factors of dt in a way inspired by the chain rule:

$$f(x) - f(x_0) = \int_{x_0}^{x} \frac{d}{\cancel{dt}} f(t)\,\cancel{dt} = \int_{x_0}^{x} df(t)$$

2.2 Definite Integral

indexintegral, definite

When the integration limits x_0, x are specified, either numerically or symbolically, the integral is called *definite*. In order to 'fully' solve a definite integral, the antiderivative $f(f)$ must be evaluated at each limit, and the answer is the difference between $f(x_0)$ and $f(x)$. For this, the 'vertical bar' notation is used:

$$\int_{x_0}^{x} df(t) = f(t)\Big|_{x_0}^{x} = f(x) - f(x_0)$$

Swapping the Limits

One can readily see that swapping the integration limits makes the integral 'run backwards', and gains an overall negative sign:

$$\int_{x}^{x_0} df(t) = f(x_0) - f(x) = -\int_{x_0}^{x} df(t)$$

Breaking the Interval

The integral remains intact if we split the interval into two or more parts. Introducing a variable a in the domain $x_0 \le a \le x$, we may write:

$$\int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{a} g(t)\,dt + \int_{a}^{x} g(t)\,dt$$

Symmetric Domain

Consider the definite integral over a symmetric domain, meaning $-x_0$ is the lower limit and x_0 is the upper limit:

$$f(x_0) - f(-x_0) = \int_{-x_0}^{x_0} g(t)\, dt$$

From studying functions, recall that even functions obey

$$f_{\text{even}}(x) - f_{\text{even}}(-x) = 0\ ,$$

and correspondingly for odd functions,

$$f_{\text{odd}}(x) + f_{\text{odd}}(-x) = 0\ ,$$

meaning

$$f_{\text{odd}}(x) - f_{\text{odd}}(-x) = 2f_{\text{odd}}(x)\ .$$

Since the integral of $g(x)$ effectively bumps up its order by one, it follows that the even-ness or oddness of function f is exactly the opposite of function g. We thus gain two cases:

$$\begin{aligned} f_{\text{even}}(x_0) - f_{\text{even}}(-x_0) &= \int_{-x_0}^{x_0} g_{\text{odd}}(t)\, dt \\ f_{\text{odd}}(x_0) - f_{\text{odd}}(-x_0) &= \int_{-x_0}^{x_0} g_{\text{even}}(t)\, dt \end{aligned}$$

The first of these results is immediately zero from the properties of even functions. In fact, the integral of any odd function over any symmetric interval, as we've shown, is *always* zero:

$$0 = \int_{-x_0}^{x_0} g_{\text{odd}}(t)\, dt \tag{1.3}$$

For the other case, we correspondingly find

$$2f_{\text{odd}}(x_0) = \int_{-x_0}^{x_0} g_{\text{even}}(t)\, dt\ ,$$

which means the integral of an even function over a symmetric interval effectively sums the same area twice. The above is also captured by

$$f_{\text{odd}}(x_0) = \int_0^{x_0} g_{\text{even}}(t)\, dt\ . \tag{1.4}$$

2.3 Integration Constant

When the lower integration limit x_0 is unspecified, the term $-f(x_0)$ is called the *integration constant*, denoted C. Setting $f(x_0) = -C$, this means:

$$\int^x f'(t)\, dt = f(x) + C$$

One way to justify the presence of the integration constant is to realize that any function $f(x) + C$ has the same derivative $f'(x)$, which is to say the absolute vertical offset of the curve has no bearing on its slope. To say this backwards, it follows that any antiderivative calculation without specific limits is only certain up to an arbitrary but non-ignorable constant C.

2.4 Indefinite Integral

indexintegral, indefinite

The integral still retains meaning if we ambiguate both integration limits by writing

$$\int f'(t)\, dt = f(x) + C\ ,$$

where C is the integration constant.

In this abstraction, the upper integration limit is always understood to be x, which kills the naming conflict in the x-variable on the right. Thus we also have

$$\int f'(x)\, dx = f(x) + C\ , \tag{1.5}$$

which is called the *indefinite integral*.

2.5 Integral Operator

In the same sense that one can apply d/dx as an operator to both sides of an equation, we can do the opposite move, which is to apply $\int dx$ across both sides of an equation as well. If

$$g(x) = \frac{d}{dx} f(x)\ ,$$

then

$$\int g(x)\, dx = \int \frac{d}{dx}\left(f(x)\right) dx = f(x) + C\ .$$

On the right, the integral and the derivative are mutually-obliterating, leaving just the enclosed function up to a constant.

Interchangeability

As a sanity check, we should be able to apply d/dx across the whole equation and recover the starting point. Explicitly, this is

$$\frac{d}{dx}\left(\int g(x)\, dx\right) = \frac{d}{dx} f(x) + \cancel{\frac{dC}{dx}}\ ,$$

which readily reduces to $g(x) = f'(x)$, provided that:

$$\frac{d}{dx}\left(\int g(x)\, dx\right) = \int \frac{d}{dx}\left(g(x)\right) dx$$

That is, it's not harmful to move the derivative operation inside the enclosure of the integral.

3 Techniques of Integration

Integral calculations are trickier than anything else in introductory calculus. Here we go through the standard bag of tricks for solving integrals by hand. (Most integrals in the wild are not solvable by hand.)

3.1 Antiderivative Exploit

The most direct way to solve an integral is pick out (by experience or by luck) the antiderivative of the function being integrated. For instance, consider

$$I = \int_0^{\sqrt{\pi/2}} x \cos\left(x^2\right) dx \, .$$

Right away, note that the function being integrated can be written as a derivative

$$x \cos\left(x^2\right) = \frac{d}{dx}\left(\frac{1}{2}\sin\left(x^2\right)\right) \, ,$$

so then

$$I = \int_0^{\sqrt{\pi/2}} \frac{d}{dx}\left(\frac{1}{2}\sin\left(x^2\right)\right) dx \, ,$$

and then the integral and derivative operators cancel, leaving only the evaluation:

$$I = \frac{1}{2}\sin\left(x^2\right)\Big|_0^{\sqrt{\pi/2}} = \frac{1}{2}(1-0) = \frac{1}{2}$$

3.2 Exponents and Roots

Powers

Starting with the power rule for differentiation

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \, ,$$

replace $n \to n+1$ for convenience and write the same rule:

$$(n+1)\,x^n = \frac{d}{dx}\left(x^{n+1}\right)$$

From this, we can apply the integral operator to derive the rule for integrating powers and roots:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \tag{1.6}$$

Going through a few exemplary cases, i.e. playing with common values of n, we generate some useful information. You are encouraged to work through each of these:

$$\int dx = x + C \tag{1.7}$$

$$\int x \, dx = \frac{1}{2}x^2 + C \tag{1.8}$$

$$\int x^2 \, dx = \frac{1}{3}x^3 + C \tag{1.9}$$

$$\int \sqrt{x} \, dx = \frac{2}{3}x^{3/2} + C \tag{1.10}$$

$$\int x^{3/2} \, dx = \frac{2}{5}x^{5/2} + C \tag{1.11}$$

$$\int x^{-2} \, dx = \frac{-1}{x} + C \tag{1.12}$$

$$\int x^{-3} \, dx = \frac{-1}{2x^2} + C \tag{1.13}$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C \tag{1.14}$$

$$\int x^{-3/2} \, dx = \frac{-2}{\sqrt{x}} + C \tag{1.15}$$

Reciprocal

One special case to power rule formula is the integral of $1/x$. Recalling that the derivative of the natural logarithm yields this result, i.e.

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \, ,$$

the following must hold:

$$\int \frac{1}{x} \, dx = \ln(x) + C \tag{1.16}$$

Exponential

Starting with the derivative rule for exponents

$$\frac{d}{dx}\left(n^x\right) = n^x \ln(n) \, ,$$

it must follow that:

$$\int n^x \, dx = \frac{n^x}{\ln(n)} + C \tag{1.17}$$

Natural Exponential

In Equation (1.17), set $n = e$ to see that the area under the natural exponential is equal to to the height of the curve itself (up to a constant):

$$\int e^x \, dx = e^x + C \tag{1.18}$$

Applied Chain Rule

Using the power rule and chain rule for derivatives, it's straightforward to derive

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}} .$$

Applying the $\int dx$ operator across both sides and simplifying leads to a useful identity:

$$\int \frac{f'(x)}{2\sqrt{f(x)}}\,dx = \sqrt{f(x)} + C \tag{1.19}$$

It takes some effort to train the eye to make use of identities such as the above. Exploring one case, suppose we have

$$f(x) = 1 \pm x^2$$

with $f'(x) = \pm 2x$. Plugging all of this in and simplifying gives a two-channel result:

$$\int \frac{\pm x}{\sqrt{1 \pm x^2}}\,dx = \sqrt{1 \pm x^2} + C \tag{1.20}$$

Problem 3

Prove that the area under a parabolic segment of base b and height h is

$$A = \frac{2}{3}bh .$$

Problem 4

Prove that the area of the 'lens' formed between the curves

$$y_1 = x^2$$
$$y_2 = ax + b$$

is

$$A = \frac{1}{6}\left(a^2 + 4b\right)^{3/2} .$$

3.3 U-Substitution

The standard integral

$$\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$$

can sometimes be made simpler by a technique called *u-substitution*, which entails choosing a function $u(x)$ and then recasting the integral in this variable.

The u-substitution can be established by multiplying $du/du = 1$ into the standard integral, i.e.

$$\int_{x_0}^{x} \frac{df}{dt}dt = \int \frac{df}{dt}\frac{du}{du}dt = \int_{u(x_0)}^{u(x)} \frac{df}{du}du ,$$

where the factor dt/dt cancels out. Importantly, note that the limits on the integral are also modified to respect $u(x)$. Once the result is attained as $f(u)$, reverse-substitute to attain $f(x)$.

A pragmatic way to choose the *correct* u-substitution can be established. Consider an indefinite integral

$$I = \int f(x)\,g(x)\,dx$$

for two functions $f(x)$, $g(x)$. Under the substitution $u = u(x)$, the above still must come out to

$$I = \int f(u)\,du ,$$

which can only mean

$$g(x) = \frac{du}{dx} .$$

That is, the function $g(x)$ must be (at least) proportional to the derivative of the substitution $u(x)$.

Exemplary Case

Consider again the definite integral

$$I = \int_0^{\sqrt{\pi/2}} x\cos\left(x^2\right)dx .$$

To solve this with u-substitution, let

$$u(x) = x^2$$

such that

$$du = 2x\,dx .$$

The limits of the integral must change to reflect the u-substitution as well. With this, the integral becomes

$$I = \int_0^{\pi/2} \frac{1}{2}\cos(u)\,du ,$$

which has a straightforward solution:

$$I = \frac{1}{2}\sin(u)\Big|_0^{\pi/2}$$

From here, one may stay in the u-domain to get the final answer, or switch back to the x-variable to recover

$$I = \frac{1}{2}\sin\left(x^2\right)\Big|_0^{\sqrt{\pi/2}} = \frac{1}{2}(1-0) = \frac{1}{2} .$$

Constant Shift

If the x-dependence in the integrand is shifted by a constant λ, i.e.

$$u(x) = x + \lambda \, ,$$

then

$$du = dx$$

always holds.

For instance, in

$$I = \int (x+3)^n \, dx \, ,$$

we can let $u = x + 3$ so the above becomes

$$I = \int u^n \, du \, ,$$

which is easy to solve using Equation (1.6) as

$$I = \frac{u^{n+1}}{n+1} + C \, .$$

Reverse-substitute to get the answer in terms of x:

$$\int (x+3)^n \, dx = \frac{(x+3)^{n+1}}{n+1} + C$$

Problem 5

Use u-substitution to prove Equation (1.20).

Problem 6

Use u-substitution to prove:

$$\int \frac{dx}{1+x} = \ln(1+x) + C \tag{1.21}$$

Problem 7

For constant a, prove:

$$\int (x-a)^{n-1} \, dx = \frac{1}{n}(x-a)^n$$

The d(sin) Shortcut

Integrals of the form

$$I = \int f(\sin(x)) \cos(x) \, dx$$

are transformed by standard u-substitution. Letting

$$u(x) = \sin(x)$$

such that

$$\frac{du}{dx} = \cos(x) \, ,$$

the above readily takes a more standard form:

$$I = \int f(\sin(x)) \cos(x) \, dx = \int f(u) \, du$$

The combination $\cos(x)\,dx$ is written $d(\sin(x))$ as a shortcut, which embeds the notions $u = \sin(x)$, $du = \cos(x)\,dx$ simultaneously:

$$\cos(x) \, dx = d(\sin(x))$$

For example, consider the indefinite integral

$$J = \int \sin^2(x) \cos(x) \, dx \, ,$$

which looks like a rather messy antiderivative to wrestle with. Applying the so-called $d \sin()$ shortcut, the integral reads

$$J = \int \sin^2(x) \, d(\sin(x)) = \int u^2 \, du \, ,$$

and the problem is now simpler in the u-variable. To finish the job, we have

$$J = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C \, .$$

3.4 Integrands with Roots

Not every integral involving a square root (or worse) can be solved by simple u-substitution. In these cases, it's worth including the exponent of the embedded root in the u-substitution.

To illustrate, consider the indefinite integral

$$I = \int \frac{x}{(x-4)^{1/3}} \, dx \, ,$$

which begs trying $u = x - 4$, but this makes an absolute mess. Instead, let us take

$$u = (x-4)^{1/3}$$

such that

$$x = u^3 + 4$$
$$dx = 3u^2 \, du \, .$$

Then, the integral looks much easier in the u-domain:

$$I = 3 \int \left(u^4 + 4u\right) du$$

Problem 8

Use the above as a starting point to prove:

$$\int \frac{x}{(x-4)^{1/3}} = \frac{3}{5}(x-4)^{2/3}(x+6) + C$$

3.5 Partial Fractions

...

3.6 Integration by Parts

Consider the product $H(x)$ of two functions $U(x)$, $V(x)$,

$$H(x) = U(x) V(x) ,$$

and take the derivative of H, minding the product rule:

$$\frac{d}{dx} H(x) = V(x) \frac{d}{dx} U(x) + U(x) \frac{d}{dx} V(x)$$

Next, apply the integral operator $\int dx$ across the whole equation:

$$\int \frac{d}{dx} H(x)\, dx = \int V(x) \frac{d}{dx} U(x)\, dx + \int U(x) \frac{d}{dx} V(x)\, dx$$

Since the integral and derivative operators are mutually annihilating, the left side is simply $H(x)$ evaluated at the integration limits. It suffices to leave the vertical bar empty while working in indefinite form:

$$\int \frac{d}{dx} H(x)\, dx = H(x) \Big| = U(x) V(x) \Big|$$

Introducing the shorthand notation

$$\frac{d}{dx} U(x) = dU$$

and similar for dV, the above is written

$$UV \Big| = \int V dU + \int U dV ,$$

where all quantities are understood to be functions of x.

The reason for doing this is, suppose you are handed an integral of the form $\int U dV$ that is difficult to solve. If we can somehow manage to identify $V(x)$, then perhaps the integral $\int V dU$ is easier than its counterpart. All of this inspires the *integration by parts* formula:

$$\int U dV = UV \Big| - \int V dU \tag{1.22}$$

Exemplary Case

To demonstrate integration by parts, consider the definite integral

$$I = \int_0^{\pi/2} x \cos(x)\, dx ,$$

which we immediately rewrite as

$$\int_0^{\pi/2} x \cos(x)\, dx = \int_0^{\pi/2} U dV .$$

Then identify

$$\begin{aligned} U &= x \\ dV &= \cos(x)\, dx , \end{aligned}$$

and we now have two 'mini problems' of determining $dU(x)$ and $V(x)$.

For this example, dU is simply equal to dx. (It's always easy to calculate dU.) As for V, we have $dV/dx = \cos(x)$, which can only mean $V(x) = \sin(x)$.

The integration by parts formula then tells us:

$$\int_0^{\pi/2} x \cos(x)\, dx = x \sin(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)\, dx$$

Notice how the 'hard' integral on the left is replaced by an 'easy' integral on the right. The answer is now straightforward:

$$I = \int_0^{\pi/2} x \cos(x)\, dx = \frac{\pi}{2} - 1$$

Natural Logarithm

The integration by parts recipe also works when there is one function in the integrand, and this is how to find the integral of the natural logarithm. Starting with

$$I = \int \ln(x)\, dx ,$$

let

$$\begin{aligned} U &= \ln(x) \\ dV &= dx \end{aligned}$$

such that

$$\begin{aligned} dU &= dx/x \\ V &= x . \end{aligned}$$

Then, we have

$$I = x \ln(x) \Big| - \int dx ,$$

simplifying to:

$$\int \ln(x)\, dx = x\ln(x) - x + C \tag{1.23}$$

Problem 9

Use u-substitution to find the integral of the shifted natural logarithm:

$$\int \ln(1+x)\, dx = (1+x)\ln(1+x) + x + C \tag{1.24}$$

Problem 10

Use integration by parts to show:

$$\int_0^\infty x^2 e^{-x} dx = 2$$

3.7 Label Trick

Consider the definite integral that attempts to calculate the area of one quarter of the unit circle:

$$A = \int_0^{\pi/2} \sin^2(\theta)\, d\theta$$

This can be attacked with integration by parts by letting

$$U = \sin(\theta)$$
$$dV = \sin(\theta)\, d\theta$$

such that

$$dU = \cos(\theta)\, d\theta$$
$$V = -\cos(\theta)\ ,$$

and then

$$A = -\cancel{\sin(\theta)\cos(\theta)\Big|_0^{\pi/2}} + \int_0^{\pi/2} \cos^2(\theta)\, d\theta\ .$$

All we've managed to show is that the function $\sin^2(\theta)$ can be replaced by $\cos^2(\theta)$ and the integral remains the same.

Now make use of the fundamental trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

to write

$$\int_0^{\pi/2} \cos^2(\theta)\, d\theta = \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin^2(\theta)\, d\theta\ .$$

The left-most and right-most integrals are both equal to A, and all of the hard work suddenly vanishes with the so-called *label trick*:

$$A = \int_0^{\pi/2} d\theta - A$$

Solving for A is a matter of algebra, and the remaining integral is trivial:

$$A = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

Tricky Logarithmic Integral

A tricky problem that you're welcome to stop reading and try on your own is the following definite integral:

$$I = \int_0^\infty \frac{\ln(x)}{1+x+x^2}\, dx$$

The key to this problem is the substitution $u = 1/x$. From this, we have $du/dx = -1/x^2$, and furthermore $\ln(1/u) = -\ln(u)$. The integration limits also end up swapping, and the integral becomes

$$I = \int_\infty^0 \frac{-\ln(u)}{1+1/u+1/u^2}\frac{-du}{u^2}\ .$$

Simplifying further, we find

$$I = \int_\infty^0 \frac{\ln(u)}{1+u+u^2}\, du\ ,$$

and swap the integration limits by paying with a negative sign:

$$I = -\int_0^\infty \frac{\ln(u)}{1+u+u^2}\, du$$

This result is exactly opposite to the problem we started with, up to a trivial change of letters. In effect, we have found

$$I = -I\ ,$$

which can *only* mean $I = 0$:

$$0 = \int_0^\infty \frac{\ln(x)}{1+x+x^2}\, dx$$

3.8 Trigonometric Integrals

Standard Functions

The integral of each trigonometric function is straightforwardly calculated using antiderivatives or other integration techniques. In indefinite form, these are:

$$\int \sin(x)\, dx = -\cos(x) + C \tag{1.25}$$

$$\int \cos(x)\, dx = \sin(x) + C \tag{1.26}$$

$$\int \tan(x)\,dx = -\int \frac{d(\cos(x))}{\cos(x)}$$
$$= -\ln(\cos(x)) + C \qquad (1.27)$$
$$\int \cot(x)\,dx = \int \frac{d(\sin(x))}{\sin(x)}$$
$$= \ln(\sin(x)) + C \qquad (1.28)$$

$$\int \sec(x)\,dx = \ln(\sec(x) + \tan(x)) + C \qquad (1.29)$$
$$\int \csc(x)\,dx = -\ln(\csc(x) + \cot(x)) + C \qquad (1.30)$$

We can also recall the derivative of each trigonometric function and make use of the $\int dx$ operator to come up with a few more:

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad (1.31)$$
$$\int \csc^2(x)\,dx = -\cot(x) + C \qquad (1.32)$$
$$\int \tan(x)\sec(x)\,dx = \sec(x) + C \qquad (1.33)$$
$$\int \cot(x)\csc(x)\,dx = -\csc(x) + C \qquad (1.34)$$

Squared Integrand

The pair of indefinite integrals

$$I_1 = \int \sin^2(x)\,dx$$
$$I_2 = \int \cos^2(x)\,dx$$

can be solved simultaneously. Using the fundamental trig identity, we see

$$I_1 + I_2 = \int \left(\sin^2(x) + \cos^2(x)\right) dx = \int dx = x \Big| \,,$$

or equivalently

$$I_1 + I_2 = x + C \,.$$

Now integrate I_1 by parts via

$$U = \sin(x)$$
$$dV = \sin(x)\,dx$$

such that

$$dU = \cos(x)\,dx$$
$$V = -\cos(x) \,,$$

and I_1 is written

$$I_1 = -\sin(x)\cos(x) \Big| + \int \cos^2(x)\,dx \,,$$

simplifying to

$$I_1 - I_2 = -\sin(x)\cos(x) + C \,.$$

With two equations and two unknowns, I_1 and I_2 can be isolated independently, resulting in

$$\int \sin^2(x)\,dx = \frac{-\sin(x)\cos(x)}{2} + \frac{x}{2} + C \qquad (1.35)$$
$$\int \cos^2(x)\,dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + C \qquad (1.36)$$

The pair of indefinite integrals

$$I_3 = \int \tan^2(x)\,dx$$
$$I_4 = \int \sec^2(x)\,dx$$

can also be solved together. Using another fundamental trig identity, find

$$I_4 - I_3 = x + C \,,$$

which means only I_3 or I_4 need be calculated and we get the other for free.

Choosing I_4, recall that the derivative of the tangent is the square of the secant, so

$$I_4 = \int \frac{d}{dx}\tan(x)\,dx = \tan(x) + C \,,$$

and conclude:

$$\int \tan^2(x)\,dx = \tan(x) - x + C \qquad (1.37)$$
$$\int \sec^2(x)\,dx = \tan(x) + C$$

Finally, the pair of indefinite integrals

$$I_5 = \int \cot^2(x)\,dx$$
$$I_6 = \int \csc^2(x)\,dx$$

can also be solved together. Using another fundamental trig identity, find

$$I_6 - I_5 = x + C \,.$$

The easiest way to proceed is to remember that the derivative of the cotangent is the negative of the square of the cosecant. Just kidding, that's not terribly easy to remember, but nonetheless the integral I_6 becomes

$$I_6 = \int \frac{d}{dx} \left(-\cot(x)\right) dx = -\cot(x) + C \ .$$

From the above we get the pair of answers:

$$\int \cot^2(x)\, dx = -\cot(x) - x + C \tag{1.38}$$

$$\int \csc^2(x)\, dx = -\cot(x) + C$$

Inverse Functions

Integrals of the inverse trigonometric functions can be tricky to find. Integration by parts works well on a few of them, such as the arctangent. For

$$I = \int \arctan(x)\, dx \ ,$$

let

$$\begin{aligned} U &= \arctan(x) \\ dV &= dx \end{aligned}$$

such that:

$$\begin{aligned} dU &= \frac{dx}{1+x^2} \\ V &= x \end{aligned}$$

With this, the integral reads

$$I = x \arctan(x) \Big| - \int \frac{x}{1+x^2}\, dx$$

The remaining integral is solved by standard u-substitution, namely $u = 1+x^2$ such that $du = 2x dx$. After simplifying, we get the answer:

$$\int \arctan(x)\, dx = x \arctan(x) - \frac{1}{2} \ln\left(1+x^2\right) + C \tag{1.39}$$

The same recipe works for several other inverse trigonometric functions, namely the arccosine, arcsine, and arccotangent:

$$\int \arccos(x)\, dx = x \cos(x) - \frac{1}{2} \ln\left(1-x^2\right) + C \tag{1.40}$$

$$\int \arcsin(x)\, dx = x \sin(x) + \frac{1}{2} \ln\left(1-x^2\right) + C \tag{1.41}$$

$$\int \text{arccot}(x)\, dx = x\, \text{arccot}(x) + \frac{1}{2} \ln\left(1+x^2\right) + C \tag{1.42}$$

Conspicuously absent from our stack of results are the integrals of the arcsecant and arccosecant. These require more than a simple u-substitution that we haven't hit yet, so stay tuned.

Reduction Formulas

For positive integer m, consider the indefinite integral

$$I = \int \sin^m(x)\, dx \ .$$

Integrating by parts, we first write

$$\begin{aligned} U &= \sin^{m-1}(x) \\ dV &= \sin(x)\, dx \end{aligned}$$

and also

$$\begin{aligned} dU &= (m-1) \sin^{m-2}(x) \cos(x)\, dx \\ V &= -\cos(x) \ . \end{aligned}$$

From this, we have

$$I = -\sin^{m-1}(x) \cos(x) \Big| + (m-1) \int \sin^{m-2}(x) \cos^2(x)\, dx \ .$$

Next, replace $\cos^2(x)$ with $1 - \sin^2(x)$ and use the label trick, giving

$$I = -\sin^{m-1}(x) \cos(x) \Big| + (m-1) \int \sin^{m-2}(x)\, dx - (m-1)\, I \ ,$$

and solving for I we arrive at a *trigonometric reduction formula*:

$$\int \sin^m(x)\, dx = \frac{-1}{m} \sin^{m-1}(x) \cos(x) \Big| + \frac{m-1}{m} \int \sin^{m-2}(x)\, dx \tag{1.43}$$

Similar reduction formulas exist for each of the elementary trig functions. Each of the following is attained by integration by parts and the label trick:

$$\int \cos^m(x)\,dx = \frac{1}{m}\cos^{m-1}(x)\sin(x)\Big| + \frac{m-1}{m}\int \cos^{m-2}(x)\,dx \quad (1.44)$$

$$\int \tan^m(x)\,dx = \frac{1}{m-1}\tan^{m-1}(x)\Big| - \int \tan^{m-2}(x)\,dx \quad (1.45)$$

$$\int \csc^m(x)\,dx = \frac{-1}{m-1}\csc^{m-2}(x)\cot(x)\Big| + \frac{m-2}{m-1}\int \csc^{m-2}(x)\,dx \quad (1.46)$$

$$\int \sec^m(x)\,dx = \frac{1}{m-1}\sec^{m-2}(x)\tan(x)\Big| + \frac{m-2}{m-1}\int \sec^{m-2}(x)\,dx \quad (1.47)$$

$$\int \cot^m(x)\,dx = \frac{-1}{m-1}\cot^{m-1}(x)\Big| - \int \cot^{m-2}(x)\,dx \quad (1.48)$$

Another reduction formula that mixes the sine and cosine can be established. Consider the case

$$I = \int \sin^m(x)\cos^n(x)\,dx\ .$$

By letting $u = \sin^{m-1}(x)$ and following the consequences, one finds

$$\int \sin^m(x)\cos^n(x)\,dx = -\frac{1}{m+n}\sin^{m-1}(x)\cos^{n+1}(x)\Big| + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)\,dx\ . \quad (1.49)$$

Note that this result reproduces Equation (1.43) for $n = 0$.

A different result is attained by letting $u = \cos^{n-1}(x)$:

$$\int \sin^m(x)\cos^n(x)\,dx = \frac{1}{m+n}\sin^{m+1}(x)\cos^{n-1}(x)\Big| + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)\,dx\ . \quad (1.50)$$

Note that this result reproduces Equation (1.44) for $m = 0$.

Mixed Wavelengths

Starting with the product formula

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)\ ,$$

suppose that α, β are multiples of an angle θ

$$\alpha = m\theta$$
$$\beta = n\theta$$

for non-equal integers m, n.

Next apply the integral operator $\int d\theta$ across the whole equation

$$2\int \sin(m\theta)\cos(n\theta)\,d\theta = \int \sin(m\theta+n\theta)\,d\theta + \int \sin(m\theta-n\theta)\,d\theta\ ,$$

and simplify:

$$\int \sin(m\theta)\cos(n\theta)\,d\theta = \frac{-\cos((m+n)\theta)}{2(m+n)} - \frac{\cos((m-n)\theta)}{2(m-n)} + C \quad (1.51)$$

More product formula exploits lead to additional mixed-wavelength integral identities:

$$\int \cos(m\theta)\cos(n\theta)\,d\theta = \frac{\sin((m+n)\theta)}{2(m+n)} + \frac{\sin((m-n)\theta)}{2(m-n)} + C \quad (1.52)$$

$$\int \sin(m\theta)\sin(n\theta)\,d\theta = \frac{-\sin((m+n)\theta)}{2(m+n)} + \frac{\sin((m-n)\theta)}{2(m-n)} + C \quad (1.53)$$

Orthogonality

Evaluating the mixed-wavelength integral identities (1.51)-(1.53), in various domains of length 2π leads to some additional information called *orthogonality relations*. (Keep in mind that m, n are different integers.)

Choosing $[-\pi : \pi]$ first, we find

$$\int_{-\pi}^{\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{-\pi}^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{-\pi}^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta = 0$$

The same results hold when the domain is changed to $[0 : 2\pi]$:

$$\int_{0}^{2\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{0}^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{0}^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta = 0$$

When the wavelengths m, n are one and the same integer m, two results switch to nonzero

$$\int_{-\pi}^{\pi} \cos^2(m\theta)\,d\theta = \int_{0}^{2\pi} \cos^2(m\theta)\,d\theta = \pi$$
$$\int_{-\pi}^{\pi} \sin^2(m\theta)\,d\theta = \int_{0}^{2\pi} \sin^2(m\theta)\,d\theta = \pi\ ,$$

and the case that mixes sine and cosine remains zero:

$$\int_{-\pi}^{\pi} \sin(m\theta)\cos(m\theta)\,d\theta$$
$$= \int_{0}^{2\pi} \sin(m\theta)\cos(m\theta)\,d\theta = 0$$

3.9 Hyperbolic Integrals

...

3.10 Trigonometric Substitution

Each of the following integrals

$$I_1 = \int \frac{dx}{x^2\sqrt{x^2+a^2}}$$
$$I_2 = \int \frac{\sqrt{a^2-x^2}}{x^2}\,dx$$
$$I_3 = \int \frac{dx}{\left(x^2-a^2\right)^{3/2}}$$

for nonzero constant a are difficult to solve by standard u-substitution or integration by parts. In fact, each requires a different trick called *trigonometric substitution*.

Tangent Substitution

When the integrand contains $x^2 + a^2$, let

$$x = a\tan(\theta)\ ,$$

so then

$$dx = a\sec^2(\theta)\,d\theta\ .$$

By standard trig identities, the quantity $x^2 + a^2$ becomes

$$x^2 + a^2 = a^2\sec^2(\theta)\ .$$

With this, the integral I_1 transforms into something we can solve:

$$I_1 = \int \frac{a\sec^2(\theta)}{a^3\tan^2(\theta)\sec(\theta)}\,d\theta = \frac{1}{a^2}\int \frac{d(\sin(\theta))}{\sin^2(\theta)}$$

<u>Problem 11</u>

Use the above as a starting point to prove:

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + C$$

Sine Substitution

When the integrand contains $a^2 - x^2$, let

$$x = a\sin(\theta)\ ,$$

so then

$$dx = a\cos(\theta)\,d\theta\ .$$

By standard trig identities, the quantity $a^2 - x^2$ becomes

$$a^2 - x^2 = a^2\cos^2(\theta)\ .$$

With the sine substitution, the integral I_2 reduces to a simpler problem:

$$I_2 = \int \frac{a^2\cos^2(\theta)}{a^2\sin^2(\theta)}\,d\theta = \int \cot^2(\theta)\,d\theta$$

<u>Problem 12</u>

Use the above as a starting point to prove:

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\arcsin\left(\frac{x}{a}\right) - \frac{\sqrt{a^2-x^2}}{x} + C$$

Secant Substitution

When the integrand contains $x^2 - a^2$, let

$$x = a \sec(\theta) \ ,$$

so then

$$dx = a \sec(\theta) \tan(\theta) \, d\theta \ .$$

By standard trig identities, the quantity $x^2 - a^2$ becomes

$$x^2 - a^2 = a^2 \tan^2(\theta) \ .$$

With the sine substitution, the integral I_3 reduces to a simpler problem:

$$I_3 = \frac{1}{a^2} \int \frac{d(\sin(\theta))}{\sin^2(\theta)} d\theta$$

Problem 13

Use the above as a starting point to prove:

$$\int \frac{dx}{(x^2 - a^2)^{3/2}} = \frac{-x}{a^2 \sqrt{x^2 - a^2}} + C$$

Trigonometric Ratios

Rational functions of sine and cosine land to a particular u-substitution:

$$u = \tan(\theta/2) \ .$$

From the trigonometric half-angle formulas, we can next write

$$\begin{aligned} \cos(\theta) &= \frac{1 - u^2}{1 + u^2} \\ \sin(\theta) &= \frac{2u}{1 + u^2} \ , \end{aligned}$$

and

$$\begin{aligned} u &= \frac{\sin(\theta)}{1 + \cos(\theta)} \\ du &= \frac{1}{2} \left(1 + u^2\right) d\theta \ . \end{aligned}$$

With this substitution, integrals of the form

$$I = \int f(\sin(\theta), \cos(\theta)) \, d\theta$$

can be written:

$$I = \int f\left(\frac{2u}{1 + u^2}, \frac{1 - u^2}{1 + u^2}\right) \frac{du}{1 + u^2}$$

In the general case, this substitution works when the function being integrated is a polynomial of two variables or a ratio of two polynomials.

To illustrate, consider the indefinite integral

$$J = \int \frac{d\theta}{3 + \cos(\theta)} \ .$$

Using the above substitutions, the integral becomes

$$J = \int \frac{du}{2 + u^2} \ .$$

Problem 14

Use the above as a starting point to prove:

$$\int \frac{d\theta}{3 + \cos(\theta)} = \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} \tan\left(\frac{\theta}{2}\right)\right) + C$$

Arcsecant and Arccosecant

The integrals of the arcsecant and the arccosecant have to be cracked with trigonometric substitution. For

$$I = \int \operatorname{arcsec}(x) \, dx \ ,$$

proceed with integration by parts to write

$$\begin{aligned} U &= \operatorname{arcsec}(x) \\ dV &= dx \end{aligned}$$

such that

$$\begin{aligned} dU &= \frac{dx}{x\sqrt{x^2 - 1}} \\ V &= x \ . \end{aligned}$$

The integral becomes

$$\int \operatorname{arcsec}(x) \, dx = x \operatorname{arcsec}(x) \Big| - \int \frac{dx}{\sqrt{x^2 - 1}} \ .$$

The new integral on the right is handled by a secant substitution. Let

$$x = \sec(\theta)$$

such that

$$dx = \sec(\theta) \tan(\theta) \, d\theta \ ,$$

and

$$\sqrt{x^2 - 1} = \tan(\theta) \ ,$$

so we have

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int \sec(\theta) \, d\theta \ .$$

The integral of the secant has a known solution, namely

$$\int \sec(\theta) \, d\theta = \ln(\sec(\theta) + \tan(\theta)) + C \ ,$$

or, in terms of the x-variable,

$$\int \sec(\theta)\, d\theta = \ln\left(x + \sqrt{x^2-1}\right) + C \,.$$

Finally, we have the answer:

$$\int \operatorname{arcsec}(x)\, dx = x \operatorname{arcsec}(x) \Big| - \ln\left(x + \sqrt{x^2-1}\right) + C \quad (1.54)$$

Problem 15

Do a similar exercise for the arccosecant:

$$\int \operatorname{arccsc}(x)\, dx = x \operatorname{arccsc}(x) \Big| + \ln\left(x + \sqrt{x^2-1}\right) + C \quad (1.55)$$

Area of the Ellipse

For the ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,,$$

the area contained in the first quadrant (a quarter of the ellipse) is given by

$$A = \int_0^a y(x)\, dx \,.$$

It also happens that the same ellipse can be described using a pair of parametric equations, particularly

$$x = a\cos(\theta)$$
$$y = b\sin(\theta) \,,$$

easily shown to reproduce the Cartesian formula. Substituting the above equations for x, y into the area integral changes the integration variable to θ:

$$A = -ab \int_{\pi/2}^{0} \sin^2(\theta)\, d\theta$$

The integral of the square of the sine is well known by known, particularly by equation (1.35). Evaluating the definite integral gives the final answer:

$$A = -ab\left(\frac{-\pi}{4}\right) = \frac{1}{4}\pi ab$$

The area of the complete ellipse is πab.

Problem 16

Show that the area of the ellipse

$$ax^2 + bxy + cy^2 = 1$$

is equal to

$$A = \frac{2\pi}{\sqrt{4ac - b^2}} \,.$$

Hint: Rotate the coordinates and write the area of the same ellipse in the rotated system.

3.11 Mirror Trick

A lesser-known technique we'll call the *mirror trick* can help with integrals such as

$$J = \int_0^{\pi/2} \frac{\sqrt{\sin(\theta)}}{\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}}\, d\theta \,.$$

For practice, consider the definite integral of a well-behaved function $g(x)$:

$$I = \int_a^b g(x)\, dx$$

By making the substitution

$$u = b + a - x$$
$$du = -dx \,,$$

we find

$$I = \int_b^a g(b + a - u)(-du) \,.$$

Of course, the integration variable itself can be swapped with any other letter, so we come up with a second equation for J involving the integral in the x-domain:

$$I = \int_a^b g(b + a - x)\, dx$$

The same idea can be applied to a different integral

$$K = \int_a^b \frac{g(x)}{g(b + a - x) + g(x)}\, dx \,,$$

which, using the same u-substitution $u = b + a - x$, becomes

$$K = \int_b^a \frac{g(b + a - u)}{g(u) + g(b + a - u)}(-du) \,,$$

or equivalently

$$K = \int_a^b \frac{g(b + a - x)}{g(x) + g(b + a - x)}\, dx \,.$$

Take the two expressions for K and take their sum,

$$2K = \int_a^b \frac{g(x) + g(b + a - x)}{g(x) + g(b + a - x)}\, dx \,,$$

and notice the entire integrand cancels, leaving

$$K = \frac{b-a}{2} \, .$$

Evidently, the result of integral K has nothing to do with the function being integrated, only the limits matter:

$$\int_a^b \frac{g(x)}{g(b+a-x)+g(x)}\,dx = \frac{b-a}{2} \tag{1.56}$$

Returning to the problem on hand, the integral J can be written

$$J = \int_0^{\pi/2} \frac{\sqrt{\sin(\theta)}}{\sqrt{\sin(\theta)}+\sqrt{\sin(\pi/2-\theta)}}\,d\theta \, .$$

Comparing this to Equation (1.56), let $a = 0$ and $b = \pi/2$ and the result is half their difference:

$$J = \frac{\pi}{4}$$

3.12 Series Expansion

Integration and series expansion play nicely together and are used often to approximate the solution to otherwise insoluble integrals.

Physical Pendulum

It's possible to show using energy conservation that a frictionless pendulum of length L and mass m in uniform gravity is governed by

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}}\sqrt{\cos(\theta)-\cos(\theta_0)} \, ,$$

where θ is the deflection of the pendulum from vertical and θ_0 represents the highest position attainable where motion momentarily stops. This is a 'separable' equation, and can be reshuffled as an indefinite integral:

$$\int \frac{d\theta}{\sqrt{\cos(\theta)-\cos(\theta_0)}} = \sqrt{\frac{2g}{L}}\int dt$$

The left side needs some preparation before proceeding. The cosine terms are replaced using the half-angle formula

$$1-\cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right) \, .$$

Also define

$$\sin(\phi) = \frac{1}{a}\sin\left(\frac{\theta}{2}\right) \, ,$$

where

$$a = \sin\left(\frac{\theta_0}{2}\right) \, ,$$

implying

$$d\theta = 2a\frac{\sqrt{1-\sin^2(\phi)}}{\sqrt{1-a^2\sin^2(\phi)}}\,d\phi \, .$$

With these substitutions, the integral on hand becomes

$$\int \frac{d\phi}{\sqrt{1-a^2\sin^2(\phi)}} = \sqrt{\frac{g}{L}}\int dt \, .$$

The left side is called an *elliptic integral*, and has no simple closed-form solution in general.

Despite the above elliptic integral, we can still use it to crank out an answer. Let $t = 0$ correspond to $\theta = 0$ and $\phi = 0$, which is the lowest position available to the pendulum. After one period of motion at $t = T$, i.e. once the angle θ has returned to zero again, and the value 2π has accumulated in ϕ. We then have a formula for the period of the motion:

$$\sqrt{\frac{g}{L}}\int_0^T dt = \int_0^{2\pi}\frac{d\phi}{\sqrt{1-a^2\sin^2(\phi)}} \, .$$

On the right, use the Taylor expansion of the radical to write

$$\frac{1}{\sqrt{1-a^2\sin^2(\phi)}} \approx 1+\frac{1}{2}a^2\sin^2(\phi) + \frac{3}{8}a^4\sin^4(\phi)+\cdots \, ,$$

which only works when $a\sin(\phi)$ is a relatively 'small' number.

While we have paid with some accuracy and generality, the thing we gain is that the right side can be integrated. Going term by term it helps to know

$$\int_0^{2\pi}\sin^2(\phi)\,d\phi = \pi$$

$$\int_0^{2\pi}\sin^4(\phi)\,d\phi = \frac{3\pi}{4} \, ,$$

attainable by elementary means or using a trigonometric reduction formula.

The integral for the period reduces to

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{a^2}{4}+\frac{9a^4}{64}+\cdots\right) \, .$$

If the initial angle θ_0 is much less than one, we further have

$$a^2 \approx \frac{\theta_0^2}{4} \, ,$$

or

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{\theta_0^2}{16}\right) .$$

From this we get the familiar period of the simple pendulum, along with a correction that accounts for more extreme initial conditions.

Problem 17

Re-work the physical pendulum calculation without oversimplifying the series coefficients to derive:

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\left(\frac{1}{2}\right)^2 a^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 a^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 a^6+\cdots\right)$$

Shifted Natural Logarithm

Starting with Equation (1.21), namely

$$\ln\left(1+x\right)+C=\int\frac{dx}{1+x} ,$$

consider the scenario $|x|<1$.

In this case, the fraction $1/\left(1+x\right)$ can be replaced via the geometric series:

$$\ln\left(1+x\right)+C=\int\left(1-x+x^2-x^3+\cdots\right)dx$$

The whole right side can be integrated quite easily:

$$\ln\left(1+x\right)+C=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

The integration constant is zero by construction, and we end up with an infinite series for the shifted natural logarithm:

$$\ln\left(1+x\right)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

This result is in fact the same thing we'd get by Taylor expanding $\ln\left(1+x\right)$ near $x=0$. Unlike the Taylor expansion however, we can now say for certain that the series approximation of $\ln\left(1+x\right)$ converges for $|x|<1$. We can be a little naughty and try $x=1$ exactly to come up with an infinite approximation for $\ln\left(2\right)$:

$$\ln\left(2\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \tag{1.57}$$

Arctangent

Recall that the derivative of the arctangent function

$$\frac{d}{dx}\arctan\left(x\right)=\frac{1}{1+x^2} ,$$

and consider the case $|x|<1$. The right side expands as a geometric series:

$$\frac{d}{dx}\arctan\left(x\right)=1-x^2+x^4-x^6+\cdots$$

Apply the $\int dx$ operator to both sides and simplify, to get, for $|x|<1$,

$$\arctan\left(x\right)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots .$$

The integration constant is easily ruled to be zero and is omitted. Nor surprisingly, this is what emerges when Taylor expanding $\arctan\left(x\right)$ near $x=0$.

The infinite expression for the arctangent can be used to come up with an expression for $\pi/4$ by setting $x=1$,

$$\frac{\pi}{4}\approx 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots , \tag{1.58}$$

called the *Leibniz formula.*

It's important to note that Equations (1.57), (1.58) each send $x=1$ to the geometric series, which may seem illegal, as this is where the geometric series is supposed to lose jurisdiction. Technically, each result is attained by letting $x\to 1$ in a formal limit, and making sure divergence does not occur.

Sine of X Squared

Innocent as it appears, the indefinite integral

$$I=\int\sin\left(x^2\right)dx$$

has no elementary solution. To make headway, replace the sine function with its exact polynomial representation, namely

$$\sin\left(x^2\right)=x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\cdots .$$

Suddenly, we see a path forward. By trading any possibility of a closed solution, we can at least deal with the right side. Integrate each term and a strange answer emerges:

$$\int\sin\left(x^2\right)dx=\frac{x^3}{3}-\frac{x^7}{42}+\frac{x^{11}}{1320}-\cdots \tag{1.59}$$

3.13 Stirling's Approximation

There is an important relationship governing very large whole numbers called *Stirling's approximation*, given by

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,.$$

While a full derivation is beyond the scope of this section, we can establish a slightly weaker version, namely

$$n! \approx \left(\frac{n}{e}\right)^n\,. \tag{1.60}$$

Derivation

To begin, write $n!$ in open form, namely

$$n! = n\,(n-1)\,(n-2)\,(n-3)\cdots(2)\,(1)\ ,$$

then take the natural log of both sides to write

$$\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots\ ,$$

and condense the right using summation notation:

$$\ln(n!) = \sum_{j=1}^{n} \ln(j)$$

Now, this almost looks like a Riemann sum if it weren't for the conspicuous absence of a Δx-like term. However, since the sum runs over whole numbers only, there is an effective $\Delta x_j = 1$ at play:

$$\ln(n!) = \sum_{j=1}^{n} \ln(j)\,\Delta x_j$$

Even though Δx_j cannot be pushed to zero, the above sum can be approximated as continuous *anyway*, but only for very large n. Working in this regime, we can replace the above with

$$\ln(n!) \approx \int_1^n \ln(x)\,dx\ ,$$

solved by

$$\ln(n!) \approx (\ln(x) - x)\Big|_1^n\ ,$$

having approximate solution

$$\ln(n!) \approx \ln(n) - n\,.$$

Apply the $\exp()$ operator to isolate the factorial term, and Equation (1.60) emerges.

Strange Product

Let us simplify the quantity

$$A = \lim_{n\to\infty}\left(\frac{(n+1)\,(n+2)\cdots(3n)}{n^{2n}}\right)^{1/n}$$

as far as possible.

One way to proceed is to take the natural log of both sides and simplify:

$$\ln(A) = \lim_{n\to\infty}\frac{\ln(n+1) + \cdots + \ln(3n) - 2n\ln(n)}{n}$$

There are $2n$ total positive terms in the sum above, so we can break apart the negative term into $2n$ parts and subtract $\ln(n)$ from each positive term to get:

$$\ln(A) = \lim_{n\to\infty}\frac{1}{n}\cdot \left(\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{2n}{n}\right)\right)$$

Simplifying, this is

$$\ln(A) = \lim_{n\to\infty}\sum_{j=1}^{2n}\ln\left(1+\frac{j}{n}\right)\frac{1}{n}\,.$$

Using the same trick that led to Stirling's approximation, argue that because the largeness of j will dominate anything to do with small j, the sum can be considered continuous with

$$x_j = j/n$$
$$\Delta x = 1/n\,.$$

In this regime, we have, approximately:

$$\ln(A) \approx \int_0^2 \ln(1+x)\,dx\ ,$$

equivalent to

$$\ln(A) \approx \int_1^3 \ln(u)\,du\ ,$$

having solution

$$\ln(A) \approx (u\ln(u) - u)\Big|_1^3\ ,$$

or

$$\ln(A) \approx 3\ln(3) - 2\ ,$$

and, finally,

$$A \approx \frac{3^3}{e^2}\,.$$

Let us now do the same calculation using Stirling's approximation. First notice A can be written

$$A = \lim_{n\to\infty} \left(\frac{1}{n^{2n}} \frac{(3n)!}{n!} \right)^{1/n} ,$$

and then Equation (1.60) tells us

$$A \approx \lim_{n\to\infty} \left(\frac{1}{n^{2n}} \frac{(3n)^{3n}}{e^{3n}} \frac{e^n}{n^n} \right)^{1/n} ,$$

reducing to $A \approx 3^3/e^2$, as expected. All n-dependence cancels out.

Strange Function

Consider the function

$$A(x) = \lim_{n\to\infty} \left(\frac{1}{n^{xn}} \frac{((x+1)\,n)!}{n!} \right)^{1/n} ,$$

where $x = 2$ reproduces the previous product.

Following similar steps, it's straightforward to show that

$$\ln(A(x)) = \lim_{n\to\infty} \sum_{j=1}^{xn} \ln\left(1 + \frac{j}{n}\right) \frac{1}{n} ,$$

or

$$\ln(A(x)) \approx \int_0^x \ln(1+t)\, dt ,$$

but let's resist solving the integral.

Attack the problem a second way using Stirling's approximation to get

$$A(x) \approx \frac{(x+1)^{x+1}}{e^x} ,$$

or

$$\ln(A(x)) = (x+1)\ln(x+1) - x .$$

With two ways to express $\ln(A)$, eliminate it to conclude

$$\int \ln(1+x)\, dx = (x+1)\ln(x+1) - x + C ,$$

which happens to be correct.

4 Integrals and Geometry

4.1 Arc Length

Integration is the tool for calculating the arc length of a differentiable curve $y = f(x)$. At a given point (x, y) on such a curve, there is a neighboring point $(x+dx, y+dy)$ connected by a straight line of length

$$dS = \sqrt{dx^2 + dy^2} .$$

The term dx can be pulled out of the radical to get

$$dS = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} ,$$

and notice the ratio dy/dx is none other than the slope $f'(x)$ of the curve being measured.

The integral over dS is the total length of the curve between a set of endpoints x_0, x_1:

$$S = \int dS = \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2}\, dx \tag{1.61}$$

Note that a similar formula can be derived by removing dy from the radical and ending up with an integral in the y-domain.

<u>Problem 18</u>

Show that the arc length of a symmetric parabolic segment of base $2a$ and height h is:

$$\begin{aligned} L &= \frac{a^2}{h} \int_0^{2h/a} \sqrt{1+x^2}\, dx \\ &= \sqrt{a^2 + 4h^2} + \frac{a^2}{2h} \ln\left(\frac{2h + \sqrt{a^2+4h^2}}{a}\right) \end{aligned}$$

Hint: You may need the secant reduction formula.

<u>Problem 19</u>

Show that the arc length of an ellipse with eccentricity e is given by the *complete elliptic integral of the second kind*:

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2(\theta)}\, d\theta$$

<u>Problem 20</u>

Show that the arc length of a hyperbola with eccentricity e is given by another elliptic integral:

$$L = 4a \int \sqrt{e^2 \cosh^2(\theta) - 1}\, d\theta$$

4.2 Volume of Revolution

A sneaky way to calculate certain three-dimensional volumes using one-dimensional integrals can be established. For this we require differentiable functions $y = f(x)$ that are greater than zero in the domain $x_0 \le x \le x_1$.

Circular Disk Method

A three-dimensional volume with axial symmetry can be produced by rotating the curve $y(x)$ about the x-axis. Each height y on the curve is swung around one full revolution to trace out a disk of area πy^2, and the total volume enclosed is the sum across the grain of many infinitely-thin disks. As an integral, such a *volume of revolution* is given by:

$$V = \int_{x_0}^{x_1} \pi \left(f(x)\right)^2 dx \tag{1.62}$$

Problem 21

Show that a cone of height H and base radius R has volume

$$V = \frac{1}{3}\pi R^2 H \, .$$

Problem 22

Use elementary methods to show that a *cone frustum* of height H with end radii R_1, R_2 has volume

$$V = \frac{1}{3}\pi \left(R_1^2 + R_1 R_2 + R_2^2\right) H \, .$$

Use the disk method with the line

$$y = \left(\frac{R_2 - R_1}{H}\right) x + R_1$$

to get the same answer.

Problem 23

A *paraboloid* is the volume formed by a parabola rotated about its axis of symmetry. Show that the volume of a paraboloid of height H and base radius R is given by

$$V = \frac{1}{2}\pi R^2 H \, .$$

Hint: Rotate the parabola $y = Hx^2/R^2$ about the y-axis and x becomes the disk radius.

Square Disk Method

Modifying the circular disk method, one can imagine summing across square disks instead. To illustrate, suppose a pyramid with square cross section has height h, length l, and width w.

We'll take the square cross section as parallel to the xy-plane, and we will integrate vertically along z. For a given height $z \le h$, the dimensions of a 'square disk' are

$$\begin{aligned} x(z) &= z\, l/h \\ y(z) &= y\, w/h \, . \end{aligned}$$

The total volume the pyramid is

$$V = \frac{lw}{h^2}\int_0^h z^2 dz = \frac{lwh}{3} \, .$$

Washer Method

Introducing a second function $g(x)$ that is less than $f(x)$ but greater than zero in the domain, we can calculate the volume of revolution trapped between the two curves. In this case, simply subtract the area of one disk from the other to form a 'washer'. The corresponding volume integral becomes:

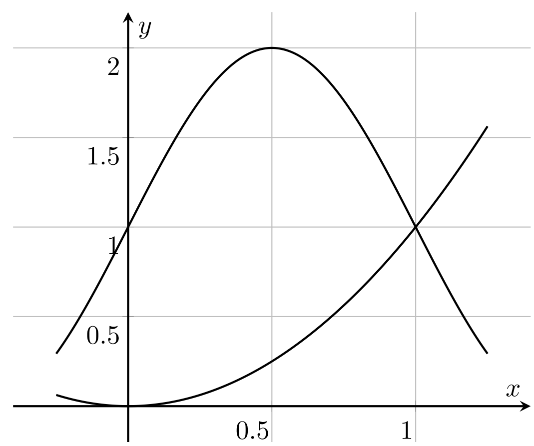
$$V = \int_{x_0}^{x_1} \pi \left(\left(f(x)\right)^2 - \left(g(x)\right)^2 \right) dx \tag{1.63}$$

Problem 24

In the domain $0 \le x \le 1$, consider the two curves

$$\begin{aligned} y_1 &= 1 + \sin(\pi x) \\ y_2 &= x^2 \end{aligned}$$

as shown. Write an expression for the volume of revolution about the x-axis and also the y-axis.



Hint: For the x-axis rotation, you should find:

$$V_x = \pi \int_0^1 \left(\left(1 + \sin(\pi x)\right)^2 - x^4 \right) dx$$

Then, with

$$\begin{aligned} x_1 &= \sqrt{y} \\ x_2 &= \frac{1}{\pi}\arcsin(y-1) \, , \end{aligned}$$

find

$$V_y = \pi \int_0^1 y\, dy + \pi \int_1^2 \left(\left(1 - x_2\right)^2 - x_2^2 \right) dy \, .$$

Cylindrical Shell Method

A different volume of revolution is attained by rotating the function $y = f(x)$ about the y-axis. In this case, a three-dimensional volume is made of many concentric cylindrical shells.

For a point x in the domain, along with a neighboring point $x + dx$, rotating about the y-axis traces a pair of circles whose radii differ by dx. The height of each circle is $f(x)$, $f(x + dx)$ respectively. This defines a cylindrical 'shell' having volume

$$\begin{aligned} dV_{\text{shell}} &= \pi (x + dx)^2 f(x + dx) \\ &\quad - \pi (x)^2 f(x) \ , \end{aligned}$$

or, in the first-order limit,

$$dV_{\text{shell}} = 2\pi x f(x)\, dx \ .$$

In essence, we see that the volume of a thin cylindrical shell is the same as that of a rectangle of thickness dx, height $f(x)$, and width $2\pi x$. The total volume is the integral of thin shells:

$$V = \int dV_{\text{shell}} = \int_{x_0}^{x_1} 2\pi x f(x)\, dx \tag{1.64}$$

Problem 25

Show that the volume of the upper half of a sphere of radius R is given by

$$V = \int_0^R 2\pi x \sqrt{R^2 - x^2}\, dx = \frac{2}{3}\pi R^3 \ .$$

Problem 26

Use the offset circle

$$(x - R)^2 + y^2 = a^2$$

to find the volume of a *torus*:

$$V = 2\int_{R-a}^{R+a} 2\pi x \sqrt{a^2 - (x - R)^2}\, dx = 2\pi^2 R a^2$$

4.3 Surface of Revolution

A technique similar to the volume of revolution can tell us the *surface area of revolution* of a solid generated by a function $y = f(x)$.

For a point x in the domain, along with a neighboring point $x + dx$, rotating about the x-axis traces a pair of circles parallel to the yz plane. The circumference of each circle is $2\pi y$, $2\pi (y + dy)$, respectively. We can take each circumference as the edges of a skinny trapezoid whose width is the arc length

$$dw = \sqrt{dx^2 + dy^2} \ ,$$

and the area of such a trapezoid is

$$dA = \pi (2y + dy)\, dw \ .$$

In the first-order limit, we can write the differential area:

$$dA = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Summing across the grain of many thin strips will cover the surface and reveal the total area of revolution for $y = f(x)$:

$$A = \int_{x_0}^{x_1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{1.65}$$

Gabriel's Horn

Consider the hyperbola

$$y = \frac{1}{x}$$

in the domain

$$1 \le x < \infty \ .$$

The volume of revolution of this particular shape is called *Gabriel's horn*, and contains an interesting 'paradox'. Computing the volume of Gabriel's horn is straightforward:

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = 1$$

Watch what happens if we try to compute the surface area:

$$A = \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}}\, dx$$

The square root term makes the integral rather ugly, but notice how its presence always scales the integrand higher. This means we can also write

$$A > \int_1^\infty 2\pi \left(\frac{1}{x}\right) dx \ ,$$

which means

$$A > 2\pi \left(\ln(\infty) - \ln(1)\right) \ .$$

What? The area is somehow infinite - the math was done correctly. But this shouldn't be, because the volume is a finite number. Some argued that filling the horn with a finite volume of paint is equivalent to painting the inside, which ought to make the area finite. Others pointed out that an infinite horn cannot be physically constructed, and that paint flows at a finite speed and would take forever to flow into the horn.

This ‘paradox’, originally brought to public attention by Torricelli, was known to seventeenth-century mathematicians, not excluding Hobbes, Wallis, and Galileo.

There really is no paradox on hand, and paint is a bad analogy. Keep in mind that paint is a three-dimensional fluid. Filling Gabriel’s horn with fluid returns to the original problem - what’s the surface area of the paint (excluding the end disc)?

Another way to illustrate the point is to compare the rates of change of the volume and surface with respect to x. Using

$$\begin{aligned} \frac{dV}{dx} &= \pi \left(\frac{1}{x}\right)^2 \\ \frac{dA}{dx} &= 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}} , \end{aligned}$$

define the rate

$$R = \frac{dV/dx}{dA/dx} ,$$

simplifying to

$$R = \frac{1}{2x\sqrt{1 + 1/x^4}} .$$

This rate vanishes in the limit $x \to \infty$, which means the area outpaces the volume in the long run.

4.4 Centroid

...

Problem 27

Show that the centroid of a parabolic segment of height h is $\bar{y} = 2h/5$.

Problem 28

Show that the centroid of a half-ellipse of base $2a$ and vertex height b is $\bar{y} = 4b/3\pi$.

5 Series Analysis

Integration is a powerful addition to the toolkit for analyzing infinite sums, particularly on the issues of convergence and divergence.

5.1 Taylor Series

The most versatile series is surely the Taylor series, which tells that a function $f(x)$ at a point x_0 is approximated by a polynomial $p(x)$ involving derivatives $f^{(q)}(x_0)$:

$$p(x) = f(x_0) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}(x_0)(x - x_0)^j + R_n(x)$$

For large n approaching infinity, the remainder term $R_n(x)$ vanishes if the series is to converge.

Derivation

To derive Taylor’s theorem, begin with the fundamental theorem of calculus, i.e. Equation (1.2), and isolate $f(x)$:

$$f(x) = f(x_0) + \int_{x_0}^{x} f^{(1)}(t)\, dt$$

Of course, the function $f^{(1)}(t)$ could itself be approximated to first order using the fundamental theorem

$$f^{(1)}(t) = f^{(1)}(x_0) + \int_{x_0}^{t} f^{(2)}(u)\, du ,$$

which begs substitution into the above, giving:

$$f(x) = f(x_0) + \int_{x_0}^{x} \left(f^{(1)}(x_0) + \int_{x_0}^{t} f^{(2)}(u)\, du \right) dt$$

After simplifying, we see the familiar first-order Taylor series term trailed by a messy integral:

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &\quad + \int_{x_0}^{x} \left(\int_{x_0}^{t} f^{(2)}(u)\, du \right) dt \end{aligned}$$

Trudging forward, take $f^{(2)}(u)$ to first order

$$f^{(2)}(u) = f^{(2)}(x_0) + \int_{x_0}^{w} f^{(3)}(w)\, dw ,$$

and substitute into the preceding integral. This first means having to solve

$$I = \int_{x_0}^{x} \left(\int_{x_0}^{t} f^{(2)}(x_0)\, du \right) dt .$$

Knowing $f^{(2)}(x_0)$ is constant, proceed using brute force to find

$$\begin{aligned} I &= f^{(2)}(x_0) \int_{x_0}^{x} \left(\int_{x_0}^{t} du \right) dt \\ &= f^{(2)}(x_0) \int_{x_0}^{x} (t - x_0)\, dt \\ &= f^{(2)}(x_0) \left(\frac{t^2}{2} - x_0 t \right) \Big|_{x_0}^{x} \\ &= \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 . \end{aligned}$$

Interestingly, this is the second-order term in the Taylor series of $f(x)$. To summarize:

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &+ \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 \\ &+ \int_{x_0}^{x} \left(\int_{x_0}^{t} \left(\int_{x_0}^{u} f^{(3)}(w)\, dw \right) du \right) dt \end{aligned}$$

Repeating the steps that got us this far, use the first-order approximation of $f^{(3)}(w)$. The obligatory integral to solve is

$$J = f^{(3)}(x_0) \int_{x_0}^{x} \left(\int_{x_0}^{t} \left(\int_{x_0}^{u} dw \right) du \right) dt \,,$$

which after a bit of grinding, comes out to

$$J = \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 \;.$$

By now we're seeing a pattern, particularly:

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &+ \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 \\ &+ \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 \\ &+ R_3(x) \;, \end{aligned}$$

where $R_3(x)$ is given by

$$\int_{x_0}^{x} \left(\int_{x_0}^{t} \left(\int_{x_0}^{u} \left(\int_{x_0}^{v} f^{(4)}(v)\, dv \right) dw \right) du \right) dt \,.$$

Remainder

In the general case, the remainder term $R_n(x)$ always contains a polynomial term plus an integral. Since the integral part ends up being higher order than n, we can always push the hard work to the next step, so to speak, and take as the remainder term:

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(x_0)(x - x_0)^{n+1}$$

6 Applied Integration

6.1 Kinematics

The equations of constant-acceleration kinematics in one dimension fall out trivially from integration. Starting with the definition

$$\frac{d}{dt} v(t) = a = \text{constant} \,,$$

integrate with respect to time to find

$$v(t) = v_0 + at \,,$$

where v_0 is the integration constant. Integrate with respect time once more to write

$$\begin{aligned} \int v(t)\, dt &= \int (v_0 + at)\, dt \\ x(t) &= x_0 + v_0 t + \frac{1}{2} a t^2 \,. \end{aligned}$$

The integration constant has been lebeled x_0.

Similarly, one can develop the regime of constant jerk by starting with

$$\frac{d}{dt} a(t) = j = \text{constant}$$

such that

$$a(t) = a_0 + jt \,.$$

Turning the same crank, one eventually finds

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} j t^3 \,.$$

6.2 The Cycloid

Let a 'generating' circle of radius R roll on the x axis. As the circle moves, the point on the rim originally at $(0, 0)$ traces the shape of a *cycloid* as shown in Figure 1.1.

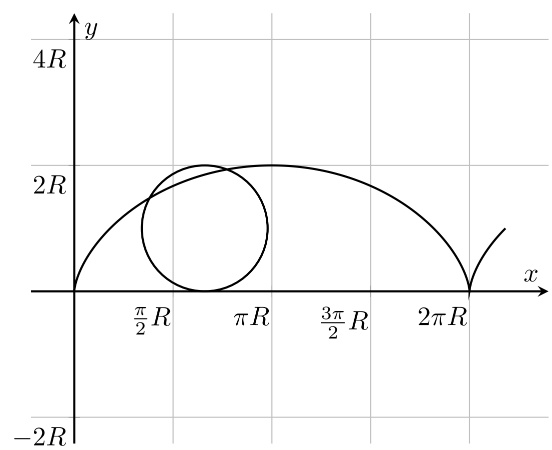


Figure 1.1: The cycloid with generating circle.

Parameterization

There is no simple expression $y(x)$ for the cycloid. Instead we introduce a parameter θ that tracks the evolution of the generating circle. In terms of θ, the shape of the cycloid is given by

$$x(\theta) = R\theta - R\sin(\theta) \tag{1.66}$$

$$y(\theta) = R - R\cos(\theta) \,. \tag{1.67}$$

The cycloid is clearly periodic in the variable θ. While θ can take on any real value and still represent a cycloid, we'll stay interested in the domain $[0 : 2\pi]$.

Velocity Envelope

Supposing θ evolves in a smooth and differentiable manner, we can take derivatives with respect to θ. For brevity, define

$$\omega(t) = \frac{d}{d\theta}\theta(t) \ ,$$

and calculate the time derivative of $x(\theta)$, $y(\theta)$ to get:

$$\begin{aligned}\frac{dx}{dt} &= R\omega - R\omega\cos(\theta)\\ \frac{dy}{dt} &= R\omega\sin(\theta)\end{aligned}$$

If we isolate the trig terms and square each equation, the fundamental trig identity can be used to derive

$$\left(\frac{dx}{dt} - R\omega\right)^2 + \left(\frac{dy}{dt}\right)^2 = (R\omega)^2 \ ,$$

which is called the *envelope* of velocities of the cycloid. Plotted in velocity space, the above depicts a circle of radius $R\omega$ centered at $(R\omega, 0)$.

Tangent Line

At a point (x_0, y_0) on the cycloid, the slope is still given by dy/dx at that point, despite using a parameterized representation of the curve. Calculating the slope is a matter of the chain rule:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1}$$

Carrying this out, we find

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1-\cos(\theta)} = \cot\left(\frac{\theta}{2}\right) \ .$$

This is enough to write down a equation for the tangent line to the cycloid:

$$y_{\text{tan}} = y_0 + \cot\left(\frac{\theta}{2}\right)(x - x_0)$$

Replacing x_0, y_0 with their representations in θ gives a neater formula, after some simplifying:

$$y_{\text{tan}} = 2R + \cot\left(\frac{\theta}{2}\right)(x - R\theta)$$

Interestingly, we see that the tangent line always passes through the point $(R\theta, 2R)$, which is the top of the generating circle as it goes along.

<u>Problem 29</u>

A stone lodged on the rim of a bicycle tire of radius R dislodges at the height of its cycloidic path. Determine its trajectory after leaving the tire. Answer:

$$\begin{aligned}x(t) &= R\pi + 2R\omega t\\ y(t) &= 2R - gt^2/2\end{aligned}$$

Normal Line

Knowing the slope at any point (x_0, y_0) on the cycloid, we can write an expression for the normal line at the same point:

$$y_{\text{norm}} = y_0 - \tan\left(\frac{\theta}{2}\right)(x - x_0)$$

Like the case for the tangent line, replacing x_0, y_0 with their representations in θ gives a neater formula, after some simplifying:

$$y_{\text{norm}} = -\tan\left(\frac{\theta}{2}\right)(x - R\theta)$$

From this, wee see that the normal line always hits the point of contact between the generating the circle and the line on which it rolls.

Arc Length

The arc length of the cycloid is straightforwardly calculated from Equation (1.61). For this, we start with

$$S = \int_0^{2\pi R}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \ ,$$

which, after substituting $x(\theta)$, $y(\theta)$ becomes

$$S = \sqrt{2}R\int_0^{2\pi}\sqrt{1-\cos(\theta)}\,d\theta \ ,$$

readily simplifying to

$$S = 2R\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)d\theta = 8R \, .$$

Area Enclosed

The area enclosed by the cycloid and the x-axis is given by the standard setup:

$$A = \int_0^{2\pi R} y\,dx = R^2\int_0^{2\pi}(1-\cos(\theta))^2\,d\theta$$

The remaining integral is straightforwardly solved, and we find the enclosed area to be three times that of the generating circle:

$$A = 3\pi R^2$$

Volume Enclosed

A cycloid revolved about the x-axis encloses a volume that we can calculate with the circular disk method, i.e. Equation (1.62). For this case, we have, after simplifying

$$V = \int_0^{2\pi} \pi R^2 \left(1 - \cos(\theta)\right)^3 d\theta \; .$$

The remaining integral is a bit tedious but isn't difficult, ending with

$$V = 5\pi^2 R^3 \; .$$

Surface Area

The surface of revolution made by revolving a cycloid about the x-axis is straightforwardly given by Equation (1.65). Here, we have

$$A = 2\sqrt{2}\pi R^2 \int_0^{2\pi} \left(1 - \cos(\theta)\right)^{3/2} d\theta \; .$$

The remaining integral is tricky to evaluate but not impossible. Leaving the details for an exercise, we ultimately find

$$A = \left(2\sqrt{2}\pi R^2\right)\left(\frac{16\sqrt{2}}{3}\right) = \frac{64}{3}\pi R^2 \; .$$

Tautochrone

Consider a cycloid flipped upside-down, described by

$$\begin{aligned} x(\theta) &= R\theta - R\sin(\theta) \\ y(\theta) &= -R + R\cos(\theta) \; . \end{aligned}$$

Pretending we have constructed a ramp in such a shape, let us analyze the sliding (not rolling) motion of a body of mass m placed at rest on the ramp.

In uniform gravity, the system respects an energy constant

$$E = \frac{1}{2}mv^2 + mgy \; ,$$

where v is the velocity of the body in motion, g is the local gravitational acceleration, and y is the height above $y = 0$. Assuming the object begins at rest, we also have

$$E = mgy_0 \; ,$$

where y_0 is the initial height of the body.

With this setup, it's useful to know the total time T required for the body to slide to the bottom of the inverted cycloid. As an integral, we have, at least provisionally,

$$T = \int_{y_0}^{-2R} dt \; ,$$

and the job is recast the integral in variables we know.

Proceed by replacing dt with something akin to arc length, namely

$$dS = v(t)\, dt \; .$$

Meanwhile, we know from geometry that

$$dS^2 = dx^2 + dy^2 \; .$$

This is enough to wrestle the time integral into something manageable:

$$T = \int_{x_0}^{R\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{\sqrt{2g\left(y_0 - y\right)}}$$

We haven't used the equations of the cycloid yet, so proceed by using

$$\begin{aligned} y_0 - y &= R\left(\cos(\theta_0) - \cos(\theta)\right) \\ dx &= R\left(1 - \cos(\theta)\right) d\theta \\ dy &= -R\sin(\theta)\, d\theta \; , \end{aligned}$$

and the above simplifies to

$$T = \sqrt{\frac{R}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos(\theta)}\, d\theta}{\sqrt{\cos(\theta_0) - \cos(\theta)}} \; .$$

Note that θ_0 corresponds to the initial position (x_0, y_0), and $\theta = \pi$ occurs when the sliding body reaches the bottom of the curve.

Ugly as it is, the time integral can be solved after making a few substitutions that are left for an exercise to the reader to find. As a hint, you should first have

$$T = \sqrt{\frac{R}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)\, d\theta}{\sqrt{\cos^2(\theta_0/2) - \cos^2(\theta/2)}} \; ,$$

and then

$$T = \sqrt{\frac{R}{g}} \int_1^0 \frac{-2du}{\sqrt{1 - u^2}} \; .$$

Keep on solving with yet another u-substitution, and the final answer comes out to

$$T = \pi\sqrt{\frac{R}{g}} \; .$$

Remarkably, the final answer $T = \pi\sqrt{R/g}$ makes no mention of the initial position (x_0, y_0) of the sliding body. This is to say that the time to slide to the bottom of a cycloid is always the same. No other known curve has this feature. The Ancient Greeks called this the *tautochrone*.

6.3 Horizontal Tank

Consider a cylindrical tank of volume $V = \pi R^2 L$, where R is the radius and L is the length. The tank is resting on its side and is filled partially with water to a depth h above the lowest point in the tank. For a given ratio h/R, what fraction of the tank's volume is filled with water?

For convenience, redefine 'down' as pointing along the positive x-axis so the empty tank fills horizontally from the right. Placing the origin at the center of the circular cross section, the line defining the water's surface has length

$$2y = 2R\sin(\theta) \; ,$$

where θ is placed conventionally. In this frame, we can forget about the length of the tank entirely.

Letting x represent the distance from the center of the tank to the water's surface, presuming $h < R$, then the x-coordinate obeys

$$x = R\cos(\theta) \; ,$$

along with

$$x + h = R \, .$$

We'll need a formula for h/R, which we take from the above:

$$\frac{h}{R} = 1 - \frac{x}{R} = 1 - \cos(\theta)$$

Per our setup, the ratio h/R picks out a unique value for θ_*, so long as $0 \leq \theta_* \leq \pi/2$.

Setting up the area integral for the cross section of water occupying the tank, we begin with

$$A = -2\int_R^{x_*} y \, dx \; ,$$

which after using the trigonometric substitutions above becomes

$$A = -2R^2 \int_0^{\theta_*} \sin^2(\theta) \, d\theta \; ,$$

simplifying further to

$$A = R^2\left(\theta_* - \frac{1}{2}\sin(2\theta_*)\right) \; .$$

Keep in mind that θ_* is nontrivially related to the ratio h/R. Using a machine equipped to run Newton's method in one dimension, we generate the following results:

Ratio $h/2R$	**Angle** θ_*	**Vol. Rat.**
0	0	0
0.05	0.45103	0.018693
0.1	0.64350	0.052044
0.15	0.79540	0.094060
0.2	0.92730	0.14238
0.25	1.0472	0.19550
0.3	1.1593	0.25232
0.35	1.2661	0.31192
0.4	1.3694	0.37353
0.45	1.4706	0.43644
0.5	$\pi/2$	0.5

The handle the case $\pi/2 \leq \theta_* \leq \pi$, write

$$\frac{h}{R} = 1 + \cos(\theta)$$

and regenerate the list of θ_*-values to find:

Ratio $h/2R$	**Angle** θ_*	**Vol. Rat.**
0.55	1.6710	0.56356
0.6	1.7722	0.62647
0.65	1.8755	0.68808
0.7	1.9823	0.74768
0.75	2.0944	0.80450
0.8	2.2144	0.85762
0.85	2.3462	0.90594
0.9	2.4981	0.94796
0.95	2.6906	0.98131
1	π	1

6.4 Mass Between Springs

Consider a point mass m in the center of two springs pulled tight and mounted distance L apart, ignoring gravity. The left spring has constant k_a, and the right has spring constant k_b, and both springs have rest length $L_0 < L/2$.

Rest Condition

When the system is not in motion, the mass will rest somewhere between the endpoints toward the stiffer spring, not necessarily at $x = L/2$. To work this out, balance all relevant forces in the x- and y-directions:

$$m\frac{d^2y}{dt^2} = F^y_{\text{net}} = 0$$

$$m\frac{d^2x}{dt^2} = F^x_{\text{net}} = F_a + F_b \; ,$$

and each left size is zero for the rest condition. Each force F_a, F_b obeys Hooke's law:

$$F_{\text{spring}} = -kx$$

Letting a constant q denote the position of the mass away from $x = L/2$, the above tells us:

$$0 = -k_a \left(-L_0 + \frac{L}{2} - q \right) + k_b \left(-L_0 + \frac{L}{2} + q \right)$$

Solving for q tells us where the system rests:

$$q = \left(\frac{k_a - k_b}{k_a + k_b} \right) \left(\frac{L}{2} - L_0 \right)$$

Looking at a few special cases, note first that q vanishes if $k_a = k_b$, giving the symmetric result. Note also that if $L/2 = L_0$, the system is under no tension at all, and q vanishes again. More curiously, if it happens that $L/2 < L_0$, this corresponds to the system being compressed rather than stretched, and the sign on q flips. That is, the offset would be away from the stiffer spring. (This situation is unstable.)

Longitudinal Vibrations

If the mass-between-springs system is perturbed in a direction that is purely longitudinal, i.e. parallel to the springs, then resulting motion is confined to one dimension. To prepare for this, define two constants

$$\begin{aligned} x_a &= -L_0 + L/2 - q \\ x_b &= -L_0 + L/2 + q \, , \end{aligned}$$

so the rest condition is written

$$0 = -k_a x_a + k_b x_b \, .$$

For the non-rest case, use Newton's second law and Hooke's law combine to write

$$m \frac{d^2}{dt^2} x\left(t\right) = -k_a \left(x_a + x\left(t\right)\right) + k_b \left(x_b - x\left(t\right)\right) \, ,$$

readily simplifying to

$$m \frac{d^2}{dt^2} x\left(t\right) = -x\left(t\right) \left(k_A + k_b\right)$$

This is a simple harmonic oscillator with effective angular frequency:

$$\omega = \sqrt{\frac{k_a + k_b}{m}}$$

Transverse Vibrations

Things get more interesting when we examine vibrations in the direction perpendicular to the springs. Taking the two spring constants as the same, i.e. $k_a = k_b = k$, an initial displacement of the mass in the y-direction results in one-dimensional motion.

In this case, we have $F^x_{\text{net}} = 0$ for the x-direction, and for the y-direction,

$$F^y_{\text{net}} = 2F_{\text{spring}} \sin\left(\theta\right) \, ,$$

where θ is the angle formed between a spring and the horizontal, and from geometry we pick out

$$\sin\left(\theta\right) = \frac{y}{\sqrt{\left(L/2\right)^2 + y^2}} \, .$$

The magnitude of the spring force is given by

$$F_{\text{spring}} = -k \left(\sqrt{\left(\frac{L}{2}\right)^2 + y^2} - L_0 \right) \, ,$$

which, as long as $L/2 \neq L_0$, has a nonzero value for $y = 0$, affirming the springs are always under tension. All together, transverse vibrations are summarized by

$$F^y_{\text{net}} = m \frac{d^2}{dt^2} y\left(t\right) = -2ky \left(1 - \frac{L_0}{\sqrt{\left(L/2\right)^2 + y^2}} \right) \, .$$

Small Vibrations

In the special case that the displacement $|y|$ is always much less than $L/2$, the above becomes

$$\begin{aligned} F^y_{\text{net}} &\approx -2ky \left(1 - \frac{2L_0}{L} \left(1 - \cancel{\frac{1}{2} \frac{4y^2}{L^2}} \right) \right) \\ &\approx -2ky \left(1 - \frac{2L_0}{L} \right) \, , \end{aligned}$$

where the square root has been eliminated by Taylor expansion.

Defining a new quantity

$$p = \frac{L}{2} - L_0 \, ,$$

the above simplifies to, of course, the equation of a harmonic oscillator

$$m \frac{d^2}{dt^2} y\left(t\right) \approx -\left(\frac{2k}{1 + L_0/p} \right) y\left(t\right) \, .$$

The angular frequency is given by

$$\omega = \sqrt{\frac{2k}{m} \left(\frac{1}{1 + L_0/p} \right)} \, ,$$

which is scaled by the tension in the springs. This is in fact a crude model for a plucked guitar string - the greater the tension, the greater the frequency of vibration.

Critical Vibrations

The problem becomes a different beast when we consider $L_0 = L/2$, meaning there is no resting tension in the system. Staying in the regime of transverse small oscillations, i.e. $|y| \ll L/2$, let us jot down a previous result without canceling the y^2-term:

$$F^y_{\text{net}} \approx -2ky\left(1 - \frac{2l_0}{L}\left(1 - \frac{1}{2}\frac{4y^2}{L^2}\right)\right)$$

Setting $2L_0 = L$, the above simplifies to

$$m\frac{d^2}{dt^2}y\left(t\right) \approx -k\left(\frac{2}{L}\right)^2\left(y\left(t\right)\right)^3 \ ,$$

which is classified as a nonlinear second-order differential equation.

Energy Constraint

Despite the scary name, we can wrestle with the above equation anyway. Letting

$$\lambda = \frac{4k}{mL^2}$$

and using the 'dot' operator as a shorthand for the time derivative, we must solve

$$\ddot{y} = -\lambda\, y^3 \ .$$

Proceed by multiplying both sides by $\dot{y}$, and condense the left using the product rule:

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2\right) = \ddot{y}\dot{y} = -\lambda\,\frac{dy}{dt}y^3$$

Multiply dt onto each side to attain a so-called 'differential form'

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2\right)dt = -\lambda\, y^3\, dy \ ,$$

which can be cleanly integrated with respect to t on the left, y on the right:

$$\frac{1}{2}\dot{y}^2 = -\frac{\lambda}{4}y^4 + C$$

This result looks very much like a conservation of energy statement. If we multiply through by a mass constant m, the left side is the kinetic energy then Cm is the total energy E. The potential energy term is proportional to y^4, not y^2, which is not a simple harmonic oscillator potential.

Initial Condition

One typical scenario for this system would have the mass released from rest at some initial value A above $y = 0$. In this case, the above equation reads

$$0 = -\frac{\lambda}{4}A^4 + C$$

at $t = 0$, and the integration constant C can be eliminated. Doing so, we get

$$\frac{1}{2}\dot{y}^2 = \frac{\lambda}{4}\left(A^4 - y^4\right) = \frac{\lambda A^4}{4}\left(1 - \left(\frac{y}{A}\right)^4\right) \ ,$$

or

$$\frac{dy}{dt} = \sqrt{\dot{y}^2} = \pm\sqrt{\lambda}\frac{A^2}{2}\sqrt{1 - \left(\frac{y}{A}\right)^4} \ ,$$

which can be separated with all y's on one side, t's on the other:

$$\frac{dy}{\sqrt{1 - \left(y/A\right)^4}} = \pm\left(\sqrt{\lambda}\frac{A^2}{2}\right)dt$$

Proceed with the substitution

$$\begin{aligned} y &= A\cos\left(\phi\right) \\ dy &= -A\sin\left(\phi\right)d\phi \ , \end{aligned}$$

and the above becomes

$$\frac{-A\cancel{\sin\left(\phi\right)}\,d\phi}{\sqrt{\cancel{\left(1 - \cos^2\left(\phi\right)\right)}\left(1 + \cos^2\left(\phi\right)\right)}} = \pm\left(\sqrt{\lambda}\frac{A^2}{2}\right)dt \ ,$$

allowing each side to be integrated:

$$\int\frac{d\phi}{\sqrt{1 + \cos^2\left(\phi\right)}} = \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\int dt$$

There are many choices for integration limits. One simple correspondence emerges by letting ϕ run from 0 to $\pi/2$, in which case the t-variable elapses a quarter-period:

$$\begin{aligned} \int_0^{\pi/4}\frac{d\phi}{\sqrt{1 + \cos^2\left(\phi\right)}} &= \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\int_0^{T/4}dt \\ &= \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\frac{1}{4}T \end{aligned}$$

The integral on the left is strictly numerical. Condensing constants and sifting out the period's relation to the amplitude gives the result:

$$AT = \text{constant}$$

7 Approximating Integrals

7.1 Riemann Sums

The predecessor to the notion of the integral and the fundamental theorem of calculus is the Riemann sum, which relates the endpoint values of a function $f(x)$ to a sum over the function's slope by

$$f(x_n) - f(x_0) \approx S = \sum_{j=0}^{n-1} f'\left(x_j^*\right) \Delta x \ ,$$

where

$$\Delta x = \frac{x_n - x_0}{n} \ ,$$

and x_j^* is any x-value within $[x_j, x_{j+1}]$, and

$$x_j = x_0 + j\Delta x \ .$$

The dimensionless integer j is the index, and n is the total number of bins in the sum.

Since there is freedom in how x_j^* is chosen, there are three standard methods called the left sum, right sum, and midpoint sum:

$$x_j^* = \begin{cases} x_j & \text{Left sum} \\ x_{j+1} & \text{Right sum} \\ (x_j + x_{j+1})/2 & \text{Midpoint sum} \end{cases}$$

Denoting S_{Left}, S_{Right} as the left and right sums respectively, the average of these yields the trapezoid rule:

$$S_{\text{Trap}} = \frac{1}{2}\left(S_{\text{Left}} + S_{\text{Right}}\right)$$

Of course, all Riemann sums are the same in the continuous limit, which is why the integral need not concern over left, right, mid, etc.

7.2 Simpson's Rule

For approximating the area under a function $f(x)$, an improvement over straight-line methods uses a quadratic function to estimate $f(x)$ at each step, known as *Simpson's rule*. To get started, propose a quadratic form

$$g(x) = Ax^2 + Bx + C \ ,$$

where the coefficients A, B, C depend on $f(x)$ in the neighborhood of x.

Now consider a point x_j somewhere in the region and write the definite integral

$$\int_{x_j-h}^{x_j+h} f(x)\, dx \approx \int_{x_j-h}^{x_j+h} g(x)\, dx = I(h) \ .$$

Without filling in the details yet, the result of such an integral is written $I(h)$, where $2h$ is the width of the integration domain. Substituting $g(x)$ into the above and turning the crank gives the form

$$I(h) = \frac{2h}{3}\left(A\left(3x_j^2 + h^2\right) + 3Bx_j + 3C\right) \ .$$

Meanwhile, examine a new quantity

$$J(h) = g(x_j - h) + 4g(x_j) + g(x_j + h) \ ,$$

which, after substituting $g(x)$, becomes

$$J(h) = \frac{3}{h} I(h) \ .$$

Evidently, the integral $I(h)$ is the same as the sum $J(h)$ up to a factor $3/h$:

$$\begin{aligned} \int_{x_j-h}^{x_j+h} f(x)\, dx &\approx \int_{x_j-h}^{x_j+h} g(x)\, dx \\ &= \frac{h}{3}\left(g(x_j - h) + 4g(x_j) + g(x_j + h)\right) \end{aligned}$$

Of course, this result only works in the neighborhood on a given x_j.

To apply this over a macroscopic interval, sum over all x_j in steps $2h$, and let $f(x)$ replace the function being evaluated. The integration region is given by

$$\frac{b-a}{n} = 2h \ ,$$

where a, b are the lower and upper limits, and n is the number of bins. The effective bin width is $2h$. In order to have $x_0 - h = a$ and $x_{n-1} + h = b$, the x_j are located via

$$\begin{aligned} x_j &= (a + h) + \frac{j}{n}(b - a) \\ x_j &= (a + h) + j(2h) \ . \end{aligned}$$

Assimilating these changes, the approximation becomes

$$\begin{aligned} &\int_a^b f(x)\, dx \approx \\ &\quad \sum_{j=0}^{n-1} \frac{h}{3}\left(f(x_j - h) + 4f(x_j) + f(x_j + h)\right) \ , \end{aligned} \tag{1.68}$$

which we may take as a final answer.

Pseudocode

One type of notation we'll employ here is *pseudocode*, which attempts to bridge a computer algorithm to human readability.

Pseudocode is read sequentially (top to bottom), paying particular mind to variables, conditions, and so on, in order to understand what a computer would do if the code were implemented in a proper language. Instructions that are indented are contained in a function, loop, or conditional. Instructions that end with an underscore (_) are continued to the next line.

As pseudocode, the Equation (1.68) can be implemented shown in the box that follows. The area being approximated is under the function $f(x) = 4x - x^2$ in the region $0 \le x \le 4$ using 15 bins. Lines of code that are indented by two spaces are 'looped over'.

```
f(x) = 4 * x - x * x

a = 0   # lower limit
b = 4   # upper limit
n = 15  # bins
h = (b - a) / (2 * n)
# Init. other variables to zero.

for j from 0 to n - 1
  xj = (a + h) + j * (2 * h)
  f1 = f(xj - h)
  f2 = f(xj)
  f3 = f(xj + h)
  simp += (h/3) * (f1 + 4*f2 + f3)
```

The approximation to the integral is held in the `simp` variable. If the above pseudocode were implemented in a suitable computation environment, one would find:

```
simp = 10.666666666666666
```

This result is indistinguishable from the exact answer to standard computation precision:

$$\int_0^4 \left(4x - x^2\right) = \frac{32}{3} = 10.666\overline{6}$$

Weighted Average Identity

Starting with Equation (1.68) for Simpson's rule, identify $2h = \Delta x$ and simplify:

$$\begin{aligned} &\int_a^b f(x)\,dx \approx \\ &\quad \frac{1}{6}\sum_{j=0}^{n-1}\left(f\left(x_j - \frac{\Delta x}{2}\right) + f\left(x_j + \frac{\Delta x}{2}\right)\right)\Delta x \\ &\qquad + \frac{1}{6}\sum_{j=0}^{n-1} 4f(x_j)\,\Delta x\,. \end{aligned}$$

The sum has been broken in two parts. The first consists of the left sum S_L and right sum S_R rules added together, which is twice the trapezoid rule S_T. The final sum involving $f(x_j)$ alone is identical to the midpoint sum S_M. Evidently, Simpson's rule is the weighted average of more elementary methods after all:

$$\int_a^b f(x)\,dx \approx \frac{1}{3}\left(S_T + 2S_M\right) \tag{1.69}$$

Setting up another program to approximate the left, right, and midpoint sums, we can also get the trapezoid sum and verify that Simpson's rule obeys the above identity. For an example problem, let us approximate the area under a curve we can verify by hand:

$$\int_{-2}^3 \left(5x^2 - x\right) dx = \frac{335}{6} = 55.8333\overline{3}$$

Then, making appropriate changes to the above program, we have:

```
f(x) = 5 * x * x - x

a = -2  # lower limit
b = 3   # upper limit
n = 15  # bins
dx = (b - a) / n
# Initialize other variables to zero.

for j from 0 to n - 1
  xj = a + j * dx
  x1 = xj + dx
  xm = (xj + x1) / 2
  left += dx * f(xj)
  right += dx * f(x1)
  mid += dx * f(xm)

# Calculate trap and simp after loop
trap = (left + right) / 2
simp = (1 / 3) * (trap + 2 * mid)
```

With the number of bins set to $n = 15$, the results of such a program turn out as:

```
left  = 52.96296296296295
right = 59.62962962962900
mid   = 55.60185185185184
trap  = 56.29629629629628
simp  = 55.83333333333332
```

Comparing each of these to the exact answer, we see the left sum underestimating, the right sum overestimating, and so on. Of course, the trailing digits in each may vary slightly, depending on the environment used. Most notably, Simpson's rule seems to get the answer (to this integral) to near-perfect precision.

Problem 30

Let $f(x) = \left(x^2 + 4x\right)^{1/3}$, and let $g(x)$ be an antiderivative of $f(x)$. If $g(5) = 7$, find $g(1) = h$.

Answer: Use the fundamental theorem to write

$$g(x) = \int_a^x \left(t^2 + 4t\right)^{1/2} dt$$

such that

$$g(5) = \int_a^5 \left(t^2 + 4t\right)^{1/2} dt = 7$$
$$g(1) = \int_a^1 \left(t^2 + 4t\right)^{1/2} dt = h\,.$$

Using both equations, we further write

$$7 = h + \int_1^5 \left(t^2 + 4t\right)^{1/2} dt\,,$$

where the remaining integral can be solved by an iterative method. Particularly, we find

$$h = 7 - 10.88222 = -3.88222\,.$$

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