# Geometric Series
# MANUSCRIPT

William F. Barnes[1]

February 10, 2024

# Contents



# Chapter 1

# Geometric Series

## 1 Introduction

An important identity that arises from playing with polynomial division is

$$x^n - a^n = (x-a)\left(\sum_{k=1}^{n} a^{k-1}x^{n-k}\right) ,$$

which can generate many handy results by choosing the proper $a$ and proper $n$ that fit a given situation.

### 1.1 Geometric Series

The derivation of an equally-important identity can begin by considering the ratio

$$\frac{x^n-1}{x-1}$$

for various values of $n$, which can be studied by taking the $a=1$-case of the above. Expanding out the cases $n=1$, $n=2$, $n=3$, etc., we find

$$\begin{aligned}\frac{x^2-1}{x-1} &= 1+x \\ \frac{x^3-1}{x-1} &= 1+x+x^2 \\ \frac{x^4-1}{x-1} &= 1+x+x^2+x^3 ,\end{aligned}$$

which suggests for arbitrary $n$:

$$\frac{x^n-1}{x-1} = 1+x+x^2+\cdots+x^{n-1} = \sum_{k=1}^{n} x^{k-1}$$

Reshuffling to put all $n$-dependence on the right, we evidently find:

$$\frac{1}{1-x} = \sum_{k=1}^{n} x^{k-1} + \frac{x^n}{1-x} \tag{1.1}$$

#### Convergence

Note that the right side contains $x$ raised to steadily increasing exponents up to $x^n$. By letting $n$ become arbitrarily large, the sum blows up to infinity unless we restrict the absolute value of $x$ to be less than one. In such a case, the above converges to the *geometric series*,

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{k=0}^{\infty} x^k \tag{1.2}$$

provided $|x| < .1$

#### Example

A basketball is dropped from 10 feet and bounces up 6 feet. On each bounce, the ball recovers 3/5 of its previous height. Bouncing forever, what is the total distance traveled by the ball?

Step 1: Add up the total distance accumulated during each movement downward:

$$D_1 = 10 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)$$

Step 2: Add up the total distance accumulated during each movement upward:

$$D_2 = 6 \cdot \left(1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots\right)$$

Step 3: Compare each infinite sequence to the geometric series, and find:

$$1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \cdots = \frac{1}{1 - 3/5} = \frac{5}{2}$$

Step 4: Assemble the total distance moved in feet:

$$D_1 + D_2 = 10 \cdot \frac{5}{2} + 6 \cdot \frac{5}{2} = 40$$

# 2 Alternate Derivations

## 2.1 Long Division Method

The most brutal way to derive the geometric series is to perform polynomial division on the quantity $1/(1-x)$:

$$\begin{array}{r|lllll}
 & 1 & x & x^2 & x^3 & \\
\hline
1 - x & 1 & & & & \\
 & 1 & -x & & & \\
\hline
 & & x & & & \\
 & & x & -x^2 & & \\
\hline
 & & & x^2 & & \\
 & & & x^2 & -x^3 & \\
\hline
 & & & & x^3 & \\
 & & & & x^3 & -x^4 \\
\hline
 & & & & & x^4
\end{array}$$

After a few terms in, it's clear that such an exercise leads to Equation (1.1) again.

## 2.2 The G-Shortcut

If all you remember is the infinite version of the geometric series but not the finite one, let

$$G = 1 + x + x^2 + x^3 + \cdots + x^n ,$$

and multiply through by $x$:

$$xG = x + x^2 + x^3 + \cdots + x^{n+1}$$

Next, take the difference $G - xG$ and divide out $(1 - x)$ from the left to get

$$G = \frac{1 \cancel{- x + x} \cancel{- x^2 + x^2} + \cdots - x^{n+1}}{1 - x} ,$$

and simplify to recover Equation (1.1):

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{x+1}}{1 - x}$$

## 2.3 Number Line Method

Consider the real numbers within the domain $[1 : 2]$, and divide the interval between 1 and 2 into $n$ equally-sized bins. From the left, we can locate the upper boundary of each bin:

$$\begin{aligned}
\text{first bin: } & 1 + 1/n \\
\text{second bin: } & 1 + 2/n \\
n - 2\text{th bin: } & 1 + (n-2)/n \\
n - 1\text{th bin: } & 1 + (n-1)/n
\end{aligned}$$

Note that the same locations can be listed from the right:

$$\begin{aligned}
\text{first bin: } & 2 - (n-1)/n \\
\text{second bin: } & 2 - (n-2)/n \\
n - 2\text{th bin: } & 2 - 2/n \\
n - 1\text{th bin: } & 2 - 1/n
\end{aligned}$$

From these, you can check that the bin representations are equivalent.

To go further, consider the real numbers within the domain $[1 + 1/n : 1 + 2/n]$, which are the two boundaries of the second bin. Divide this interval into $n$ equal 'new' bins. From the left, we can locate the upper boundary of the first new bin as

$$\text{first bin: } 1 + \frac{1 + 1/n}{n} = 1 + \frac{1}{n} + \frac{1}{n^2} .$$

From the right, locate the upper boundary of the second new bin:

$$\text{second bin: } 2 - \frac{n-2}{n} - \frac{n-2}{n^2}$$

Supposing some value $z$ lies within the second new bin, it follows that

$$1 + \frac{1}{n} + \frac{1}{n^2} < z < 2 - \frac{n-2}{n} - \frac{n-2}{n^2} .$$

While the above is sufficient to continue, it's worthwhile to simplify the right side to get

$$1 + \frac{1}{n} + \frac{1}{n^2} < z < 1 + \frac{1}{n} + \frac{2}{n^2} .$$

There may be enough to spot a pattern. To be sure, consider the real numbers within the domain

$$\left[1+\frac{1}{n}+\frac{1}{n^2}:1+\frac{1}{n}+\frac{2}{n^2}\right] .$$

Divide this into $n$ bins and find the boundaries of the second 'new new' bin, which results in

$$L < z < R \, ,$$

where:

$$\begin{aligned} L &= 1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3} \\ R &= 2-\frac{n-2}{n}-\frac{n-2}{n^2}-\frac{n-2}{n^3} \\ &= 1+\frac{1}{n}+\frac{1}{n^2}+\frac{2}{n^3} \end{aligned}$$

Looking at the $L$ and $R$ terms above, it's clear what will happen if we execute $q$ iterations of this game, namely, each sum accumulates a new term with increasing powers of $n$ in the denominator up to $1/n^q$. The variable $z$ gets squeezed into a smaller interval.

Furthermore, note that $R$ can be rewritten in terms of $L$ via

$$R = 2-(n-2)(L-1) \, ,$$

which also means

$$L < z < 2-(n-2)(L-1) \, .$$

Since $L$ is less than the entire right side, we can skip over $z$ and go with

$$L < 2-(n-2)(L-1) \, ,$$

readily simplifying to

$$L < \frac{1}{1-1/n} \, .$$

Letting $1/n = x$, we finally get something very much like the geometric series:

$$1+x+x^2+x^3+\cdots+x^q < \frac{1}{1-x}$$

In the limit $q \to \infty$, this result is indistinguishable from Equation (1.2).

# 3 Manipulations

## 3.1 Squaring the Geometric Series

It's possible to multiply converging infinite sums together and get a new infinite sum. The simplest of these simply multiplies the geometric series into itself. Doing this carefully, we find

$$\begin{aligned} \left(\frac{1}{1-x}\right)^2 = 1+x+x^2+x^3+\cdots \\ +x+x^2+x^3+x^4+\cdots \\ +x^2+x^3+x^4+x^5+\cdots \, , \end{aligned}$$

simplifying to:

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4+\cdots \tag{1.3}$$

As an exercise in brute force algebra, higher powers can be handled as well:

$$\frac{1}{(1-x)^3} = 1+3x+6x^2+10x^3+15x^4+\cdots \tag{1.4}$$

$$\frac{1}{(1-x)^4} = 1+4x+10x^2+20x^3+35x^4+\cdots \tag{1.5}$$

**Relation to Pascal's Triangle**

Pausing a moment on the sequence of coefficients going with the above results, namely

$$\begin{aligned} &\{1,2,3,4,5,\dots\} \\ &\{1,3,6,10,15,\dots\} \\ &\{1,4,10,20,35,\dots\} \, , \end{aligned}$$

notice these sequences are already present in the (standard left-aligned) Pascal's triangle:

$$\begin{array}{lllllll} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

Choosing the $n$th column and reading down the triangle predicts the expansion of $1/(1-x)^n$.

## 3.2 Negative Argument

In the geometric series, setting $x \to -x$ has the effect of reversing the sign on all odd-powered terms while leaving even-powered terms the same. With this, we get a slew of results for free:

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\cdots \tag{1.6}$$

$$\frac{1}{(1+x)^2} = 1-2x+3x^2-4x^3+5x^4-\cdots \tag{1.7}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots \quad (1.8)$$

$$\frac{1}{(1+x)^4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \cdots \quad (1.9)$$

These can be predicted by the compliment Pascal triangle based on subtraction rather than addition:

$$\begin{array}{lllllll}
1 & & & & & & \\
1 & -1 & & & & & \\
1 & -2 & 1 & & & & \\
1 & -3 & 3 & -1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
1 & -5 & 10 & -10 & 5 & -1 & \\
1 & -6 & 15 & -20 & 15 & -6 & 1
\end{array}$$

Choosing the $n$th column and reading downward predicts the expansion of $1/(1+x)^n$.

### 3.3 Squared Argument

Consider the sum

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$$

for $|x| < 1$. Using Equations (1.3), (1.6) to expand the right side, we find, after simplifying:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots \quad (1.10)$$

Evidently, replacing $x \to x^2$ is as straightforward as it looks. All powers are doubled. In a similar way, we can try $x \to -x^2$ to establish

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots . \quad (1.11)$$

Note also that the sum of the above results essentially repeats the problem with all powers doubled again:

$$\frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + \cdots \quad (1.12)$$

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + \cdots \quad (1.13)$$

### 3.4 Pascal Transform

Introducing the shift $z = 1 - x$, the geometric series becomes

$$\frac{1}{z} = 1 + (1-z) + (1-z)^2 + (1-z)^3 + \cdots ,$$

which converges for $0 < z < 2$.

For a seemingly roundabout exercise, note that each term on the right is a polynomial $(1-z)^n$ with all integer powers of $n$ present. If each of these is expanded out, the right side of the equation ends up containing the entirety of the Pascal triangle based on the quantity $(1-z)$.

However, Equations (1.6) - (1.9) already claim the columns of the complement Pascal triangle. Using this information to replace the right side entirely gives a curious representation of $1/z$:

$$\frac{1}{z} = \frac{1}{1+z} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)^3} + \cdots \quad (1.14)$$

In light of this shifty move, the above no longer converges for $0 < z < 2$ as before the so-called Pascal transform. Proceed by letting $y = 1/(1+z)$, and the above becomes

$$\frac{1}{1/y - 1} = y + y^2 + y^3 + \cdots ,$$

and add 1 to both sides:

$$\frac{1/y}{1/y - 1} = \frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

Evidently, Equation (1.9) still embeds the geometric series in the $y$-variable, which converges for all $|y| < 1$. Since $z$ is dependent on $y$, the restriction on $z$ is therefore:

$$\left|\frac{1}{1+z}\right| < 1$$

In other words, all $z > 0$ lead to convergence, and so do all $z < -2$.

This allows for some interesting relationships between the real numbers, particularly neighboring fractions, for instance:

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

## 4 Repeating Decimals

The geometric series helps make sense of decimal numbers whose digits eventually repeat. Consider a number of the format

$$N = 0.abcd \ldots qabcd \ldots q ,$$

where the sequence $abcd \ldots q$ is $Q$ digits in length. As a sum, $N$ can be written

$$N = \left(\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots + \frac{q}{10^Q}\right) \times \left(1 + \frac{1}{10^Q} + \frac{1}{10^{2Q}} + \frac{1}{10^{3Q}} + \cdots\right) ,$$

which has been factored into a product of two terms.

For the first term we can define the shorthand

$$N' = \frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots + \frac{q}{10^Q}$$

as the truncation of $N$ before the sequence repeats. The second term is precisely a geometric series:

$$1 + \frac{1}{10^Q} + \frac{1}{10^{2Q}} + \frac{1}{10^{3Q}} + \cdots = \frac{1}{1 - 10^{-Q}}$$

Reconstituting $N$ from these items, we have something that allows repeating decimals to be written in closed form:

$$N = \frac{N'}{1 - 10^{-Q}}$$

To have an example, the decimal $0.12312323\ldots$ has $N' = 0.123$ with $Q = 3$:

$$0.123123123\cdots = \frac{0.123}{1 - 10^{-3}} = \frac{123}{1000 - 1} = \frac{123}{999}$$

For another example, the special case $N' = 0.9$ with $Q = 1$ tells us

$$0.999\cdots = \frac{0.9}{1 - 10^{-1}} = \frac{9}{10 - 1} = \frac{9}{9} = 1\ ,$$

which means $0.999\ldots$ repeating forever is indistinguishable from 1:

$$0.999\cdots = 1$$

# 5 Zeno's Paradox

## 5.1 The Paradox

An ancient 'paradox' originating in Greece began with Zeno of Elia, as recalled by Aristotle:

*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*

This sounds fine, but then the ancient Greeks take the argument off the rails:

*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.*

According to Zeno, to reach a destination, an object must go half-way first, but to reach the half-way point, it has to reach the quarter-way point, and so on. The object in turn may never reach its destination, and even worse, it's not clear where the object gets stuck, or if the motion ever starts at all.

## 5.2 Linear Motion

Had the Greeks known about the geometric series, particularly the notion of convergence, then maybe there would have been no paradox, as we can make easy work of the situation.

Consider the one-dimensional motion of any object with constant velocity $V$ that takes time $T$ to move distance $X$, or

$$X = VT\ .$$

### Spatial Sum

To pose the problem as the Greeks may have, suppose that the interval $X$ were divided into sections of decreasing size starting with $X/2$, and then $X/4$, $X/8$, $X/16$, etc. Zeno claims that their sum can't tally to $X$, but let us check:

$$\begin{aligned} \text{Spatial sum} &= \frac{X}{2} + \frac{X}{4} + \frac{X}{8} + \frac{X}{16} + \cdots \\ &= \frac{X}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) \end{aligned}$$

The parenthesized sum is a geometric series equivalent to $1/\left(1 - 1/2\right) = 2$, leaving

$$\text{Spatial sum} = X\ ,$$

simple as that.

### Temporal Sum

The paradox can be resolved in the time variable as well. For this, relate the distances $X/2$, $X/4$, $X/8$, etc. to the time required to traverse each:

$$\begin{aligned} X/2 &= Vt_1 \\ X/4 &= Vt_2 \\ X/8 &= Vt_3 \\ X/2^j &= Vt_j \end{aligned}$$

Then, the total time is

$$\text{Temporal sum} = \sum_{j=1}^{\infty} t_j = \frac{X}{V} \sum_{j=1}^{\infty} \left(\frac{1}{2^j}\right)\ ,$$

and the remaining sum is the same as above and resolves to one. In conclusion we find

$$\text{Temporal sum} = \frac{X}{V} = T$$

and no evidence of a paradox.

## 6 Infinite Sum Analysis

The geometric series can help make sense of infinite sums that, at face value, don't appear to be penetrable. To demonstrate, consider the infinite sum

$$A = \sum_{k=0}^{\infty} \frac{k}{2^k}$$

The first term in the sum is identically zero, to it's harmless to start the index at one instead of zero:

$$A = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

Next let $n = k - 1$ and the sum becomes

$$A = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{n}{2^n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} .$$

On the right, the first sum is simply $A$ again. The second sum is a geometric series that resolves to one. Thus

$$A = \frac{1}{2} A + 1 ,$$

which can only mean

$$A = 2 .$$

The same trick works on harder sums. For instance, suppose

$$B = \sum_{k=0}^{\infty} \frac{k^2}{2^k} .$$

Observing that the first term in the sum is zero, and using the same substitution $n = k - 1$ leads to

$$B = \frac{1}{2} \sum_{n=0}^{\infty} \frac{n^2 + 2n + 1}{2^n} = \frac{B}{2} + A + 1 ,$$

which is only solved by

$$B = 6 .$$