# Integral Calculus
# MANUSCRIPT

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# Chapter 1

# Integral Calculus

## 1 Area Under a Curve

### 1.1 Review

The workhorse equation of differential calculus is undoubtedly the definition of the derivative of a function $f(x)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} ,$$

which gives the instantaneous slope of the function at $x_0$.

An extension of the derivative comes in the form of Taylor's theorem, which attempts to approximate the function $f(x)$ near a given point $x_0$:

$$f(x) \approx f(x_0) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}(x_0)(x - x_0)^q$$

Of course, Taylor's theorem embeds the first derivative as its first-order case.

As it turns out, all of this derivative-play, i.e. differential calculus, is only half of the total picture. There is, in fact, another important relationship between the function $f(x)$ and its slope $f'(x)$ that is the inverse to the notion of the derivative.

### 1.2 Motivation

Working in the general case, consider a point $x_0$ in the domain of a 'well-behaved' function $f(x)$, and also consider a point $x_1$ that is arbitrarily close to $x_0$. By the derivative formula, we can surely write

$$\lim_{x_1 \to x_0} f(x_1) - f(x_0) = f'(x_0) \lim_{x_1 \to x_0} (x_1 - x_0) .$$

Also, consider another point $x_2$ that is arbitrarily close to $x_1$, which means

$$\lim_{x_2 \to x_1} f(x_2) - f(x_1) = f'(x_1) \lim_{x_2 \to x_1} (x_2 - x_1) ,$$

and just to start a pattern, consider yet another point $x_3$ obeying

$$\lim_{x_3 \to x_2} f(x_3) - f(x_2) = f'(x_2) \lim_{x_3 \to x_2} (x_3 - x_2) \ .$$

For definiteness, let's have the $x$-variables relate by

$$x_0 < x_1 < x_2 < x_3 < \cdots < x_n \ ,$$

assuming the pattern keeps going. It also helps to label the interval between each:

$$\Delta x_j = x_{j+1} - x_j$$

Proceed boldly by taking the sum of the equations written above. Doing the right-hand side first, we have

$$\begin{aligned} \text{RHS} &= f'(x_0)\,\Delta x_0 + f'(x_1)\,\Delta x_1 \\ &\quad + f'(x_2)\,\Delta x_2 + \cdots \ , \end{aligned}$$

which can be written concisely as a sum

$$\text{RHS} = \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j$$

that goes out to $n$ total terms.

As for the left-hand side, we have

$$\begin{aligned} \text{LHS} &= \lim_{x_1 \to x_0} f(x_1) - f(x_0) \\ &\quad + \lim_{x_2 \to x_1} f(x_2) - f(x_1) \\ &\quad + \lim_{x_3 \to x_2} f(x_3) - f(x_2) \\ &\quad + \cdots + \lim_{x_n \to x_{n-1}} f(x_n) - f(x_{n-1}) \\ &= f(x_n) - f(x_0) \end{aligned}$$

Notice how any given $f(x_j)$ present in the above has a negative counterpart, thus most terms in the above cancel out in pairs. This obliterates any notion of 'limit' on the left, as only the difference $f(x_n) - f(x_0)$ remains.

Putting the left side and the right side together, we seem to have discovered

$$f(x_n) - f(x_0) \approx \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j \ . \tag{1.1}$$

On the left is simply the difference of a function at two points in its domain. The right, however, seems to be the total *area* of $n$ rectangles, with the $j$th rectangle having height $f'(x_j)$ and width $\Delta x_j$.

As a whole, Equation (1.1) suggests a way to approximate the area under the curve $f'(x)$ between the endpoints $x_0$, $x_n$. The tricky part, in general, is finding whatever function $f(x)$ corresponds to the slope $f'(x)$, i.e. the notorious *antiderivative*.

## 1.3 Riemann Sums

At face value, Equation (1.1) can be implemented 'as-is' to approximate the area under $f'(x)$. To clean up the notation, make the substitution $f'(x) = g(x)$, and write the above as

$$f(x_n) - f(x_0) \approx S = \sum_{j=0}^{n-1} g\left(x_j^*\right) \Delta x \ ,$$

where the argument sent to $g(x)$ is denoted $x_j^*$. Furthermore, the subscript on $\Delta x_j$ has been dropped with the understanding that each $\Delta x_j$ is one and the same length given by

$$\Delta x = \frac{x_n - x_0}{n} \ ,$$

implying

$$x_j = x_0 + j\Delta x \ .$$

### Left, Right, Midpoint Sum

The reason $x_j^*$ gets special attention is there are no natural restrictions on where $x_j^*$ occurs within the interval $\Delta x_j$. Right off the bat, there are three obvious options

$$x_j^* = \begin{cases} x_j & \text{Left sum} \\ x_{j+1} & \text{Right sum} \\ (x_j + x_{j+1})/2 & \text{Midpoint sum} \end{cases} \ ,$$

which sample from $g(x)$ differently. Explicitly, these mean:

$$\begin{aligned} \frac{S_{\text{Left}}}{\Delta x} &= g(x_0) + g(x_0 + \Delta x) \\ &\quad + g(x_0 + 2\Delta x) + \cdots + g(x_n - \Delta x) \\ \frac{S_{\text{Right}}}{\Delta x} &= g(x_0 + \Delta x) + g(x_0 + 2\Delta x) \\ &\quad + g(x_0 + 3\Delta x) + \cdots + g(x_n) \\ \frac{S_{\text{Mid}}}{\Delta x} &= g\left(x_0 + \frac{\Delta x}{2}\right) + g\left(x_0 + \frac{3\Delta x}{2}\right) \\ &\quad + g\left(x_0 + \frac{5\Delta x}{2}\right) + \cdots + g\left(x_n - \frac{\Delta x}{2}\right) \end{aligned}$$

Example 1

Using the midpoint sum rule with $n = 10$ bins, approximate the area under the function

$$g(x) = 5x + 2$$

in the domain

$$-2 \le x \le 3 \ .$$

Let $x_0 = -2$, let $x_n = 3$, and $n = 10$ so that

$$\Delta x = \frac{x_n - x_0}{n} = \frac{3-(-2)}{10} = \frac{1}{2} \, .$$

At step $j$ in the sum we further have

$$x_j = x_0 + j\Delta x = -2 + \frac{j}{2} \, .$$

To prepare for the midpoint sum, note that

$$x_j^* = \frac{x_j + x_{j+1}}{2} = \frac{-7}{4} + \frac{j}{2} \, .$$

The midpoint sum $S_M$ is given by

$$S_M = \sum_{j=0}^{n-1} g\left(x_j^*\right)\Delta x = \sum_{j=0}^{9}\left(5\left(\frac{-7}{4}+\frac{j}{2}\right)+2\right)\frac{1}{2} \, ,$$

which simplifies nicely:

$$\begin{aligned} S_M &= \frac{1}{2}\sum_{j=0}^{9}\left(2-\frac{35}{4}\right)+\frac{5}{4}\sum_{j=0}^{9} j \\ &= \frac{1}{2}(10)\left(2-\frac{35}{4}\right)+\frac{5}{4}(45) \\ &= \frac{45}{2} = 22.5 \end{aligned}$$

Using the midpoint rule, 22.5 happens to be the exact solution to the stated problem, regardless of how many bins we choose. This brings out a special relationship between the midpoint rule and straight lines: the approximation is perfect.

<u>Example 2</u>

Using the right sum rule with any number $n$ bins, approximate the area under the function

$$g(x) = 4x - x^2$$

in the domain

$$0 \le x \le 4 \, .$$

Let $x_0 = 0$, let $x_n = 4$, and $n = 10$ so that

$$\Delta x = \frac{x_n - x_0}{n} = \frac{4-0}{n} = \frac{4}{n} \, .$$

At step $j$ in the sum we further have

$$x_j = x_0 + j\Delta x = \frac{4j}{n} \, .$$

To prepare for the right sum rule, note that

$$x_{j+1} = \frac{4j}{n} + \frac{4}{n} \, .$$

Then, the right sum rule is

$$\begin{aligned} S_R &= \sum_{j=0}^{n-1} f'\left(x_{j+1}\right)\Delta x \\ &= \sum_{j=0}^{n-1}\left(4\left(\frac{4j}{n}+\frac{4}{n}\right)-\left(\frac{4j}{n}+\frac{4}{n}\right)^2\right)\frac{4}{n} \end{aligned}$$

Let $k = j+1$ and simplify the right side to get

$$S_R = \frac{4^3}{n^3}\left(n\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2\right) \, .$$

By analyzing the remaining sums, it's straightforward to show that

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \, , \end{aligned}$$

and the sum simplifies to

$$S_R = \frac{32}{3}\left(1-\frac{1}{n^2}\right)$$

This result contains a factor of $n$, allowing the exactness of $S_R$ to be tuned. Note that $1/n^2$ vanishes for sufficiently large $n$, telling us the exact area under the curve is $32/3$.

### Trapezoid Rule

An improvement over rectangle-based methods is the average the left- and right rules, which effectively turns rectangles into trapezoids, giving (you guessed it) the *trapezoid rule*:

$$\begin{aligned} S_{\text{Trap}} &= \frac{1}{2}\left(S_{\text{Left}} + S_{\text{Right}}\right) \\ &= \frac{1}{2}\sum_{j=0}^{n-1}\left(g\left(x_j\right)+g\left(x_{j+1}\right)\right)\Delta x \\ &= \frac{g\left(x_0\right)+g\left(x_n\right)}{2}\Delta x + \sum_{j=1}^{n-1} g\left(x_j\right)\Delta x \end{aligned}$$

Without summation notation, the above reads

$$\begin{aligned} \frac{S_{\text{Trap}}}{\Delta x} &= \frac{g\left(x_0\right)}{2} + g\left(x_0+\Delta x\right) \\ &\quad + g\left(x_0+2\Delta x\right)+\cdots+\frac{g\left(x_n\right)}{2} \, . \end{aligned}$$

# 2 The Integral

There is a regime where all versions of the Riemann sum converge to the same answer, and that is when we impose the limit $\Delta x \to 0$ and simultaneously $n \to \infty$. In this limit, the entire picture gets squeezed together, and the area under a curve is approximated by an infinite number of vertical lines. In other words, the Riemann sum becomes an exact solution to the area under the curve $f'(x)$:

$$f(x_n) - f(x_0) = \lim_{n\to\infty} \sum_{j=0}^{n-1} f'(x_j)\,\Delta x_j \,.$$

## 2.1 Integral Notation

To update the above with cleaner notation, the summation is replaced by the 'integral', literally a giant 'S', via

$$\lim_{\Delta x \to 0} \sum \Delta x \to \int dx \,,$$

which also replaces $\Delta x$ with $dx$. The limits on the sum turn into *integration limits*, one 'lower' limit and one 'upper' limit:

$$\lim_{\Delta x\to 0} \sum_{j=0}^{n-1} \Delta x \to \int_{x_0}^{x_n} dx$$

All $j$-subscripts have also been dropped, as $x$ is now understood to be a continuous variable inside the integral.

## 2.2 Fundamental Theorem

Using integral notation, the above is written

$$f(x_n) - f(x_0) = \int_{x_0}^{x_n} f'(x)\,dx \,.$$

While this is workable, it's customary to drop the $n$-subscript fdrom $f(x_n)$, and this term becomes $f(x)$. To prevent a naming conflict on the right, swap the integration variable from $x$ to $t$:

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt \,. \tag{1.2}$$

This result is called the *fundamental theorem of calculus*, which is the full inversion of the definition of the derivative.

## 2.3 Role of the Antiderivative

A less tautological way to write Equation (1.2) is

$$f(x) - f(x_0) = \int_{x_0}^{x} g(t)\,dt \,,$$

where $f'(t)$ is renamed to some given or otherwise evident function $g(t)$. The left-side function $f(x)$ is considered unknown.

In order to 'solve' the integral, $g(t)$ must be expressed as the derivative of something else, which means to find the antiderivative of $g(t)$. The 'something else' in this case has already been named, particularly $f(t)$:

$$f(x) - f(x_0) = \int_{x_0}^{x} \frac{d}{dt}\left(f(t)\right) dt$$

The ability to evaluate an integral usually comes down to the ability to find the antiderivative of the function being integrated. This can be quite the chore, if not impossible.

With the proper antiderivative in place, the derivative and the integral on the right mutually annihilate, leaving $f(t)$ alone evaluated at the integration limits, i.e., the quantity $f(x) - f(x_0)$. One way to think of this is to cancel the factors of $dt$ in a way inspired by the chain rule:

$$f(x) - f(x_0) = \int_{x_0}^{x} \frac{d}{\cancel{dt}} f(t)\,\cancel{dt} = \int_{x_0}^{x} df(t)$$

## 2.4 Definite Integral

When the integration limits $x_0$, $x$ are specified, either numerically or symbolically, the integral is called *definite*. In order to 'fully' solve a definite integral, the antiderivative $f(f)$ must be evaluated at each limit, and the answer is the difference between $f(x_0)$ and $f(x)$. For this, the 'vertical bar' notation is used:

$$\int_{x_0}^{x} df(t) = f(t)\Big|_{x_0}^{x} = f(x) - f(x_0)$$

#### Swapping the Limits

One can readily see that swapping the integration limits makes the integral 'run backwards', and gains an overall negative sign:

$$\int_{x}^{x_0} df(t) = f(x_0) - f(x) = -\int_{x_0}^{x} df(t)$$

#### Breaking the Interval

The integral remains intact if we split the interval into two or more parts. Introducing a variable $a$ in the domain $x_0 \le a \le x$, we may write:

$$\int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{a} g(t)\,dt + \int_{a}^{x} g(t)\,dt$$

## 2.5 Symmetric Domain

Consider the definite integral over a symmetric domain, meaning $-x_0$ is the lower limit and $x_0$ is the upper limit:

$$f(x_0) - f(-x_0) = \int_{-x_0}^{x_0} g(t)\, dt$$

From studying functions, recall that even functions obey

$$f_{\text{even}}(x) - f_{\text{even}}(-x) = 0 ,$$

and correspondingly for odd functions,

$$f_{\text{odd}}(x) + f_{\text{odd}}(-x) = 0 ,$$

meaning

$$f_{\text{odd}}(x) - f_{\text{odd}}(-x) = 2f_{\text{odd}}(x) \ .$$

Since the integral of $g(x)$ effectively bumps up its order by one, it follows that the even-ness or oddness of function $f$ is exactly the opposite of function $g$. We thus gain two cases:

$$\begin{aligned} f_{\text{even}}(x_0) - f_{\text{even}}(-x_0) &= \int_{-x_0}^{x_0} g_{\text{odd}}(t)\, dt \\ f_{\text{odd}}(x_0) - f_{\text{odd}}(-x_0) &= \int_{-x_0}^{x_0} g_{\text{even}}(t)\, dt \end{aligned}$$

The first of these results is immediately zero from the properties of even functions. In fact, the integral of any odd function over any symmetric interval, as we've shown, is *always* zero:

$$0 = \int_{-x_0}^{x_0} g_{\text{odd}}(t)\, dt \tag{1.3}$$

For the other case, we correspondingly find

$$2f_{\text{odd}}(x_0) = \int_{-x_0}^{x_0} g_{\text{even}}(t)\, dt ,$$

which means the integral of an even function over a symmetric interval effectively sums the same area twice. The above is also captured by

$$f_{\text{odd}}(x_0) = \int_{0}^{x_0} g_{\text{even}}(t)\, dt . \tag{1.4}$$

## 2.6 Integration Constant

When the lower integration limit $x_0$ is unspecified, the term $-f(x_0)$ is called the *integration constant*, denoted $C$. Setting $f(x_0) = -C$, this means:

$$\int^{x} f'(t)\, dt = f(x) + C$$

One way to justify the presence of the integration constant is to realize that any function $f(x) + C$ has the same derivative $f'(x)$, which is to say the absolute vertical offset of the curve has no bearing on its slope. To say this backwards, it follows that any antiderivative calculation without specific limits is only certain up to an arbitrary but non-ignorable constant $C$.

## 2.7 Indefinite Integral

The integral still retains meaning if we ambiguate both integration limits by writing

$$\int f'(t)\, dt = f(x) + C ,$$

where $C$ is the integration constant.

In this abstraction, the upper integration limit is always understood to be $x$, which kills the naming conflict in the $x$-variable on the right. Thus we also have

$$\int f'(x)\, dx = f(x) + C , \tag{1.5}$$

which is called the *indefinite integral*.

## 2.8 Integral Operator

In the same sense that one can apply $d/dx$ as an operator to both sides of an equation, we can do the opposite move, which is to apply $\int dx$ across both sides of an equation as well. If

$$g(x) = \frac{d}{dx} f(x) \ ,$$

then

$$\int g(x)\, dx = \int \frac{d}{dx}(f(x))\, dx = f(x) + C .$$

On the right, the integral and the derivative are mutually-obliterating, leaving just the enclosed function up to a constant.

### Interchangeability

As a sanity check, we should be able to apply $d/dx$ across the whole equation and recover the starting point. Explicitly, this is

$$\frac{d}{dx}\left(\int g(x)\, dx\right) = \frac{d}{dx} f(x) + \cancel{\frac{dC}{dx}} ,$$

which readily reduces to $g(x) = f'(x)$, provided that:

$$\frac{d}{dx}\left(\int g(x)\, dx\right) = \int \frac{d}{dx}(g(x))\, dx$$

That is, it's not harmful to move the derivative operation inside the enclosure of the integral.

# 3 Techniques of Integration

Integral calculations are trickier than anything else in introductory calculus. Here we go through the standard bag of tricks for solving integrals by hand. (Most integrals in the wild are not solvable by hand.)

## 3.1 Antiderivative Exploit

The most direct way to solve an integral is pick out (by experience or by luck) the antiderivative of the function being integrated. For instance, consider

$$I = \int_0^{\sqrt{\pi/2}} x \cos\left(x^2\right) dx \,.$$

Right away, note that the function being integrated can be written as a derivative

$$x \cos\left(x^2\right) = \frac{d}{dx}\left(\frac{1}{2}\sin\left(x^2\right)\right) \,,$$

so then

$$I = \int_0^{\sqrt{\pi/2}} \frac{d}{dx}\left(\frac{1}{2}\sin\left(x^2\right)\right) dx \,,$$

and then the integral and derivative operators cancel, leaving only the evaluation:

$$I = \frac{1}{2}\sin\left(x^2\right)\Big|_0^{\sqrt{\pi/2}} = \frac{1}{2}(1-0) = \frac{1}{2}$$

## 3.2 Exponents and Roots

### Powers

Starting with the power rule for differentiation

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \,,$$

replace $n \to n+1$ for convenience and write the same rule:

$$(n+1)\, x^n = \frac{d}{dx}\left(x^{n+1}\right)$$

From this, we can apply the integral operator to derive the rule for integrating powers and roots:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \tag{1.6}$$

Going through a few exemplary cases, i.e. playing with common values of $n$, we generate some useful information. You are encouraged to work through each of these:

$$\int dx = x + C \tag{1.7}$$

$$\int x \, dx = \frac{1}{2}x^2 + C \tag{1.8}$$

$$\int x^2 \, dx = \frac{1}{3}x^3 + C \tag{1.9}$$

$$\int \sqrt{x} \, dx = \frac{2}{3}x^{3/2} + C \tag{1.10}$$

$$\int x^{3/2} \, dx = \frac{2}{5}x^{5/2} + C \tag{1.11}$$

$$\int x^{-2} \, dx = \frac{-1}{x} + C \tag{1.12}$$

$$\int x^{-3} \, dx = \frac{-1}{2x^2} + C \tag{1.13}$$

$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C \tag{1.14}$$

$$\int x^{-3/2} \, dx = \frac{-2}{\sqrt{x}} + C \tag{1.15}$$

### Reciprocal

One special case to power rule formula is the integral of $1/x$. Recalling that the derivative of the natural logarithm yields this result, i.e.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \,,$$

the following must hold:

$$\int \frac{1}{x} \, dx = \ln(x) + C \tag{1.16}$$

### Exponential

Starting with the derivative rule for exponents

$$\frac{d}{dx}\left(n^x\right) = n^x \ln(n) \,,$$

it must follow that:

$$\int n^x \, dx = \frac{n^x}{\ln(n)} + C \tag{1.17}$$

**Applied Chain Rule**

Using the power rule and chain rule for derivatives, it's straightforward to derive

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}} \, .$$

Applying the $\int dx$ operator across both sides and simplifying leads to a useful identity:

$$\int \frac{f'(x)}{2\sqrt{f(x)}}\, dx = \sqrt{f(x)} + C \tag{1.18}$$

It takes some effort to train the eye to make use of identities such as the above. Exploring one case, suppose we have

$$f(x) = 1 \pm x^2$$

with $f'(x) = \pm 2x$. Plugging all of this in and simplifying gives a two-channel result:

$$\int \frac{\pm x}{\sqrt{1 \pm x^2}}\, dx = \sqrt{1 \pm x^2} + C \tag{1.19}$$

Problem 1

Prove that the area under a parabolic segment of base $b$ and height $h$ is

$$A = \frac{2}{3} bh \, .$$

Problem 2

Prove that the area of the 'lens' formed between the curves

$$y_1 = x^2$$
$$y_2 = ax + b$$

is

$$A = \frac{1}{6}\left(a^2 + 4b\right)^{3/2} \, .$$

## 3.3 U-Substitution

The standard integral

$$\int_{x_0}^{x} f'(t)\, dt = f(x) - f(x_0)$$

can sometimes be made simpler by a technique called *u-substitution*, which entails choosing a function $u(x)$ and then recasting the integral in this variable.

The $u$-substitution can be established by multiplying $du/du = 1$ into the standard integral, i.e.

$$\int_{x_0}^{x} \frac{df}{dt} dt = \int \frac{df}{dt}\frac{du}{du} dt = \int_{u(x_0)}^{u(x)} \frac{df}{du} du \, ,$$

where the factor $dt/dt$ cancels out. Importantly, note that the limits on the integral are also modified to respect $u(x)$. Once the result is attained as $f(u)$, reverse-substitute to attain $f(x)$.

A pragmatic way to choose the *correct* $u$-substitution can be established. Consider an indefinite integral

$$I = \int f(x)\, g(x)\, dx$$

for two functions $f(x)$, $g(x)$. Under the substitution $u = u(x)$, the above still must come out to

$$I = \int f(u)\, du \, ,$$

which can only mean

$$g(x) = \frac{du}{dx} \, .$$

That is, the function $g(x)$ must be (at least) proportional to the derivative of the substitution $u(x)$.

**Exemplary Case**

Consider again the definite integral

$$I = \int_0^{\sqrt{\pi/2}} x \cos\left(x^2\right) dx \, .$$

To solve this with $u$-substitution, let

$$u(x) = x^2$$

such that

$$du = 2x\, dx \, .$$

The limits of the integral must change to reflect the $u$-substitution as well. With this, the integral becomes

$$I = \int_0^{\pi/2} \frac{1}{2} \cos(u)\, du \, ,$$

which has a straightforward solution:

$$I = \frac{1}{2} \sin(u) \Big|_0^{\pi/2}$$

From here, one may stay in the $u$-domain to get the final answer, or switch back to the $x$-variable to recover

$$I = \frac{1}{2} \sin\left(x^2\right) \Big|_0^{\sqrt{\pi/2}} = \frac{1}{2}(1-0) = \frac{1}{2} \, .$$

**Constant Shift**

If the $x$-dependence in the integrand is shifted by a constant $\lambda$, i.e.

$$u(x) = x + \lambda \,,$$

then

$$du = dx$$

always holds.

For instance, in

$$I = \int (x+3)^n \, dx \,,$$

we can let $u = x + 3$ so the above becomes

$$I = \int u^n \, du \,,$$

which is easy to solve using Equation (1.6) as

$$I = \frac{u^{n+1}}{n+1} + C \,.$$

Reverse-substitute to get the answer in terms of $x$:

$$\int (x+3)^n \, dx = \frac{(x+3)^{n+1}}{n+1} + C$$

Problem 3

Use $u$-substitution to prove Equation (1.19).

Problem 4

Use $u$-substitution to prove:

$$\int \frac{dx}{1+x} = \ln(1+x) + C \tag{1.20}$$

Problem 5

For a constant $a$, prove:

$$\int (x-a)^{n-1} \, dx = \frac{1}{n} (x-a)^n$$

**The d(sin) Shortcut**

Integrals of the form

$$I = \int f(\sin(x)) \cos(x) \, dx$$

are transformed by standard $u$-substitution. Letting

$$u(x) = \sin(x)$$

such that

$$\frac{du}{dx} = \cos(x) \,,$$

the above readily takes a more standard form:

$$I = \int f(\sin(x)) \cos(x) \, dx = \int f(u) \, du$$

The combination $\cos(x) \, dx$ is written $d(\sin(x))$ as a shortcut, which embeds the notions $u = \sin(x)$, $du = \cos(x) \, dx$ simultaneously:

$$\cos(x) \, dx = d(\sin(x))$$

For example, consider the indefinite integral

$$J = \int \sin^2(x) \cos(x) \, dx \,,$$

which looks like a rather messy antiderivative to wrestle with. Applying the so-called $d \sin()$ shortcut, the integral reads

$$J = \int \sin^2(x) \, d(\sin(x)) = \int u^2 \, du \,,$$

and the problem is now simpler in the $u$-variable. To finish the job, we have

$$J = \frac{1}{3} u^3 + C = \frac{1}{3} \sin^3(x) + C \,.$$

## 3.4 Integrands with Roots

Not every integral involving a square root (or worse) can be solved by simple $u$-substitution. In these cases, it's worth including the exponent of the embedded root in the $u$-substitution.

To illustrate, consider the indefinite integral

$$I = \int \frac{x}{(x-4)^{1/3}} \, dx \,,$$

which begs trying $u = x - 4$, but this makes an absolute mess. Instead, let us take

$$u = (x-4)^{1/3}$$

such that

$$x = u^3 + 4$$
$$dx = 3u^2 \, du \,.$$

Then, the integral looks much easier in the $u$-domain:

$$I = 3 \int \left(u^4 + 4u\right) du$$

Problem 6

Use the above as a starting point to prove:

$$\int \frac{x}{(x-4)^{1/3}} = \frac{3}{5} (x-4)^{2/3} (x+6) + C$$

## 3.5 Partial Fractions

...

## 3.6 Integration by Parts

Consider the product $H(x)$ of two functions $U(x)$, $V(x)$,

$$H(x) = U(x) V(x) \ ,$$

and take the derivative of $H$, minding the product rule:

$$\frac{d}{dx} H(x) = V(x) \frac{d}{dx} U(x) + U(x) \frac{d}{dx} V(x)$$

Next, apply the integral operator $\int dx$ across the whole equation:

$$\int \frac{d}{dx} H(x) \, dx = \int V(x) \frac{d}{dx} U(x) \, dx + \int U(x) \frac{d}{dx} V(x) \, dx$$

Since the integral and derivative operators are mutually annihilating, the left side is simply $H(x)$ evaluated at the integration limits. It suffices to leave the vertical bar empty while working in indefinite form:

$$\int \frac{d}{dx} H(x) \, dx = H(x) \Big| = U(x) V(x) \Big|$$

Introducing the shorthand notation

$$\frac{d}{dx} U(x) = dU$$

and similar for $dV$, the above is written

$$UV \Big| = \int V dU + \int U dV \ ,$$

where all quantities are understood to be functions of $x$.

The reason for doing this is, suppose you are handed an integral of the form $\int U dV$ that is difficult to solve. If we can somehow manage to identify $V(x)$, then perhaps the integral $\int V dU$ is easier than its counterpart. All of this inspires the *integration by parts* formula:

$$\int U dV = UV \Big| - \int V dU \tag{1.21}$$

#### Exemplary Case

To demonstrate integration by parts, consider the definite integral

$$I = \int_0^{\pi/2} x \cos(x) \, dx \ ,$$

which we immediately rewrite as

$$\int_0^{\pi/2} x \cos(x) \, dx = \int_0^{\pi/2} U dV \ .$$

Then identify

$$\begin{aligned} U &= x \\ dV &= \cos(x) \, dx \ , \end{aligned}$$

and we now have two 'mini problems' of determining $dU(x)$ and $V(x)$.

For this example, $dU$ is simply equal to $dx$. (It's always easy to calculate $dU$.) As for $V$, we have $dV/dx = \cos(x)$, which can only mean $V(x) = \sin(x)$.

The integration by parts formula then tells us:

$$\int_0^{\pi/2} x \cos(x) \, dx = x \sin(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x) \, dx$$

Notice how the 'hard' integral on the left is replaced by an 'easy' integral on the right. The answer is now straightforward:

$$I = \int_0^{\pi/2} x \cos(x) \, dx = \frac{\pi}{2} - 1$$

#### Natural Logarithm

The integration by parts recipe also works when there is one function in the integrand, and this is how to find the integral of the natural logarithm. Starting with

$$I = \int \ln(x) \, dx \ ,$$

let

$$\begin{aligned} U &= \ln(x) \\ dV &= dx \end{aligned}$$

such that

$$\begin{aligned} dU &= dx/x \\ V &= x \ . \end{aligned}$$

Then, we have

$$I = x \ln(x) \Big| - \int dx \ ,$$

simplifying to:

$$\int \ln(x)\,dx = x\ln(x) - x + C \tag{1.22}$$

Problem 7

Use $u$-substitution to find the integral of the shifted natural logarithm:

$$\int \ln(1+x)\,dx = (1+x)\ln(1+x) + x + C \tag{1.23}$$

## 3.7 Label Trick

Consider the definite integral that attempts to calculate the area of one quarter of the unit circle:

$$A = \int_0^{\pi/2} \sin^2(\theta)\,d\theta$$

This can be attacked with integration by parts by letting

$$U = \sin(\theta)$$
$$dV = \sin(\theta)\,d\theta$$

such that

$$dU = \cos(\theta)\,d\theta$$
$$V = -\cos(\theta)\ ,$$

and then

$$A = \cancel{-\sin(\theta)\cos(\theta)\Big|_0^{\pi/2}} + \int_0^{\pi/2} \cos^2(\theta)\,d\theta\ .$$

All we've managed to show is that the function $\sin^2(\theta)$ can be replaced by $\cos^2(\theta)$ and the integral remains the same.

Now make use of the fundamental trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

to write

$$\int_0^{\pi/2} \cos^2(\theta)\,d\theta = \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin^2(\theta)\,d\theta\ .$$

The left-most and right-most integrals are both equal to $A$, and all of the hard work suddenly vanishes with the so-called *label trick*:

$$A = \int_0^{\pi/2} d\theta - A$$

Solving for $A$ is a matter of algebra, and the remaining integral is trivial:

$$A = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

### Tricky Logarithmic Integral

A tricky problem that you're welcome to stop reading and try on your own is the following definite integral:

$$I = \int_0^{\infty} \frac{\ln(x)}{1+x+x^2}\,dx$$

The key to this problem is the substitution $u = 1/x$. From this, we have $du/dx = -1/x^2$, and furthermore $\ln(1/u) = -\ln(u)$. The integration limits also end up swapping, and the integral becomes

$$I = \int_{\infty}^{0} \frac{-\ln(u)}{1+1/u+1/u^2}\frac{-du}{u^2}\ .$$

Simplifying further, we find

$$I = \int_{\infty}^{0} \frac{\ln(u)}{1+u+u^2}\,du\ ,$$

and swap the integration limits by paying with a negative sign:

$$I = -\int_0^{\infty} \frac{\ln(u)}{1+u+u^2}\,du$$

This result is exactly opposite to the problem we started with, up to a trivial change of letters. In effect, we have found

$$I = -I\ ,$$

which can *only* mean $I = 0$:

$$0 = \int_0^{\infty} \frac{\ln(x)}{1+x+x^2}\,dx$$

## 3.8 Trigonometric Integrals

### Standard Functions

The integral of each trigonometric function is straightforwardly calculated using antiderivatives or other integration techniques. In indefinite form, these are:

$$\int \sin(x)\,dx = -\cos(x) + C \tag{1.24}$$

$$\int \cos(x)\,dx = \sin(x) + C \tag{1.25}$$

$$\int \tan(x)\,dx = -\int \frac{d(\cos(x))}{\cos(x)} = -\ln(\cos(x)) + C \tag{1.26}$$

$$\int \cot(x)\,dx = \int \frac{d(\sin(x))}{\sin(x)} = \ln(\sin(x)) + C \tag{1.27}$$

$$\int \sec(x)\, dx = \ln(\sec(x) + \tan(x)) + C \tag{1.28}$$

$$\int \csc(x)\, dx = -\ln(\csc(x) + \cot(x)) + C \tag{1.29}$$

We can also recall the derivative of each trigonometric function and make use of the $\int dx$ operator to come up with a few more:

$$\int \sec^2(x)\, dx = \tan(x) + C \tag{1.30}$$

$$\int \csc^2(x)\, dx = -\cot(x) + C \tag{1.31}$$

$$\int \tan(x)\sec(x)\, dx = \sec(x) + C \tag{1.32}$$

$$\int \cot(x)\csc(x)\, dx = -\csc(x) + C \tag{1.33}$$

**Squared Integrand**

The pair of indefinite integrals

$$I_1 = \int \sin^2(x)\, dx$$
$$I_2 = \int \cos^2(x)\, dx$$

can be solved simultaneously. Using the fundamental trig identity, we see

$$I_1 + I_2 = \int \left(\sin^2(x) + \cos^2(x)\right) dx = \int dx = x\bigg| \,,$$

or equivalently

$$I_1 + I_2 = x + C \,.$$

Now integrate $I_1$ by parts via

$$U = \sin(x)$$
$$dV = \sin(x)\, dx$$

such that

$$dU = \cos(x)\, dx$$
$$V = -\cos(x) \;,$$

and $I_1$ is written

$$I_1 = -\sin(x)\cos(x)\bigg| + \int \cos^2(x)\, dx \,,$$

simplifying to

$$I_1 - I_2 = -\sin(x)\cos(x) + C \,.$$

With two equations and two unknowns, $I_1$ and $I_2$ can be isolated independently, resulting in

$$\int \sin^2(x)\, dx = \frac{-\sin(x)\cos(x)}{2} + \frac{x}{2} + C \tag{1.34}$$

$$\int \cos^2(x)\, dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + C \tag{1.35}$$

The pair of indefinite integrals

$$I_3 = \int \tan^2(x)\, dx$$
$$I_4 = \int \sec^2(x)\, dx$$

can also be solved together. Using another fundamental trig identity, find

$$I_4 - I_3 = x + C \,,$$

which means only $I_3$ or $I_4$ need be calculated and we get the other for free.

Choosing $I_4$, recall that the derivative of the tangent is the square of the secant, so

$$I_4 = \int \frac{d}{dx}\tan(x)\, dx = \tan(x) + C \,,$$

and conclude:

$$\int \tan^2(x)\, dx = \tan(x) - x + C \tag{1.36}$$

$$\int \sec^2(x)\, dx = \tan(x) + C$$

Finally, the pair of indefinite integrals

$$I_5 = \int \cot^2(x)\, dx$$
$$I_6 = \int \csc^2(x)\, dx$$

can also be solved together. Using another fundamental trig identity, find

$$I_6 - I_5 = x + C \,.$$

The easiest way to proceed is to remember that the derivative of the cotangent is the negative of the square of the cosecant. Just kidding, that's not terribly easy to remember, but nonetheless the integral $I_6$ becomes

$$I_6 = \int \frac{d}{dx}\left(-\cot(x)\right) dx = -\cot(x) + C \,.$$

From the above we get the pair of answers:

$$\int \cot^2(x)\, dx = -\cot(x) - x + C \tag{1.37}$$

$$\int \csc^2(x)\, dx = -\cot(x) + C$$

### Inverse Functions

Integrals of the inverse trigonometric functions can be tricky to find. Integration by parts works well on a few of them, such as the arctangent. For

$$I = \int \arctan(x)\, dx \ ,$$

let

$$\begin{aligned} U &= \arctan(x) \\ dV &= dx \end{aligned}$$

such that:

$$\begin{aligned} dU &= \frac{dx}{1+x^2} \\ V &= x \end{aligned}$$

With this, the integral reads

$$I = x \arctan(x) \bigg| - \int \frac{x}{1+x^2}\, dx$$

The remaining integral is solved by standard $u$-substitution, namely $u = 1+x^2$ such that $du = 2xdx$. After simplifying, we get the answer:

$$\begin{aligned} \int \arctan(x)\, dx &= x \arctan(x) \\ &\quad - \frac{1}{2} \ln\left(1+x^2\right) + C \qquad (1.38) \end{aligned}$$

The same recipe works for several other inverse trigonometric functions, namely the arccosine, arcsine, and arccotangent:

$$\begin{aligned} \int \arccos(x)\, dx &= x \cos(x) \\ &\quad - \frac{1}{2} \ln\left(1-x^2\right) + C \qquad (1.39) \end{aligned}$$

$$\begin{aligned} \int \arcsin(x)\, dx &= x \sin(x) \\ &\quad + \frac{1}{2} \ln\left(1-x^2\right) + C \qquad (1.40) \end{aligned}$$

$$\begin{aligned} \int \operatorname{arccot}(x)\, dx &= x \operatorname{arccot}(x) \\ &\quad + \frac{1}{2} \ln\left(1+x^2\right) + C \qquad (1.41) \end{aligned}$$

Conspicuously absent from our stack of results are the integrals of the arcsecant and arccosecant. These require more than a simple $u$-substitution that we haven't hit yet, so stay tuned.

### Reduction Formulas

For positive integer $m$, consider the indefinite integral

$$I = \int \sin^m(x)\, dx \ .$$

Integrating by parts, we first write

$$\begin{aligned} U &= \sin^{m-1}(x) \\ dV &= \sin(x)\, dx \end{aligned}$$

and also

$$\begin{aligned} dU &= (m-1) \sin^{m-2}(x) \cos(x)\, dx \\ V &= -\cos(x) \ . \end{aligned}$$

From this, we have

$$\begin{aligned} I = &-\sin^{m-1}(x) \cos(x) \bigg| \\ &+ (m-1) \int \sin^{m-2}(x) \cos^2(x)\, dx \ . \end{aligned}$$

Next, replace $\cos^2(x)$ with $1 - \sin^2(x)$ and use the label trick, giving

$$\begin{aligned} I = &-\sin^{m-1}(x) \cos(x) \bigg| \\ &+ (m-1) \int \sin^{m-2}(x)\, dx - (m-1)\, I \ , \end{aligned}$$

and solving for $I$ we arrive at a *trigonometric reduction formula*:

$$\begin{aligned} \int \sin^m(x)\, dx &= \frac{-1}{m} \sin^{m-1}(x) \cos(x) \bigg| \\ &\quad + \frac{m-1}{m} \int \sin^{m-2}(x)\, dx \qquad (1.42) \end{aligned}$$

Similar reduction formulas exist for each of the elementary trig functions. Each of the following is attained by integration by parts and the label trick:

$$\begin{aligned} \int \cos^m(x)\, dx &= \frac{1}{m} \cos^{m-1}(x) \sin(x) \bigg| \\ &\quad + \frac{m-1}{m} \int \cos^{m-2}(x)\, dx \qquad (1.43) \end{aligned}$$

$$\begin{aligned} \int \tan^m(x)\, dx &= \frac{1}{m-1} \tan^{m-1}(x) \bigg| \\ &\quad - \int \tan^{m-2}(x)\, dx \qquad (1.44) \end{aligned}$$

$$\begin{aligned} \int \csc^m(x)\, dx &= \frac{-1}{m-1} \csc^{m-2}(x) \cot(x) \bigg| \\ &\quad + \frac{m-2}{m-1} \int \csc^{m-2}(x)\, dx \qquad (1.45) \end{aligned}$$

$$\int \sec^m(x)\,dx = \frac{1}{m-1}\sec^{m-2}(x)\tan(x)\Big| + \frac{m-2}{m-1}\int \sec^{m-2}(x)\,dx \quad (1.46)$$

$$\int \cot^m(x)\,dx = \frac{-1}{m-1}\cot^{m-1}(x)\Big| - \int \cot^{m-2}(x)\,dx \quad (1.47)$$

Another reduction formula that mixes the sine and cosine can be established. Consider the case

$$I = \int \sin^m(x)\cos^n(x)\,dx \,.$$

By letting $u = \sin^{m-1}(x)$ and following the consequences, one finds

$$\int \sin^m(x)\cos^n(x)\,dx = -\frac{1}{m+n}\sin^{m-1}(x)\cos^{n+1}(x)\Big| + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)\,dx \,. \quad (1.48)$$

Note that this result reproduces Equation (1.42) for $n = 0$.

A different result is attained by letting $u = \cos^{n-1}(x)$:

$$\int \sin^m(x)\cos^n(x)\,dx = \frac{1}{m+n}\sin^{m+1}(x)\cos^{n-1}(x)\Big| + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)\,dx \,. \quad (1.49)$$

Note that this result reproduces Equation (1.43) for $m = 0$.

### Mixed Wavelengths

Starting with the product formula

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta) \;,$$

suppose that $\alpha$, $\beta$ are multiples of an angle $\theta$

$$\alpha = m\theta$$
$$\beta = n\theta$$

for non-equal integers $m$, $n$.

Next apply the integral operator $\int d\theta$ across the whole equation

$$2\int \sin(m\theta)\cos(n\theta)\,d\theta = \int \sin(m\theta+n\theta)\,d\theta + \int \sin(m\theta-n\theta)\,d\theta \,,$$

and simplify:

$$\int \sin(m\theta)\cos(n\theta)\,d\theta = \frac{-\cos((m+n)\,\theta)}{2(m+n)} - \frac{\cos((m-n)\,\theta)}{2(m-n)} + C \quad (1.50)$$

More product formula exploits lead to additional mixed-wavelength integral identities:

$$\int \cos(m\theta)\cos(n\theta)\,d\theta = \frac{\sin((m+n)\,\theta)}{2(m+n)} + \frac{\sin((m-n)\,\theta)}{2(m-n)} + C \quad (1.51)$$

$$\int \sin(m\theta)\sin(n\theta)\,d\theta = \frac{-\sin((m+n)\,\theta)}{2(m+n)} + \frac{\sin((m-n)\,\theta)}{2(m-n)} + C \quad (1.52)$$

### Orthogonality

Evaluating the mixed-wavelength integral identities (1.50)-(1.52), in various domains of length $2\pi$ leads to some additional information called *orthogonality relations*. (Keep in mind that $m$, $n$ are different integers.)

Choosing $[-\pi : \pi]$ first, we find

$$\int_{-\pi}^{\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{-\pi}^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{-\pi}^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta = 0$$

The same results hold when the domain is changed to $[0 : 2\pi]$:

$$\int_{0}^{2\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{0}^{2\pi} \cos(m\theta)\cos(n\theta)\,d\theta = 0$$
$$\int_{0}^{2\pi} \sin(m\theta)\sin(n\theta)\,d\theta = 0$$

When the wavelengths $m$, $n$ are one and the same integer $m$, two results switch to nonzero

$$\int_{-\pi}^{\pi} \cos^2(m\theta)\, d\theta = \int_0^{2\pi} \cos^2(m\theta)\, d\theta = \pi$$
$$\int_{-\pi}^{\pi} \sin^2(m\theta)\, d\theta = \int_0^{2\pi} \sin^2(m\theta)\, d\theta = \pi\ ,$$

and the case that mixes sine and cosine remains zero:

$$\int_{-\pi}^{\pi} \sin(m\theta)\cos(m\theta)\, d\theta = \int_0^{2\pi} \sin(m\theta)\cos(m\theta)\, d\theta = 0$$

## 3.9 Trigonometric Substitution

Each of the following integrals

$$I_1 = \int \frac{dx}{x^2\sqrt{x^2+a^2}}$$
$$I_2 = \int \frac{\sqrt{a^2-x^2}}{x^2}\, dx$$
$$I_3 = \int \frac{dx}{\left(x^2-a^2\right)^{3/2}}$$

for nonzero constant $a$ are difficult to solve by standard $u$-substitution or integration by parts. In fact, each requires a different trick called *trigonometric substitution*.

### Tangent Substitution

When the integrand contains $x^2 + a^2$, let

$$x = a\tan(\theta)\ ,$$

so then

$$dx = a\sec^2(\theta)\, d\theta\ .$$

By standard trig identities, the quantity $x^2 + a^2$ becomes

$$x^2 + a^2 = a^2\sec^2(\theta)\ .$$

With this, the integral $I_1$ transforms into something we can solve:

$$I_1 = \int \frac{a\sec^2(\theta)}{a^3\tan^2(\theta)\sec(\theta)}\, d\theta = \frac{1}{a^2}\int \frac{d(\sin(\theta))}{\sin^2(\theta)}$$

Problem 8

Use the above as a starting point to prove:

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2 x} + C$$

### Sine Substitution

When the integrand contains $a^2 - x^2$, let

$$x = a\sin(\theta)\ ,$$

so then

$$dx = a\cos(\theta)\, d\theta\ .$$

By standard trig identities, the quantity $a^2 - x^2$ becomes

$$a^2 - x^2 = a^2\cos^2(\theta)\ .$$

With the sine substitution, the integral $I_2$ reduces to a simpler problem:

$$I_2 = \int \frac{a^2\cos^2(\theta)}{a^2\sin^2(\theta)}\, d\theta = \int \cot^2(\theta)\, d\theta$$

Problem 9

Use the above as a starting point to prove:

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\, dx = -\arcsin\left(\frac{x}{a}\right) - \frac{\sqrt{a^2-x^2}}{x} + C$$

### Secant Substitution

When the integrand contains $x^2 - a^2$, let

$$x = a\sec(\theta)\ ,$$

so then

$$dx = a\sec(\theta)\tan(\theta)\, d\theta\ .$$

By standard trig identities, the quantity $x^2 - a^2$ becomes

$$x^2 - a^2 = a^2\tan^2(\theta)\ .$$

With the sine substitution, the integral $I_3$ reduces to a simpler problem:

$$I_3 = \frac{1}{a^2}\int \frac{d(\sin(\theta))}{\sin^2(\theta)} d\theta$$

Problem 10

Use the above as a starting point to prove:

$$\int \frac{dx}{\left(x^2-a^2\right)^{3/2}} = \frac{-x}{a^2\sqrt{x^2-a^2}} + C$$

**Trigonometric Ratios**

Rational functions of sine and cosine land to a particular $u$-substitution:

$$u = \tan(\theta/2) \ .$$

From the trigonometric half-angle formulas, we can next write

$$\cos(\theta) = \frac{1-u^2}{1+u^2}$$
$$\sin(\theta) = \frac{2u}{1+u^2} \ ,$$

and

$$u = \frac{\sin(\theta)}{1+\cos(\theta)}$$
$$du = \frac{1}{2}\left(1+u^2\right) d\theta \ .$$

With this substitution, integrals of the form

$$I = \int f\left(\sin(\theta), \cos(\theta)\right) d\theta$$

can be written:

$$I = \int f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \frac{du}{1+u^2}$$

In the general case, this substitution works when the function being integrated is a polynomial of two variables or a ratio of two polynomials.

To illustrate, consider the indefinite integral

$$J = \int \frac{d\theta}{3+\cos(\theta)} \ .$$

Using the above substitutions, the integral becomes

$$J = \int \frac{du}{2+u^2} \ .$$

Problem 11

Use the above as a starting point to prove:

$$\int \frac{d\theta}{3+\cos(\theta)} = \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} \tan\left(\frac{\theta}{2}\right)\right) + C$$

**Arcsecant and Arccosecant**

The integrals of the arcsecant and the arccosecant have to be cracked with trigonometric substitution. For

$$I = \int \operatorname{arcsec}(x)\, dx \ ,$$

proceed with integration by parts to write

$$U = \operatorname{arcsec}(x)$$
$$dV = dx$$

such that

$$dU = \frac{dx}{x\sqrt{x^2-1}}$$
$$V = x \ .$$

The integral becomes

$$\int \operatorname{arcsec}(x)\, dx = x \operatorname{arcsec}(x) \Big| - \int \frac{dx}{\sqrt{x^2-1}} \ .$$

The new integral on the right is handled by a secant substitution. Let

$$x = \sec(\theta)$$

such that

$$dx = \sec(\theta)\tan(\theta)\, d\theta \ ,$$

and

$$\sqrt{x^2-1} = \tan(\theta) \ ,$$

so we have

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \sec(\theta)\, d\theta \ .$$

The integral of the secant has a known solution, namely

$$\int \sec(\theta)\, d\theta = \ln\left(\sec(\theta) + \tan(\theta)\right) + C \ ,$$

or, in terms of the $x$-variable,

$$\int \sec(\theta)\, d\theta = \ln\left(x + \sqrt{x^2-1}\right) + C \ .$$

Finally, we have the answer:

$$\int \operatorname{arcsec}(x)\, dx = x \operatorname{arcsec}(x) \Big| - \ln\left(x + \sqrt{x^2-1}\right) + C \quad (1.53)$$

Problem 12

Do a similar exercise for the arccosecant:

$$\int \operatorname{arccsc}(x)\, dx = x \operatorname{arccsc}(x) \Big| + \ln\left(x + \sqrt{x^2-1}\right) + C \quad (1.54)$$

**Area of the Ellipse**

For the ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, ,$$

the area contained in the first quadrant (a quarter of the ellipse) is given by

$$A = \int_0^a y\,(x)\,dx \, .$$

It also happens that the same ellipse can be described using a pair of parametric equations, particularly

$$\begin{aligned} x &= a\cos(\theta) \\ y &= b\sin(\theta) \ , \end{aligned}$$

easily shown to reproduce the Cartesian formula. Substituting the above equations for $x$, $y$ into the area integral changes the integration variable to $\theta$:

$$A = -ab\int_{\pi/2}^{0} \sin^2(\theta)\,d\theta$$

The integral of the square of the sine is well known by known, particularly by equation (1.34). Evaluating the definite integral gives the final answer:

$$A = -ab\left(\frac{-\pi}{4}\right) = \frac{1}{4}\pi ab$$

The area of the complete ellipse is $\pi ab$.

Problem 13

Show that the area of the ellipse

$$ax^2 + bxy + cy^2 = 1$$

is equal to

$$A = \frac{2\pi}{\sqrt{4ac - b^2}} \, .$$

Hint: Rotate the coordinates and write the area of the same ellipse in the rotated system.

## 3.10 Mirror Trick

A lesser-known technique we'll call the *mirror trick* can help with integrals such as

$$J = \int_0^{\pi/2} \frac{\sqrt{\sin(\theta)}}{\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}}\,d\theta \, .$$

For practice, consider the definite integral of a well-behaved function $g(x)$:

$$I = \int_a^b g(x)\,dx$$

By making the substitution

$$\begin{aligned} u &= b + a - x \\ du &= -dx \, , \end{aligned}$$

we find

$$I = \int_b^a g(b + a - u)(-du) \ .$$

Of course, the integration variable itself can be swapped with any other letter, so we come up with a second equation for $J$ involving the integral in the $x$-domain:

$$I = \int_a^b g(b + a - x)\,dx$$

The same idea can be applied to a different integral

$$K = \int_a^b \frac{g(x)}{g(b + a - x) + g(x)}\,dx \, ,$$

which, using the same $u$-substitution $u = b + a - x$, becomes

$$K = \int_b^a \frac{g(b + a - u)}{g(u) + g(b + a - u)}(-du) \ ,$$

or equivalently

$$K = \int_a^b \frac{g(b + a - x)}{g(x) + g(b + a - x)}\,dx \, .$$

Take the two expressions for $K$ and take their sum,

$$2K = \int_a^b \frac{g(x) + g(b + a - x)}{g(x) + g(b + a - x)}\,dx \, ,$$

and notice the entire integrand cancels, leaving

$$K = \frac{b - a}{2} \, .$$

Evidently, the result of integral $K$ has nothing to do with the function being integrated, only the limits matter:

$$\int_a^b \frac{g(x)}{g(b + a - x) + g(x)}\,dx = \frac{b - a}{2} \tag{1.55}$$

Returning to the problem on hand, the integral $J$ can be written

$$J = \int_0^{\pi/2} \frac{\sqrt{\sin(\theta)}}{\sqrt{\sin(\theta)} + \sqrt{\sin(\pi/2 - \theta)}}\,d\theta \, .$$

Comparing this to Equation (1.55), let $a = 0$ and $b = \pi/2$ and the result is half their difference:

$$J = \frac{\pi}{4}$$

## 3.11 Series Expansion

Integration and series expansion play nicely together and are used often to approximate the solution to otherwise insoluble integrals.

### Physical Pendulum

It's possible to show using energy conservation that a frictionless pendulum of length $L$ and mass $m$ in uniform gravity is governed by

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}}\sqrt{\cos(\theta) - \cos(\theta_0)} \,,$$

where $\theta$ is the deflection of the pendulum from vertical and $\theta_0$ represents the highest position attainable where motion momentarily stops. This is a 'separable' equation, and can be reshuffled as an indefinite integral:

$$\int \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}} = \sqrt{\frac{2g}{L}} \int dt$$

The left side needs some preparation before proceeding. The cosine terms are replaced using the half-angle formula

$$1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right) \,.$$

Also define

$$\sin(\phi) = \frac{1}{a}\sin\left(\frac{\theta}{2}\right) \,,$$

where

$$a = \sin\left(\frac{\theta_0}{2}\right) \,,$$

implying

$$d\theta = 2a\frac{\sqrt{1 - \sin^2(\phi)}}{\sqrt{1 - a^2\sin^2(\phi)}}\, d\phi \,.$$

With these substitutions, the integral on hand becomes

$$\int \frac{d\phi}{\sqrt{1 - a^2\sin^2(\phi)}} = \sqrt{\frac{g}{L}} \int dt \,.$$

The left side is called an *elliptic integral*, and has no simple closed-form solution in general.

Despite the above elliptic integral, we can still use it to crank out an answer. Let $t = 0$ correspond to $\theta = 0$ and $\phi = 0$, which is the lowest position available to the pendulum. After one period of motion at $t = T$, i.e. once the angle $\theta$ has returned to zero again, and the value $2\pi$ has accumulated in $\phi$. We then have a formula for the period of the motion:

$$\sqrt{\frac{g}{L}}\int_0^T dt = \int_0^{2\pi} \frac{d\phi}{\sqrt{1 - a^2\sin^2(\phi)}} \,.$$

On the right, use the Taylor expansion of the radical to write

$$\frac{1}{\sqrt{1 - a^2\sin^2(\phi)}} \approx 1 + \frac{1}{2}a^2\sin^2(\phi) + \frac{3}{8}a^4\sin^4(\phi) + \cdots \,,$$

which only works when $a\sin(\phi)$ is a relatively 'small' number.

While we have paid with some accuracy and generality, the thing we gain is that the right side can be integrated. Going term by term it helps to know

$$\int_0^{2\pi} \sin^2(\phi)\, d\phi = \pi$$
$$\int_0^{2\pi} \sin^4(\phi)\, d\phi = \frac{3\pi}{4} \,,$$

attainable by elementary means or using a trigonometric reduction formula.

The integral for the period reduces to

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{a^2}{4} + \frac{9a^4}{64} + \cdots\right) \,.$$

If the initial angle $\theta_0$ is much less than one, we further have

$$a^2 \approx \frac{\theta_0^2}{4} \,,$$

or

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta_0^2}{16}\right) \,.$$

From this we get the familiar period of the simple pendulum, along with a correction that accounts for more extreme initial conditions.

### Shifted Natural Logarithm

Starting with Equation (1.20), namely

$$\ln(1+x) + C = \int \frac{dx}{1+x} \,,$$

consider the scenario $|x| < 1$.

In this case, the fraction $1/(1+x)$ can be replaced via the geometric series:

$$\ln(1+x) + C = \int \left(1 - x + x^2 - x^3 + \cdots\right) dx$$

The whole right side can be integrated quite easily:

$$\ln(1+x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

The integration constant is zero by construction, and we end up with an infinite series for the shifted natural logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

This result is in fact the same thing we'd get by Taylor expanding $\ln(1+x)$ near $x = 0$. Unlike the Taylor expansion however, we can now say for certain that the series approximation of $\ln(1+x)$ converges for $|x| < 1$. We can be a little naughty and try $x = 1$ exactly to come up with an infinite approximation for $\ln(2)$:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{1.56}$$

**Arctangent**

Recall that the derivative of the arctangent function

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \ ,$$

and consider the case $|x| < 1$. The right side expands as a geometric series:

$$\frac{d}{dx}\arctan(x) = 1 - x^2 + x^4 - x^6 + \cdots$$

Apply the $\int dx$ operator to both sides and simplify, to get, for $|x| < 1$,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \ .$$

The integration constant is easily ruled to be zero and is omitted. Nor surprisingly, this is what emerges when Taylor expanding $\arctan(x)$ near $x = 0$.

The infinite expression for the arctangent can be used to come up with an expression for $\pi/4$ by setting $x = 1$,

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \ , \tag{1.57}$$

called the *Leibniz formula*.

It's important to note that Equations (1.56), (1.57) each send $x = 1$ to the geometric series, which may seem illegal, as this is where the geometric series is supposed to lose jurisdiction. Technically, each result is attained by letting $x \to 1$ in a formal limit, and making sure divergence does not occur.

**Sine of X Squared**

Innocent as it appears, the indefinite integral

$$I = \int \sin\left(x^2\right) dx$$

has no elementary solution. To make headway, replace the sine function with its exact polynomial representation, namely

$$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \ .$$

Suddenly, we see a path forward. By trading any possibility of a closed solution, we can at least deal with the right side. Integrate each term and a strange answer emerges:

$$\int \sin\left(x^2\right) dx = \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \cdots \tag{1.58}$$

## 3.12 Stirling's Approximation

There is an important relationship governing very large whole numbers called *Stirling's approximation*, given by

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \ .$$

While a full derivation is beyond the scope of this section, we can establish a slightly weaker version, namely

$$n! \approx \left(\frac{n}{e}\right)^n \ . \tag{1.59}$$

**Derivation**

To begin, write $n!$ in open form, namely

$$n! = n(n-1)(n-2)(n-3)\cdots(2)(1) \ ,$$

then take the natural log of both sides to write

$$\ln(n!) = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots \ ,$$

and condense the right using summation notation:

$$\ln(n!) = \sum_{j=1}^{n} \ln(j)$$

Now, this almost looks like a Riemann sum if it weren't for the conspicuous absence of a $\Delta x$-like term. However, since the sum runs over whole numbers only, there is an effective $\Delta x_j = 1$ at play:

$$\ln(n!) = \sum_{j=1}^{n} \ln(j)\,\Delta x_j$$

Even though $\Delta x_j$ cannot be pushed to zero, the above sum can be approximated as continuous *anyway*, but only for very large $n$. Working in this regime, we can replace the above with

$$\ln(n!) \approx \int_1^n \ln(x)\,dx\,,$$

solved by

$$\ln(n!) \approx (\ln(x) - x)\Big|_1^n\,,$$

having approximate solution

$$\ln(n!) \approx \ln(n) - n\,.$$

Apply the $\exp()$ operator to isolate the factorial term, and Equation (1.59) emerges.

### Strange Product

Let us simplify the quantity

$$A = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\cdots(3n)}{n^{2n}}\right)^{1/n}$$

as far as possible.

One way to proceed is to take the natural log of both sides and simplify:

$$\ln(A) = \lim_{n\to\infty}\frac{\ln(n+1)+\cdots+\ln(3n)-2n\ln(n)}{n}$$

There are $2n$ total positive terms in the sum above, so we can break apart the negative term into $2n$ parts and subtract $\ln(n)$ from each positive term to get:

$$\ln(A) = \lim_{n\to\infty}\frac{1}{n}\cdot \left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{2n}{n}\right)\right)$$

Simplifying, this is

$$\ln(A) = \lim_{n\to\infty}\sum_{j=1}^{2n}\ln\left(1+\frac{j}{n}\right)\frac{1}{n}\,.$$

Using the same trick that led to Stirling's approximation, argue that because the largeness of $j$ will dominate anything to do with small $j$, the sum can be considered continuous with

$$x_j = j/n$$
$$\Delta x = 1/n\,.$$

In this regime, we have, approximately:

$$\ln(A) \approx \int_0^2 \ln(1+x)\,dx\,,$$

equivalent to

$$\ln(A) \approx \int_1^3 \ln(u)\,du\,,$$

having solution

$$\ln(A) \approx (u\ln(u) - u)\Big|_1^3\,,$$

or

$$\ln(A) \approx 3\ln(3) - 2\,,$$

and, finally,

$$A \approx \frac{3^3}{e^2}\,.$$

Let us now do the same calculation using Stirling's approximation. First notice $A$ can be written

$$A = \lim_{n\to\infty}\left(\frac{1}{n^{2n}}\frac{(3n)!}{n!}\right)^{1/n}\,,$$

and then Equation (1.59) tells us

$$A \approx \lim_{n\to\infty}\left(\frac{1}{n^{2n}}\frac{(3n)^{3n}}{e^{3n}}\frac{e^n}{n^n}\right)^{1/n}\,,$$

reducing to $A \approx 3^3/e^2$, as expected. All $n$-dependence cancels out.

### Strange Function

Consider the function

$$A(x) = \lim_{n\to\infty}\left(\frac{1}{n^{xn}}\frac{((x+1)\,n)!}{n!}\right)^{1/n}\,,$$

where $x = 2$ reproduces the previous product.

Following similar steps, it's straightforward to show that

$$\ln(A(x)) = \lim_{n\to\infty}\sum_{j=1}^{xn}\ln\left(1+\frac{j}{n}\right)\frac{1}{n}\,,$$

or

$$\ln(A(x)) \approx \int_0^x \ln(1+t)\,dt\,,$$

but let's resist solving the integral.

Attack the problem a second way using Stirling's approximation to get

$$A(x) \approx \frac{(x+1)^{x+1}}{e^x}\,,$$

or

$$\ln(A(x)) = (x+1)\ln(x+1) - x\,.$$

With two ways to express $\ln(A)$, eliminate it to conclude

$$\int \ln(1+x)\,dx = (x+1)\ln(x+1) - x + C\,,$$

which happens to be correct.

# 4 Integrals and Geometry

## 4.1 Arc Length

Integration is the tool for calculating the arc length of a differentiable curve $y = f(x)$. At a given point $(x, y)$ on such a curve, there is a neighboring point $(x + dx, y + dy)$ connected by a straight line of length

$$dS = \sqrt{dx^2 + dy^2} .$$

The term $dx$ can be pulled out of the radical to get

$$dS = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} ,$$

and notice the ratio $dy/dx$ is none other than the slope $f'(x)$ of the curve being measured.

The integral over $dS$ is the total length of the curve between a set of endpoints $x_0$, $x_1$:

$$S = \int dS = \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2}\, dx \tag{1.60}$$

Note that a similar formula can be derived by removing $dy$ from the radical and ending up with an integral in the $y$-domain.

Problem 1

Show that the arc length of a symmetric parabolic segment of base $2a$ and height $h$ is:

$$\begin{aligned} L &= \frac{a^2}{h} \int_0^{2h/a} \sqrt{1 + x^2}\, dx \\ &= \sqrt{a^2 + 4h^2} + \frac{a^2}{2h} \ln\left(\frac{2h + \sqrt{a^2 + 4h^2}}{a}\right) \end{aligned}$$

Hint: You may need the secant reduction formula.

Problem 2

Show that the arc length of an ellipse with eccentricity $e$ is given by the *complete elliptic integral of the second kind*:

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2(\theta)}\, d\theta$$

Problem 3

Show that the arc length of a hyperbola with eccentricity $e$ is given by another elliptic integral:

$$L = 4a \int \sqrt{e^2 \cosh^2(\theta) - 1}\, d\theta$$

## 4.2 Volume of Revolution

A sneaky way to calculate certain three-dimensional volumes using one-dimensional integrals can be established. For this we require differentiable functions $y = f(x)$ that are greater than zero in the domain $x_0 \le x \le x_1$.

### Circular Disk Method

A three-dimensional volume with axial symmetry can be produced by rotating the curve $y(x)$ about the $x$-axis. Each height $y$ on the curve is swung around one full revolution to trace out a disk of area $\pi y^2$, and the total volume enclosed is the sum across the grain of many infinitely-thin disks. As an integral, such a *volume of revolution* is given by:

$$V = \int_{x_0}^{x_1} \pi (f(x))^2\, dx \tag{1.61}$$

Problem 4

Show that a cone of height $H$ and base radius $R$ has volume

$$V = \frac{1}{3}\pi R^2 H .$$

Problem 5

Use elementary methods to show that a *cone frustum* of height $H$ with end radii $R_1$, $R_2$ has volume

$$V = \frac{1}{3}\pi \left(R_1^2 + R_1 R_2 + R_2^2\right) H .$$

Use the disk method with the line

$$y = \left(\frac{R_2 - R_1}{H}\right) x + R_1$$

to get the same answer.

Problem 6

A *paraboloid* is the volume formed by a parabola rotated about its axis of symmetry. Show that the volume of a paraboloid of height $H$ and base radius $R$ is given by

$$V = \frac{1}{2}\pi R^2 H .$$

Hint: Rotate the parabola $y = Hx^2/R^2$ about the $y$-axis and $x$ becomes the disk radius.

### Square Disk Method

Modifying the circular disk method, one can imagine summing across square disks instead. To illustrate, suppose a pyramid with square cross section has height $h$, length $l$, and width $w$.

We'll take the square cross section as parallel to the $xy$-plane, and we will integrate vertically along $z$. For a given height $z \le h$, the dimensions of a 'square disk' are

$$\begin{aligned} x(z) &= z\, l/h \\ y(z) &= y\, w/h \,. \end{aligned}$$

The total volume the pyramid is

$$V = \frac{lw}{h^2} \int_0^h z^2 dz = \frac{lwh}{3} \,.$$

**Washer Method**

Introducing a second function $g(x)$ that is less than $f(x)$ but greater than zero in the domain, we can calculate the volume of revolution trapped between the two curves. In this case, simply subtract the area of one disk from the other to form a 'washer'. The corresponding volume integral becomes:

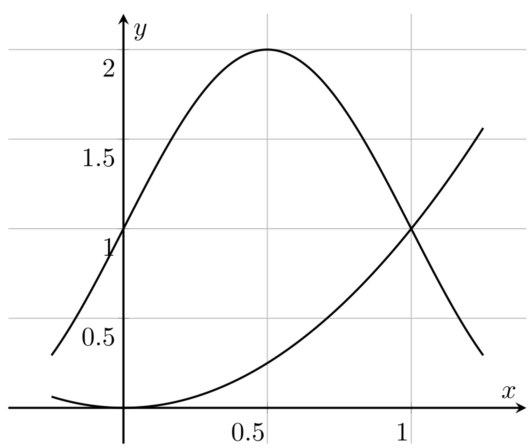
$$V = \int_{x_0}^{x_1} \pi \left( (f(x))^2 - (g(x))^2 \right) dx \tag{1.62}$$

Problem 7

In the domain $0 \le x \le 1$, consider the two curves

$$\begin{aligned} y_1 &= 1 + \sin(\pi x) \\ y_2 &= x^2 \end{aligned}$$

as shown. Write an expression for the volume of revolution about the $x$-axis and also the $y$-axis.




Hint: For the $x$-axis rotation, you should find:

$$V_x = \pi \int_0^1 \left( (1 + \sin(\pi x))^2 - x^4 \right) dx$$

Then, with

$$\begin{aligned} x_1 &= \sqrt{y} \\ x_2 &= \frac{1}{\pi} \arcsin(y-1) \ , \end{aligned}$$

find

$$V_y = \pi \int_0^1 y\, dy + \pi \int_1^2 \left( (1 - x_2)^2 - x_2^2 \right) dy \,.$$

**Cylindrical Shell Method**

A different volume of revolution is attained by rotating the function $y = f(x)$ about the $y$-axis. In this case, a three-dimensional volume is made of many concentric cylindrical shells.

For a point $x$ in the domain, along with a neighboring point $x + dx$, rotating about the $y$-axis traces a pair of circles whose radii differ by $dx$. The height of each circle is $f(x)$, $f(x + dx)$ respectively. This defines a cylindrical 'shell' having volume

$$\begin{aligned} dV_{\text{shell}} = \pi\,&(x + dx)^2 f(x + dx) \\ &- \pi\,(x)^2 f(x) \ , \end{aligned}$$

or, in the first-order limit,

$$dV_{\text{shell}} = 2\pi x f(x)\, dx \,.$$

In essence, we see that the volume of a thin cylindrical shell is the same as that of a rectangle of thickness $dx$, height $f(x)$, and width $2\pi x$. The total volume is the integral of thin shells:

$$V = \int dV_{\text{shell}} = \int_{x_0}^{x_1} 2\pi x f(x)\, dx \tag{1.63}$$

Problem 8

Show that the volume of the upper half of a sphere of radius $R$ is given by

$$V = \int_0^R 2\pi x \sqrt{R^2 - x^2}\, dx = \frac{2}{3}\pi R^3 \,.$$

Problem 9

Use the offset circle

$$(x - R)^2 + y^2 = a^2$$

to find the volume of a *torus*:

$$V = 2 \int_{R-a}^{R+a} 2\pi x \sqrt{a^2 - (x - R)^2}\, dx = 2\pi^2 R a^2$$

## 4.3 Surface of Revolution

A technique similar to the volume of revolution can tell us the *surface area of revolution* of a solid generated by a function $y = f(x)$.

For a point $x$ in the domain, along with a neighboring point $x + dx$, rotating about the $x$-axis traces a pair of circles parallel to the $yz$ plane. The circumference of each circle is $2\pi y$, $2\pi(y + dy)$, respectively. We can take each circumference as the edges of a skinny trapezoid whose width is the arc length

$$dw = \sqrt{dx^2 + dy^2}\ ,$$

and the area of such a trapezoid is

$$dA = \pi(2y + dy)\, dw\ .$$

In the first-order limit, we can write the differential area:

$$dA = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Summing across the grain of many thin strips will cover the surface and reveal the total area of revolution for $y = f(x)$:

$$A = \int_{x_0}^{x_1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{1.64}$$

**Gabriel's Horn**

Consider the hyperbola

$$y = \frac{1}{x}$$

in the domain

$$1 \leq x < \infty\ .$$

The volume of revolution of this particular shape is called *Gabriel's horn*, and contains an interesting 'paradox'. Computing the volume of Gabriel's horn is straightforward:

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = 1$$

Watch what happens if we try to compute the surface area:

$$A = \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}}\, dx$$

The square root term makes the integral rather ugly, but notice how its presence always scales the integrand higher. This means we can also write

$$A > \int_1^\infty 2\pi \left(\frac{1}{x}\right)\ , dx$$

which means

$$A > 2\pi\left(\ln(\infty) - \ln(1)\right)\ .$$

What? The area is somehow infinite - the math was done correctly. But this shouldn't be, because the volume is a finite number. Some argued that filling the horn with a finite volume of paint is equivalent to painting the inside, which ought to make the area finite. Others pointed out that an infinite horn cannot be physically constructed, and that paint flows at a finite speed and would take forever to flow into the horn.

This 'paradox' was known to seventeenth-century mathematicians, not excluding Hobbes, Wallis, and Galileo, originally brought to public attention by Torricelli.

There really is no paradox on hand, and paint is a bad analogy. Keep in mind that paint is a three-dimensional fluid. Filling Gabriel's horn with fluid returns to the original problem - what's the surface area of the paint (excluding the end disc)?

Another way to illustrate the point is to compare the rates of change of the volume and surface with respect to $x$. Using

$$\frac{dV}{dx} = \pi\left(\frac{1}{x}\right)^2$$
$$\frac{dA}{dx} = 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \frac{1}{x^4}}\ ,$$

define the rate

$$R = \frac{dV/dx}{dA/dx}\ ,$$

simplifying to

$$R = \frac{1}{2x\sqrt{1 + 1/x^4}}\ .$$

This rate vanishes in the limit $x \to \infty$, which means the area outpaces the volume in the long run.

## 4.4 Centroid

...

<u>Problem 10</u>

Show that the centroid of a parabolic segment of height $h$ is $\bar{y} = 2h/5$.

<u>Problem 11</u>

Show that the centroid of a half-ellipse of base $2a$ and vertex height $b$ is $\bar{y} = 4b/3\pi$.

## 4.5 The Cycloid

### Definition

Let a 'generating' circle of radius $R$ roll on the $x$ axis. As the circle moves, the point on the rim originally at $(0, 0)$ traces the shape of a *cycloid* as shown in Figure 1.1.

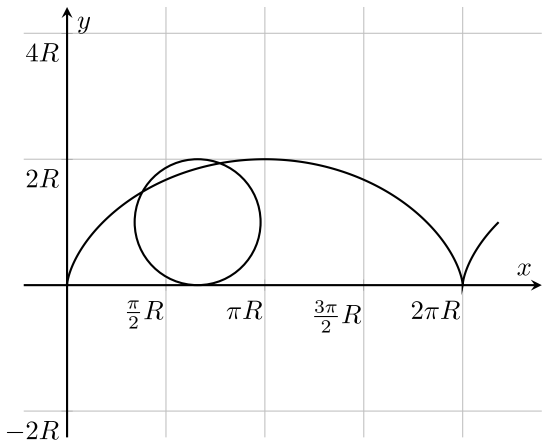



Figure 1.1: The cycloid with generating circle.

### Parameterization

There is no simple expression $y(x)$ for the cycloid. Instead we introduce a parameter $\theta$ that tracks the evolution of the generating circle. In terms of $\theta$, the shape of the cycloid is given by

$$x(\theta) = R\theta - R\sin(\theta) \tag{1.65}$$
$$y(\theta) = R - R\cos(\theta) \ . \tag{1.66}$$

The cycloid is clearly periodic in the variable $\theta$. While $\theta$ can take on any real value and still represent a cycloid, we'll stay interested in the domain $[0 : 2\pi]$.

### Velocity Envelope

Supposing $\theta$ evolves in a smooth and differentiable manner, we can take derivatives with respect to $\theta$. For brevity, define

$$\omega(t) = \frac{d}{d\theta}\theta(t) \ ,$$

and calculate the time derivative of $x(\theta)$, $y(\theta)$ to get:

$$\frac{dx}{dt} = R\omega - R\omega\cos(\theta)$$
$$\frac{dy}{dt} = R\omega\sin(\theta)$$

If we isolate the trig terms and square each equation, the fundamental trig identity can be used to derive

$$\left(\frac{dx}{dt} - R\omega\right)^2 + \left(\frac{dy}{dt}\right)^2 = (R\omega)^2 \ ,$$

which is called the *envelope* of velocities of the cycloid. Plotted in velocity space, the above depicts a circle of radius $R\omega$ centered at $(R\omega, 0)$.

### Tangent Line

At a point $(x_0, y_0)$ on the cycloid, the slope is still given by $dy/dx$ at that point, despite using a parameterized representation of the curve. Calculating the slope is a matter of the chain rule:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1}$$

Carrying this out, we find

$$\frac{dy}{dx} = \frac{\sin(\theta)}{1 - \cos(\theta)} = \cot\left(\frac{\theta}{2}\right) \ .$$

This is enough to write down a equation for the tangent line to the cycloid:

$$y_{\text{tan}} = y_0 + \cot\left(\frac{\theta}{2}\right)(x - x_0)$$

Replacing $x_0$, $y_0$ with their representations in $\theta$ gives a neater formula, after some simplifying:

$$y_{\text{tan}} = 2R + \cot\left(\frac{\theta}{2}\right)(x - R\theta)$$

Interestingly, we see that the tangent line always passes through the point $(R\theta, 2R)$, which is the top of the generating circle as it goes along.

Problem 12

A stone lodged on the rim of a bicycle tire of radius $R$ dislodges at the height of its cycloidic path. Determine its trajectory after leaving the tire. Answer:

$$x(t) = R\pi + 2R\omega t$$
$$y(t) = 2R - gt^2/2$$

### Normal Line

Knowing the slope at any point $(x_0, y_0)$ on the cycloid, we can write an expression for the normal line at the same point:

$$y_{\text{norm}} = y_0 - \tan\left(\frac{\theta}{2}\right)(x - x_0)$$

Like the case for the tangent line, replacing $x_0$, $y_0$ with their representations in $\theta$ gives a neater formula, after some simplifying:

$$y_{\text{norm}} = -\tan\left(\frac{\theta}{2}\right)(x - R\theta)$$

From this, wee see that the normal line always hits the point of contact between the generating the circle and the line on which it rolls.

### Arc Length

The arc length of the cycloid is straightforwardly calculated from Equation (1.60). For this, we start with

$$S = \int_0^{2\pi R} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\ ,$$

which, after substituting $x(\theta)$, $y(\theta)$ becomes

$$S = \sqrt{2}R \int_0^{2\pi} \sqrt{1 - \cos(\theta)}\, d\theta\ ,$$

readily simplifying to

$$S = 2R \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 8R\ .$$

### Area Enclosed

The area enclosed by the cycloid and the $x$-axis is given by the standard setup:

$$A = \int_0^{2\pi R} y\, dx = R^2 \int_0^{2\pi} (1 - \cos(\theta))^2\, d\theta$$

The remaining integral is straightforwardly solved, and we find the enclosed area to be three times that of the generating circle:

$$A = 3\pi R^2$$

### Volume Enclosed

A cycloid revolved about the $x$-axis encloses a volume that we can calculate with the circular disk method, i.e. Equation (1.61). For this case, we have, after simplifying

$$V = \int_0^{2\pi} \pi R^2 (1 - \cos(\theta))^3\, d\theta\ .$$

The remaining integral is a bit tedious but isn't difficult, ending with

$$V = 5\pi^2 R^3\ .$$

### Surface Area

The surface of revolution made by revolving a cycloid about the $x$-axis is straightforwardly given by Equation (1.64). Here, we have

$$A = 2\sqrt{2}\pi R^2 \int_0^{2\pi} (1 - \cos(\theta))^{3/2}\, d\theta\ .$$

The remaining integral is tricky to evaluate but not impossible. Leaving the details for an exercise, we ultimately find

$$A = \left(2\sqrt{2}\pi R^2\right)\left(\frac{16\sqrt{2}}{3}\right) = \frac{64}{3}\pi R^2\ .$$

### Tautochrone

Consider a cycloid flipped upside-down, described by

$$\begin{aligned} x(\theta) &= R\theta - R\sin(\theta) \\ y(\theta) &= -R + R\cos(\theta)\ . \end{aligned}$$

Pretending we have constructed a ramp in such a shape, let us analyze the sliding (not rolling) motion of a body of mass $m$ placed at rest on the ramp.

In uniform gravity, the system respects an energy constant

$$E = \frac{1}{2}mv^2 + mgy\ ,$$

where $v$ is the velocity of the body in motion, $g$ is the local gravitational acceleration, and $y$ is the height above $y = 0$. Assuming the object begins at rest, we also have

$$E = mgy_0\ ,$$

where $y_0$ is the initial height of the body.

With this setup, it's useful to know the total time $T$ required for the body to slide to the bottom of the inverted cycloid. As an integral, we have, at least provisionally,

$$T = \int_{y_0}^{-2R} dt\ ,$$

and the job is recast the integral in variables we know.

Proceed by replacing $dt$ with something akin to arc length, namely

$$dS = v(t)\, dt\ .$$

Meanwhile, we know from geometry that

$$dS^2 = dx^2 + dy^2\ .$$

This is enough to wrestle the time integral into something manageable:

$$T = \int_{x_0}^{R\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{\sqrt{2g(y_0 - y)}}$$

We haven't used the equations of the cycloid yet, so proceed by using

$$\begin{aligned} y_0 - y &= R\left(\cos\left(\theta_0\right) - \cos\left(\theta\right)\right) \\ dx &= R\left(1 - \cos\left(\theta\right)\right) d\theta \\ dy &= -R \sin\left(\theta\right) d\theta \,, \end{aligned}$$

and the above simplifies to

$$T = \sqrt{\frac{R}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos\left(\theta\right)}\, d\theta}{\sqrt{\cos\left(\theta_0\right) - \cos\left(\theta\right)}} \,.$$

Note that $\theta_0$ corresponds to the initial position $(x_0, y_0)$, and $\theta = \pi$ occurs when the sliding body reaches the bottom of the curve.

Ugly as it is, the time integral can be solved after making a few substitutions that are left for an exercise to the reader to find. As a hint, you should first have

$$T = \sqrt{\frac{R}{g}} \int_{\theta_0}^{\pi} \frac{\sin\left(\theta/2\right) d\theta}{\sqrt{\cos^2\left(\theta_0/2\right) - \cos^2\left(\theta/2\right)}} \,,$$

and then

$$T = \sqrt{\frac{R}{g}} \int_{1}^{0} \frac{-2du}{\sqrt{1 - u^2}} \,.$$

Keep on solving with yet another $u$-substitution, and the final answer comes out to

$$T = \pi \sqrt{\frac{R}{g}} \,.$$

Remarkably, the final answer $T = \pi\sqrt{R/g}$ makes no mention of the initial position $(x_0, y_0)$ of the sliding body. This is to say that the time to slide to the bottom of a cycloid is always the same. No other known curve has this feature. The Ancient Greeks called this the *tautochrone*.

# 5 Series Analysis

Integration is a powerful addition to the toolkit for analyzing infinite sums, particularly on the issues of convergence and divergence.

## 5.1 Taylor Series

The most versatile series is surely the Taylor series, which tells that a function $f(x)$ at a point $x_0$ is approximated by a polynomial $p(x)$ involving derivatives $f^{(q)}(x_0)$:

$$p\left(x\right) = f\left(x_0\right) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}\left(x_0\right) \left(x - x_0\right)^j + R_n\left(x\right)$$

For large $n$ approaching infinity, the remainder term $R_n(x)$ vanishes if the series is to converge.

**Derivation**

To derive Taylor's theorem, begin with the fundamental theorem of calculus, i.e. Equation (1.2), and isolate $f(x)$:

$$f\left(x\right) = f\left(x_0\right) + \int_{x_0}^{x} f^{(1)}\left(t\right) dt$$

Of course, the function $f^{(1)}(t)$ could itself be approximated to first order using the fundamental theorem

$$f^{(1)}\left(t\right) = f^{(1)}\left(x_0\right) + \int_{x_0}^{t} f^{(2)}\left(u\right) du \,,$$

which begs substitution into the above, giving:

$$f\left(x\right) = f\left(x_0\right) + \int_{x_0}^{x} \left(f^{(1)}\left(x_0\right) + \int_{x_0}^{t} f^{(2)}\left(u\right) du\right) dt$$

After simplifying, we see the familiar first-order Taylor series term trailed by a messy integral:

$$\begin{aligned} f\left(x\right) = &\, f\left(x_0\right) + f^{(1)}\left(x_0\right)\left(x - x_0\right) \\ &+ \int_{x_0}^{x} \left(\int_{x_0}^{t} f^{(2)}\left(u\right) du\right) dt \end{aligned}$$

Trudging forward, take $f^{(2)}(u)$ to first order

$$f^{(2)}\left(u\right) = f^{(2)}\left(x_0\right) + \int_{x_0}^{w} f^{(3)}\left(w\right) dw \,,$$

and substitute into the preceding integral. This first means having to solve

$$I = \int_{x_0}^{x} \left(\int_{x_0}^{t} f^{(2)}\left(x_0\right) du\right) dt \,.$$

Knowing $f^{(2)}(x_0)$ is constant, proceed using brute force to find

$$\begin{aligned} I &= f^{(2)}\left(x_0\right) \int_{x_0}^{x} \left(\int_{x_0}^{t} du\right) dt \\ &= f^{(2)}\left(x_0\right) \int_{x_0}^{x} \left(t - x_0\right) dt \\ &= f^{(2)}\left(x_0\right) \left(\frac{t^2}{2} - x_0 t\right) \Bigg|_{x_0}^{x} \\ &= \frac{1}{2} f^{(2)}\left(x_0\right) \left(x - x_0\right)^2 \,. \end{aligned}$$

Interestingly, this is the second-order term in the Taylor series of $f(x)$. To summarize:

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &+ \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 \\ &+ \int_{x_0}^{x} \left( \int_{x_0}^{t} \left( \int_{x_0}^{u} f^{(3)}(w)\, dw \right) du \right) dt \end{aligned}$$

Repeating the steps that got us this far, use the first-order approximation of $f^{(3)}(w)$. The obligatory integral to solve is

$$J = f^{(3)}(x_0) \int_{x_0}^{x} \left( \int_{x_0}^{t} \left( \int_{x_0}^{u} dw \right) du \right) dt \,,$$

which after a bit of grinding, comes out to

$$J = \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 \;.$$

By now we're seeing a pattern, particularly:

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &+ \frac{1}{2} f^{(2)}(x_0)(x - x_0)^2 \\ &+ \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 \\ &+ R_3(x) \;, \end{aligned}$$

where $R_3(x)$ is given by

$$\int_{x_0}^{x} \left( \int_{x_0}^{t} \left( \int_{x_0}^{u} \left( \int_{x_0}^{v} f^{(4)}(v)\, dv \right) dw \right) du \right) dt \,.$$

**Remainder**

In the general case, the remainder term $R_n(x)$ always contains a polynomial term plus an integral. Since the integral part ends up being higher order than $n$, we can always push the hard work to the next step, so to speak, and take as the remainder term:

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(x_0)(x - x_0)^{n+1}$$

# 6 Mass Between Springs

Consider a point mass $m$ in the center of two springs pulled tight and mounted distance $L$ apart, ignoring gravity. The left spring has constant $k_a$, and the right has spring constant $k_b$, and both springs have rest length $L_0 < L/2$.

## 6.1 Rest Condition

When the system is not in motion, the mass will rest somewhere between the endpoints toward the stiffer spring, not necessarily at $x = L/2$. To work this out, balance all relevant forces in the $x$- and $y$-directions:

$$\begin{aligned} m\frac{d^2 y}{dt^2} &= F^y_{\text{net}} = 0 \\ m\frac{d^2 x}{dt^2} &= F^x_{\text{net}} = F_a + F_b \;, \end{aligned}$$

and each left size is zero for the rest condition. Each force $F_a$, $F_b$ obeys Hooke's law:

$$F_{\text{spring}} = -kx$$

Letting a constant $q$ denote the position of the mass away from $x = L/2$, the above tells us:

$$0 = -k_a \left( -L_0 + \frac{L}{2} - q \right) + k_b \left( -L_0 + \frac{L}{2} + q \right)$$

Solving for $q$ tells us where the system rests:

$$q = \left( \frac{k_a - k_b}{k_a + k_b} \right) \left( \frac{L}{2} - L_0 \right)$$

Looking at a few special cases, note first that $q$ vanishes if $k_a = k_b$, giving the symmetric result. Note also that if $L/2 = L_0$, the system is under no tension at all, and $q$ vanishes again. More curiously, if it happens that $L/2 < L_0$, this corresponds to the system being compressed rather than stretched, and the sign on $q$ flips. That is, the offset would be away from the stiffer spring. (This situation is unstable.)

## 6.2 Longitudinal Vibrations

If the mass-between-springs system is perturbed in a direction that is purely longitudinal, i.e. parallel to the springs, then resulting motion is confined to one dimension. To prepare for this, define two constants

$$\begin{aligned} x_a &= -L_0 + L/2 - q \\ x_b &= -L_0 + L/2 + q \;, \end{aligned}$$

so the rest condition is written

$$0 = -k_a x_a + k_b x_b \;.$$

For the non-rest case, use Newton's second law and Hooke's law combine to write

$$m\frac{d^2}{dt^2} x(t) = -k_a (x_a + x(t)) + k_b (x_b - x(t)) \;,$$

readily simplifying to

$$m\frac{d^2}{dt^2} x(t) = -x(t)(k_A + k_b)$$

This is a simple harmonic oscillator with effective angular frequency:

$$\omega = \sqrt{\frac{k_a + k_b}{m}}$$

## 6.3 Transverse Vibrations

Things get more interesting when we examine vibrations in the direction perpendicular to the springs. Taking the two spring constants as the same, i.e. $k_a = k_b = k$, an initial displacement of the mass in the $y$-direction results in one-dimensional motion.

In this case, we have $F^x_{\text{net}} = 0$ for the $x$-direction, and for the $y$-direction,

$$F^y_{\text{net}} = 2F_{\text{spring}} \sin(\theta) \ ,$$

where $\theta$ is the angle formed between a spring and the horizontal, and from geometry we pick out

$$\sin(\theta) = \frac{y}{\sqrt{(L/2)^2 + y^2}} \ .$$

The magnitude of the spring force is given by

$$F_{\text{spring}} = -k\left(\sqrt{\left(\frac{L}{2}\right)^2 + y^2} - L_0\right) \ ,$$

which, as long as $L/2 \neq L_0$, has a nonzero value for $y = 0$, affirming the springs are always under tension. All together, transverse vibrations are summarized by

$$F^y_{\text{net}} = m\frac{d^2}{dt^2}y(t) = -2ky\left(1 - \frac{L_0}{\sqrt{(L/2)^2 + y^2}}\right) \ .$$

### Small Vibrations

In the special case that the displacement $|y|$ is always much less than $L/2$, the above becomes

$$\begin{aligned} F^y_{\text{net}} &\approx -2ky\left(1 - \frac{2L_0}{L}\left(1 - \cancel{\frac{1}{2}\frac{4y^2}{L^2}}\right)\right) \\ &\approx -2ky\left(1 - \frac{2L_0}{L}\right) \ , \end{aligned}$$

where the square root has been eliminated by Taylor expansion.

Defining a new quantity

$$p = \frac{L}{2} - L_0 \ ,$$

the above simplifies to, of course, the equation of a harmonic oscillator

$$m\frac{d^2}{dt^2}y(t) \approx -\left(\frac{2k}{1 + L_0/p}\right) y(t) \ .$$

The angular frequency is given by

$$\omega = \sqrt{\frac{2k}{m}\left(\frac{1}{1 + L_0/p}\right)} \ ,$$

which is scaled by the tension in the springs. This is in fact a crude model for a plucked guitar string - the greater the tension, the greater the frequency of vibration.

## 6.4 Critical Vibrations

The problem becomes a different beast when we consider $L_0 = L/2$, meaning there is no resting tension in the system. Staying in the regime of transverse small oscillations, i.e. $|y| \ll L/2$, let us jot down a previous result without canceling the $y^2$-term:

$$F^y_{\text{net}} \approx -2ky\left(1 - \frac{2l_0}{L}\left(1 - \frac{1}{2}\frac{4y^2}{L^2}\right)\right)$$

Setting $2L_0 = L$, the above simplifies to

$$m\frac{d^2}{dt^2}y(t) \approx -k\left(\frac{2}{L}\right)^2 (y(t))^3 \ ,$$

which is classified as a nonlinear second-order differential equation.

### Energy Constraint

Despite the scary name, we can wrestle with the above equation anyway. Letting

$$\lambda = \frac{4k}{mL^2}$$

and using the 'dot' operator as a shorthand for the time derivative, we must solve

$$\ddot{y} = -\lambda\, y^3 \ .$$

Proceed by multiplying both sides by $\dot{y}$, and condense the left using the product rule:

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2\right) = \ddot{y}\dot{y} = -\lambda\,\frac{dy}{dt}y^3$$

Multiply $dt$ onto each side to attain a so-called 'differential form'

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}^2\right) dt = -\lambda\, y^3\, dy \ ,$$

which can be cleanly integrated with respect to $t$ on the left, $y$ on the right:

$$\frac{1}{2}\dot{y}^2 = -\frac{\lambda}{4}y^4 + C$$

This result looks very much like a conservation of energy statement. If we multiply through by a mass constant $m$, the left side is the kinetic energy then $Cm$ is the total energy $E$. The potential energy term is proportional to $y^4$, not $y^2$, which is not a simple harmonic oscillator potential.

### Initial Condition

One typical scenario for this system would have the mass released from rest at some initial value $A$ above $y = 0$. In this case, the above equation reads

$$0 = -\frac{\lambda}{4} A^4 + C$$

at $t = 0$, and the integration constant $C$ can be eliminated. Doing so, we get

$$\frac{1}{2}\dot{y}^2 = \frac{\lambda}{4}\left(A^4 - y^4\right) = \frac{\lambda A^4}{4}\left(1 - \left(\frac{y}{A}\right)^4\right) ,$$

or

$$\frac{dy}{dt} = \sqrt{\dot{y}^2} = \pm\sqrt{\lambda}\frac{A^2}{2}\sqrt{1 - \left(\frac{y}{A}\right)^4} ,$$

which can be separated with all $y$'s on one side, $t$'s on the other:

$$\frac{dy}{\sqrt{1 - (y/A)^4}} = \pm\left(\sqrt{\lambda}\frac{A^2}{2}\right) dt$$

Proceed with the substitution

$$\begin{aligned} y &= A\cos(\phi) \\ dy &= -A\sin(\phi)\, d\phi , \end{aligned}$$

and the above becomes

$$\frac{-A \cancel{\sin(\phi)}\, d\phi}{\sqrt{\cancel{\left(1 - \cos^2(\phi)\right)}\left(1 + \cos^2(\phi)\right)}} = \pm\left(\sqrt{\lambda}\frac{A^2}{2}\right) dt ,$$

allowing each side to be integrated:

$$\int \frac{d\phi}{\sqrt{1 + \cos^2(\phi)}} = \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\int dt$$

There are many choices for integration limits. One simple correspondence emerges by letting $\phi$ run from 0 to $\pi/2$, in which case the $t$-variable elapses a quarter-period:

$$\begin{aligned} \int_0^{\pi/4} \frac{d\phi}{\sqrt{1 + \cos^2(\phi)}} &= \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\int_0^{T/4} dt \\ &= \mp\left(\sqrt{\lambda}\frac{A}{2}\right)\frac{1}{4}T \end{aligned}$$

The integral on the left is strictly numerical, and the general form of the above doesn't change when different limits are chosen. Condensing constants and separating out period's relation to the amplitude gives a nifty result:

$$AT = \text{constant}$$