Factoring
MANUSCRIPT

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Chapter 1

Factoring

1 Factoring Quadratics

Factoring quadratic expressions, which is the reverse job of polynomial multiplication, is among the most tortured exercises in mathematics education. It is often the student's first encounter with something presented as not *purely* formulaic, which is to mean factoring can still go wrong without making a technical mistake. It's the student's first real brush with the algebraic abyss.

Un-Foiling an Equation

Statements such as

$$(2x + 3)(4x - 1) = 8x^2 + 10x - 3$$

are easily understood reading left-to-right.

On the left are *four* terms grouped by signs and parentheses, but on the right, after carrying out a swift FOIL operation, are *three* terms. Despite the algebraic balance of the equation above, there's still a sense that information is lost in going left to right. For this reason, it's not wholly obvious how to un-distribute a polynomial into separate products, i.e. build the equation right-to-left.

There are at least half a dozen 'common' methods for factoring quadratic expressions, some more useful than others in certain regimes. Following is a brief survey of some of these.

1.1 Box Method

The most grotesque means of factoring is surely the *box method*, usually confined to pre-college pedagogy.

To factor the quadratic expression

$$2x^2 - x - 6 \,,$$

first draw a box, and then put the first and third terms along the diagonal as shown:

$2x^2$	
	-6

Off to the side, multiply the first and third coefficients to get -12. List every factorization of this number:

$$-12 = \begin{cases} \pm 1 \times \mp 12 \\ \pm 2 \times \mp 6 \\ \pm 3 \times \mp 4 \end{cases}$$

The pair of factors that sums to the expression's *middle* coefficient, namely -1, is the pair we need. The box is then updated:

$2x^2$	$-4x$
$3x$	-6

Finally, look across each row and each column of the box. In the margin of each, write the greatest common factor occurring in that row or column:

x	-2	
$2x^2$	$-4x$	$2x$
$3x$	-6	3

The final result is read from the margins:

$$2x^2 - x - 6 = (x-2)(2x+3)$$

The box method is popular for numerous reasons, one being that it's easy for an instructor to prepare factoring worksheets. Correctness can be judged based on whether the student writes the correct term in the correct box. Easy to fill in, easy to grade.

On the other hand, one immediately sees the departure from mathematical thinking inherent to the box method. Once the box is drawn, the student is asked to set aside the tools of algebraic procedure, giving way to something akin to spreadsheet heuristics.

Problem 1

Use the box method to show:

$$6x^2 + 11x + 4 = (3x+4)(2x+1)$$

Hint:

$3x$	4	
$6x^2$	$8x$	$2x$
$3x$	4	1

Problem 2

Use the box method to show:

$$3x^2 - 2x - 8 = (x-2)(3x+4)$$

Hint:

x	-2	
$3x^2$	$-6x$	$3x$
$4x$	-8	4

1.2 Split-Term Method

A relatively quick means of factoring a quadratic expression involves splitting the middle term.

For the trinomial

$$2x^2 + 7x + 3 ,$$

first record the product of the two outer coefficients, namely $(2)(3) = 6$. The task becomes splitting the middle coefficient (7) into two numbers whose product is 6, which is straightforwardly done:

$$2x^2 + 7x + 3 = 2x^2 + \underbrace{6x + x}_{7x} + 3$$

We then proceed to write

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x(x+3) + (x+3) \\ &= (2x+1)(x+3) \ , \end{aligned}$$

and the problem is solved after some regrouping of terms.

The split-term method works well when it's easy to divine *how* the middle term should be split. Like most common methods, this can depend on a bit of luck, and works best when the coefficients involved are tame integers.

1.3 Completing the Square

Completing the square is a robust tool for factoring quadratic expressions.

Supposing we're given

$$x^2 + 6x + 2 ,$$

the first move is to look at the middle coefficient, namely 6, and divide that by two. This number becomes the term inside a 'square' factor as follows:

$$x^2 + 6x + 2 = (x+3)^2 - 3^2 + 2$$

Note that a constant -3^2 is introduced to keep algebraic balance.

To proceed, seek solutions to

$$(x+3)^2 - 3^2 + 2 = 0 ,$$

resulting in

$$\begin{aligned} x_1 &= -3 + \sqrt{7} \\ x_2 &= -3 - \sqrt{7} , \end{aligned}$$

so the final answer is written

$$x^2 + 6x + 2 = \left(x + 3 - \sqrt{7}\right)\left(x + 3 + \sqrt{7}\right) .$$

1.4 Algebraic Identities

The limit case of a lucky factoring problem is one that maps perfectly onto a known algebraic identity. Committing identities to memory such as

$$\begin{aligned} a^2 + 2ab + b^2 &= (a+b)^2 \\ a^2 - 2ab + b^2 &= (a-b)^2 \\ a^2 - b^2 &= (a+b)(a-b) \ , \end{aligned}$$

quick work can be made of problems that fit them, i.e.:

$$\begin{aligned} x^2 + 10x + 25 &= (x+5)^2 \\ x^2 - 6x + 9 &= (x-3)^2 \\ 9x^2 - 16 &= (3x+4)(3x-4) \end{aligned}$$

Normalizing a Quadratic

Factoring a quadratic expression is undoubtedly simpler when the leading coefficient A doesn't complicate things, epitomized by $A = 1$. When this isn't the case, one can always factor A from the whole expression

$$Ax^2 + Bx + C = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right) ,$$

and focus on the parenthesized quantity, proceeding as if there is no leading coefficient. Of course, the coefficients B and C are then modified by A, which can make the problem much less penetrable.

Unit Leading Coefficient

In the special case that $A = 1$, we have

$$x^2 + Bx + C = (x - x_1)(x - x_2) ,$$

implying

$$x_1 + x_2 = -B$$
$$x_1 \cdot x_2 = C ,$$

and furthermore

$$x_1 = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right)$$
$$x_2 = \frac{1}{2}\left(-B - \sqrt{B^2 - 4C}\right) .$$

Unit First and Third

In the special case that the first and third coefficients are equal to one, we have

$$x^2 + Bx + 1 = (x - x_1)(x - x_2) ,$$

implying

$$x_1 + x_2 = -B$$
$$x_1 \cdot x_2 = 1 .$$

With $x_1 \cdot x_2 = 1$ established, we can define a variable

$$z = \frac{1}{x}$$

and seek solutions to a modified expression

$$1 + \frac{B}{z} + z^2 = 0 ,$$

equivalent to

$$z^2 + Bz + 1 = 0 ,$$

exactly what we started with, up to a change of variable. Solving one equation for x solves the modified equation (should you encounter one) for z, and vice versa.

1.5 Quadratic Formula

One of the surest ways to factor a quadratic equation is using the quadratic formula. For this, we have

$$Ax^2 + Bx + C = A\left(x - x_{\text{int}_+}\right)\left(x - \text{int}_-\right) ,$$

where $x_{\text{int}_\pm}$ is given by:

$$x_{\text{int}_\pm} = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

Problem 3

Derive the quadratic formula by completing the square on:

$$Ax^2 + Bx + C = 0$$

Hint:

$$\left(\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2 = \frac{B^2}{4A} - C$$

1.6 Method of Transform

Now we develop a technique for factoring quadratic expressions that is not in the common teachings.

Starting with a general quadratic expression

$$Ax^2 + Bx + C ,$$

split the middle term B into the sum of two unknown variables f_1 and f_2:

$$Ax^2 + Bx + C = Ax^2 + f_1 x + f_2 x + C \tag{1.1}$$

We may recast the right side in factored form by (for lack of a more precise term) 'completing the rectangle'

$$Ax^2 + Bx + C = \frac{1}{C}(f_1 x + C)(f_2 x + C) \tag{1.2}$$

or equivalently:

$$Ax^2 + Bx + C = \frac{1}{A}(Ax + f_1)(Ax + f_2) \tag{1.3}$$

Transform Kernel

For the above forms to remain consistent, the restriction on $f_{1,2}$ emerges:

$$f_1 + f_2 = B$$
$$f_1 \cdot f_2 = AC$$

This is the quantification of the factoring mantra: we must find two numbers whose sum is B and whose product is AC. That is, we've shifted the burden to finding the two terms f_1, f_2.

As an aside, it's straightforwardly shown that $f_{1,2}$ relate to the discriminant by

$$\mathcal{D} = (f_1 - f_2)^2 .$$

Like the non-transformed case, the two solutions $f_{1,2}$ are the same when the quantity $B^2 - 4AC$ is zero.

Transformed Quadratic

As a system of two equations and two unknowns, the restriction on $f_{1,2}$ combine to a single equation which is itself quadratic:

$$f^2 - Bf + AC = 0 \tag{1.4}$$

Interestingly, this exposes a deeper quality of factoring problems, in that every problem represented by Equations (1.1) - (1.3) embeds the simpler problem represented by Equation (1.4).

Unit Leading Coefficient

In the special case that $A = 1$, the method of transform returns something *almost* tautological (i.e. useless) in the sense that

$$x^2 + Bx + C$$

implies

$$f^2 - Bf + C = 0 \,.$$

The left-hand expressions are the same up to a minus sign on the active variable, so we identify

$$f = -x \,.$$

That is, if we find solutions to f, their negations are the solutions to x.

Factoring by Transform

To summarize, following is a heuristic for factoring quadratics by method of transform.

- Given

  $$Ax^2 + Bx + C \,,$$

  determine $-B$ and the product AC.
- Find two solutions $f_{1,2}$ to

  $$f^2 - Bf + AC = 0 \,.$$

- Complete the rectangle using

  $$Ax^2 + Bx + C = Ax^2 + f_1 x + f_2 x + C$$

  or similar, and simplify.

Problem 4
Factor:

$$6x^2 + 11x + 4$$

Answer: Identify $B = 11$ and $AC = 24$ to write

$$f^2 - 11f + 24 = 0 \,,$$

from which we discern

$$(f - 8)(f - 3) = 0 \,.$$

With two solutions for f, proceed using the split-term method:

$$\begin{aligned} 6x^2 + 11x + 4 &= 6x^2 + 8x + 3x + 4 \\ &= 2x(3x + 4) + (3x + 4) \\ &= (2x + 1)(3x + 4) \end{aligned}$$

Problem 5
Factor:

$$3x^2 - 2x - 8$$

Answer: Identify $B = -2$ and $AC = -24$ to write

$$f^2 + 2f - 24 = 0 \,,$$

from which we discern

$$(f + 6)(f - 4) = 0 \,.$$

With two solutions for f, proceed using the split-term method:

$$\begin{aligned} 3x^2 - 2x - 8 &= 3x^2 - 6x + 4x - 8 \\ &= 3x(x - 2) + 4(x - 2) \\ &= (3x + 4)(x - 2) \end{aligned}$$

Problem 6
Factor:

$$15x^2 + 14x - 8$$

Answer: Identify $B = 14$ and $AC = -120$ to write

$$f^2 - 14f - 120 = 0 \,,$$

from which we discern

$$(f + 6)(f - 20) = 0 \,.$$

With two solutions for f, proceed by completing the rectangle:

$$\begin{aligned} 15x^2 + 14x - 8 &= \frac{-1}{8}(-6x - 8)(20x - 8) \\ &= (5x - 2)(3x + 4) \end{aligned}$$

Problem 7
Factor:

$$36x^2 - 121y^2$$

Answer: Identify $B = 0$ and $AC = -36 \cdot 121y^2$ to write

$$f^2 = 36 \cdot 121y^2 \,,$$

from which we discern

$$f = \pm 66y \,.$$

With two solutions for f, proceed by completing the rectangle:

$$\begin{aligned} 36x^2 - 121y^2 &= \frac{1}{A}\left(Ax + 66y\right)\left(Ax - 66y\right) \\ &= \left(6x + 11y\right)\left(6x - 11y\right) \end{aligned}$$

<u>Problem 8</u>

Factor:

$$x^2 + 3x - 28$$

Answer: Identify $B = 3$ and $AC = -28$ to write

$$f^2 - 3f - 28 = 0 \,,$$

reminiscent of the original problem, up the the variable change $f = -x$. Proceed by standard means to find $f_1 = 4$, $f_2 = -7$. The solutions in x are the negatives of these, so we finally have

$$x^2 + 3x - 28 = \left(x - 4\right)\left(x + 7\right) \ .$$

Use any factoring method to prove the following:

<u>Problem 9</u>

$$4x^2 + 15x + 9 = \left(4x + 3\right)\left(x + 3\right)$$

<u>Problem 10</u>

$$6x^2 + 23x - 4 = \left(6x - 1\right)\left(x + 4\right)$$

<u>Problem 11</u>

$$10x^2 + x - 3 = \left(5x + 3\right)\left(2x - 1\right)$$

<u>Problem 12</u>

$$15x^2 - 7x - 4 = \left(3x + 1\right)\left(5x - 4\right)$$

<u>Problem 13</u>

$$4x^2 + 2x - 12 = 2\left(2x - 3\right)\left(x - 2\right)$$

<u>Problem 14</u>

$$8x^2 - 2xy - 3y^2 = \left(2x + y\right)\left(4x - 3y\right)$$

<u>Problem 15</u>

$$\left(x + 4\right)^3 - 9x - 36 = \left(x + 4\right)\left(x + 1\right)\left(x + 7\right)$$

<u>Problem 16</u>

$$x^2 + 2x + 1 - y^2 = \left(x + 1 + y\right)\left(x + 1 - y\right)$$

<u>Problem 17</u>

$$x^2 - y^2 + 12y - 36 = \left(x + y - 6\right)\left(x - y + 6\right)$$

<u>Problem 18</u>

$$3x^3 + 5x^2y + xy^2 - y^3 = \left(3x - y\right)\left(x + y\right)^2$$

<u>Problem 19</u>

Find the GCF and LCM of:

$$\begin{aligned} &x^2y - xy^2 \\ &3x - 3y \\ &x^2 - 2xy + y^2 \end{aligned}$$

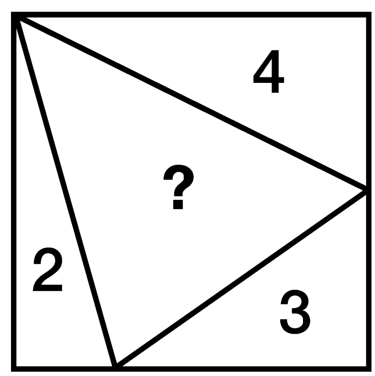
Answer:

$$\begin{aligned} x^2 - xy^2 &= xy\left(x - y\right) \\ 3x - 2y &= 3\left(x - y\right) \\ x^2 - 2xy + y^2 &= \left(x - y\right)^2 \end{aligned}$$

$$\begin{aligned} \text{GCF} &= \left(x - y\right) \\ \text{LCM} &= 3xy\left(x - y\right)^2 \end{aligned}$$

<u>Problem 20</u>

Determine the unknown area contained in the square as shown.



Answer: Starting on the left side and reading counter-clockwise, the four perimeter segments are (A), $(B + C)$, $(D + E)$, (A). Note also that

$$\begin{aligned} AB &= 4 \\ CD &= 6 \\ AE &= 8 \,. \end{aligned}$$

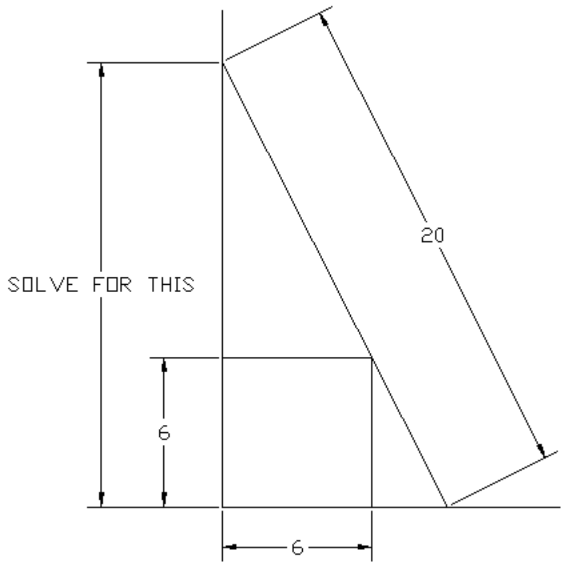
Use $\left(A - B\right)\left(A - E\right) = CD$ to arrive at

$$A^2 + \frac{32}{A^2} = 18 \,,$$

which resolves to $A = 4$, making the unknown area equal to 7.

Problem 21
Source: stirlingsouth.com

A 20-foot ladder leans on a perpendicular wall such that it touches the edge of a $6 \times 6 \times 6$-foot cube flatly pushed against the wall, as seen in the figure. Find the vertical height of the ladder above the cube.



Answer: Denote x as the unknown vertical component of the ladder's projection, and y as the unknown horizontal component. By area arguments, or by considering similar triangles, observe that $xy = 36$. Further, the Pythagorean theorem dictates

$$(x+6)^2 + (y+6)^2 = 20^2 ,$$

where completing the square in the variable $x + y$ eventually gives

$$x + y = -6 + 2\sqrt{109} ,$$

resolving to $x \approx 11.840$, $y \approx 3.0405$.

Problem 22

Consider the system of equations

$$\begin{aligned} 2x^2 + 3y &= 19 \\ 4x - y &= 3 . \end{aligned}$$

Eliminate y, x separately to derive:

$$\begin{aligned} x^2 + 6x &= 14 \\ y^2 + 30y &= 143 \end{aligned}$$

Solve each of these for x, y, respectively. Find two ordered-pair solutions to the system. Answer: $(1.7953, 4.1833)$, $(-7.7958, -34.183)$

2 Factoring Cubics

A *cubic expression* is a *third-order* polynomial:

$$Ax^3 + Bx^2 + Cx + D$$

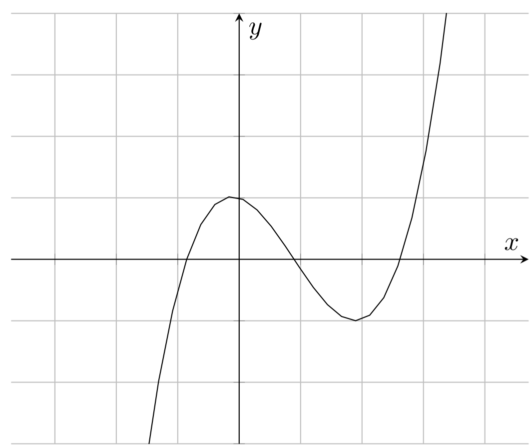
As a generalized quadratic expression, one might wonder how a cubic expression could be factored, and whether there exists an order-three analog to the quadratic formula. As it turns out, this answer isn't so trivial to come by. Unlike quadratic expressions, whose mysteries were shooed away millennia ago, it took until the sixteenth century to crack the problem of cubics.

2.1 Cubic Equations

To approach this problem we first construct a *cubic equation*

$$y = Ax^3 + Bx^2 + Cx + D ,$$

qualitatively appearing as follows:



In the above, the coefficients are chosen such that

$$\begin{aligned} A &= 1/2 \\ B &= -4/3 \\ C &= -1/3 \\ D &= 1 \end{aligned}$$

The leading coefficient A determines the overall steepness of the curve. For $A > 0$, the curve grows upward for increasing x, while dipping (very) negative for decreasing x. Far from the origin, this is true regardless of B, C, D, as the x^3-term grows much faster than the lower-order terms. For $A < 0$, similar comments apply.

The coefficient D plays the role of the y-intercept, controlling the vertical placement of the plot on the

Cartesian plane. The coefficients B and C are responsible for the overall structure near $x = 0$. In the general case, a cubic equation flaunts two vertex points and up to three x-intercepts. The number of x-intercepts can vary depending on the values of B, C, D, but there is always at least one.

Factoring Cubic Equations

We now develop a method to factor any cubic expression, which amounts to looking for x-intercepts in the cubic equation

$$y = Ax^3 + Bx^2 + Cx + D .$$

2.2 Depressed Cubic

To begin, we employ a trick to do away with the x^2-term by making the curious substitution

$$x = z - \frac{B}{3A} .$$

Letting the algebra carry forth, we find

$$\begin{aligned} 0 = z^3 + z\left(-\frac{B^2}{3A^2} + \frac{C}{A}\right) \\ + 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A} , \end{aligned}$$

which is a bit ugly, but conveniently omits a z^2-term. Proceed by setting

$$\begin{aligned} b &= -\frac{B^2}{3A^2} + \frac{C}{A} \\ -c &= 2\left(\frac{B}{3A}\right)^3 - \frac{C}{A}\left(\frac{B}{3A}\right) + \frac{D}{A} \end{aligned}$$

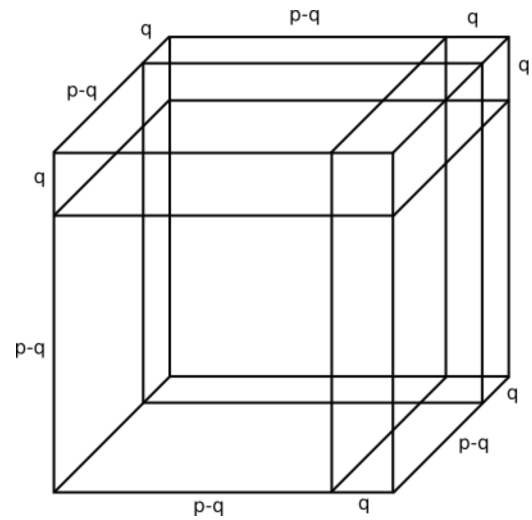
to arrive at the equation of the *depressed cubic*:

$$z^3 + bz = c$$

Henceforth we'll work in the variable z, keeping in mind that we get back to x using the initial subsitution.

Geometric Form of the Cubic

Using geometry as an aid to solving cubic equations is a trick attributed to Gerolamo Cardano in 1545. Tracing Cardano's steps, begin with a cube of side p, and then introduce three planes inside the cube, parallel to the top, right, and back faces. Set each plane length q from the cube's respective sides as shown.



The total volume inside the total cube consists of the 'primary' cube of side $p - q$, three slabs of volume $q\,(p-q)^2$, three bars of volume $q^2\,(p-q)$, and a small cube of side q. Meanwhile, the total volume is simply p^3. Equating the total volumes, we write

$$p^3 = (p-q)^3 + 3q^2\,(p-q) + 3\,(p-q)^2\,q + q^3 ,$$

readily simplifying to

$$(p-q)^3 + 3pq\,(p-q) = p^3 - q^3 .$$

Looking closely, note the above is a depressed cubic equation. Identifying

$$\begin{aligned} p - q &= z \\ 3pq &= b \\ p^3 - q^3 &= c , \end{aligned}$$

we recover the form $z^3 + bz = c$.

Ferro-Tartaglia Formula

Picking up from Cardano's geometric formulation of the depressed cubic problem

$$z^3 + bz = c ,$$

eliminate q between the latter two equations to get

$$p^6 - cp^3 - \left(\frac{b}{3}\right)^3 = 0 ,$$

which is in fact a quadratic equation in the variable p^3, easily isolated by the quadratic formula:

$$p^3 = \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$$

This automatically gives us q^3, specifically

$$q^3 = p^3 - c = -\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} ,$$

allowing a solution for z to be written:

$$z_0 = p - q$$
$$z_0 = \left(\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}\right)^{1/3} + \left(\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}\right)^{1/3}$$

The above is associated with Italian mathematician N. F. Tartaglia (1500-1557), although its discovery is credited to another Italian mathematician S. del Ferro (1465-1526).

In conclusion, we have the Ferro-Tartaglia formula as

$$z_0 = \left(\frac{c}{2} + q\right)^{1/3} + \left(\frac{c}{2} - q\right)^{1/3} ,$$

where

$$q = \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} .$$

Problem 23

Find one solution to:

$$z^3 - \frac{z}{3} - \frac{2}{27} = 0$$

Step 1: Identify the above as a depressed cubic equation and pick out coefficients:

$$3pq = b = -\frac{1}{3}$$
$$p^3 - q^3 = c = \frac{2}{27}$$

Step 2: Solve for p^3, and write p and q:

$$p^3 = \frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$$
$$p^3 = \frac{1}{27} + \sqrt{\left(\frac{1}{27}\right)^2 - \left(\frac{1}{9}\right)^3} = \frac{1}{27}$$
$$p = \frac{1}{3}$$
$$q = -\frac{1}{3}$$

Step 3: Write the solution for z in terms of p and q:

$$z = p - q = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

2.3 Completing the Cubic Solution

In the general case, the geometric approach to solving a depressed cubic equation

$$z^3 + bz = c$$

produces one solution, however there should exist (up to) three total solutions to the equation. Labeling the known solution as z_0, it follows that $(z - z_0)$ can be factored out of the depressed cubic equation to get

$$z^3 + bz - c = (z - z_0)\left(z^2 + z_0 z + \frac{c}{z_0}\right) .$$

Remaining solutions to the depressed cubic equation are given by

$$0 = z^2 + z_0 z + \frac{c}{z_0} ,$$

an easy application of the quadratic formula.

Problem 24

Use the known solution $z_0 = 2/3$ to continue factoring the expression:

$$z^3 - \frac{z}{3} - \frac{2}{27}$$

Step 1: Substitute $x_0 = 2/3$ and $c = -2/27$ into the quadratic equation for z and simplify:

$$z = \frac{2/3}{2}\left(-1 \pm \sqrt{1 - \frac{4 \cdot 2/27}{(2/3)^3}}\right) = -\frac{1}{3}$$

Step 2: Pick out any new solution(s) gained. In this case, we have two copies of the same number:

$$z_1 = z_2 = -\frac{1}{3}$$

Step 3: Write the final form:

$$z^3 - \frac{x}{3} - \frac{2}{27} = \left(z - \frac{2}{3}\right)\left(z + \frac{1}{3}\right)^2$$

Depressed Cubic in Disguise

Consider the rather exotic quantity

$$\left(7 + \sqrt{50}\right)^{1/3} + \left(7 - \sqrt{50}\right)^{1/3} ,$$

which might seem impossible to evaluate, namely because $7 - \sqrt{50}$ is surely negative, hinting of complex numbers. Proceeding with caution, let us store the whole quantity in a variable x, and then calculate x^3:

$$\begin{aligned} x^3 &= \left(\left(7+\sqrt{50}\right)^{1/3} + \left(7-\sqrt{50}\right)^{1/3}\right)^3 \\ &= 14 + 3\left(\left(7+\sqrt{50}\right)^{1/3}\left(7-\sqrt{50}\right)^{1/3}\right) \\ &\quad \cdot \left(\left(7+\sqrt{50}\right)^{1/3} + \left(7-\sqrt{50}\right)^{1/3}\right) \\ &= 14 + 3\left(-1\right)^{1/3} x \end{aligned}$$

Evidently then, we find

$$x^3 = -3x + 14 \,,$$

the equation of a depressed cubic. By standard means we find the solution to be $x = 2$, which finally means:

$$\left(7+\sqrt{50}\right)^{1/3} + \left(7-\sqrt{50}\right)^{1/3} = 2$$

2.4 Negative Radicals

It will turn out that the Ferro-Tartaglia formula isn't so straightforwardly applied in all cases, especially from the point of view of a 1500s mathematician. To demonstrate, let us factor the expression:

$$x^3 + 5x^2 - 2x - 24$$

The first target is the depressed cubic form. Start with the transformation $x = z - B/3A$ and simplify to find

$$\begin{aligned} b &= -\frac{31}{3} \\ c &= \frac{308}{27} \,, \end{aligned}$$

and the corresponding depressed cubic equation reads

$$z^3 + z\left(\frac{-31}{3}\right) = \frac{308}{27} \,.$$

With b and c, a solution z_0 to the depressed cubic is generated from the Ferro-Tartaglia formula. Applying this, we inevitably hit

$$q = \sqrt{\frac{c^2}{4} + \frac{b^3}{27}} = \sqrt{-\frac{25}{3}} \,.$$

Alarmingly, there is a minus sign inside the square root term.

Bombelli's Wild Thought

The courage to work with an equation that contains a negative term inside a square root came first to Rafael Bombelli, now known as his *wild thought.* To replicate Bombelli's wild thought, define a pair of numbers U, V where

$$\begin{aligned} \left(\frac{c}{2} + q\right)^{1/3} &= U + \sqrt{-1}V \\ \left(\frac{c}{2} - q\right)^{1/3} &= U - \sqrt{-1}V \,, \end{aligned}$$

such that the sum

$$z_0 = 2U$$

becomes an identity.

To proceed, raise the top equation to the third power to write

$$\frac{c}{2} + q = U\left(U^2 - 3V^2\right) + \sqrt{-1}V\left(3U^2 - V^2\right) \,,$$

which was enough for Bombelli to see the solution. By separating the 'real' term without the factor of $\sqrt{-1}$ from its 'imaginary' counterpart, we end up with a pair of equations where *neither* makes reference to $\sqrt{-1}$. Moreover, Bombelli went on to seek integer solutions for U, V, which isn't always easy, or always possible.

Continuing the example on hand, we have

$$\frac{c}{2} + q = \frac{154}{27} + \sqrt{-\frac{25}{3}} \,,$$

and comparing this to the expanded form in terms of U, V, gives

$$\begin{aligned} \frac{154}{27} &= U\left(U^2 - 3V^2\right) \\ \sqrt{\frac{25}{3}} &= V\left(3U^2 - V^2\right) \,. \end{aligned}$$

To make headway on the problem, some fiddling leads one to the substitution

$$\begin{aligned} 2\sqrt{3}V &= \tilde{V} \\ 6U &= \tilde{U} \,, \end{aligned}$$

and the above becomes

$$\begin{aligned} 11 \cdot 14 \cdot 8 &= \tilde{U}\left(\tilde{U} + 3\tilde{V}\right)\left(\tilde{U} - 3\tilde{V}\right) \\ 1 \cdot 12 \cdot 10 &= \tilde{V}\left(\tilde{U} + \tilde{V}\right)\left(\tilde{U} - \tilde{V}\right) \,, \end{aligned}$$

solved by two integers

$$\begin{aligned} \tilde{U} &= 11 \\ \tilde{V} &= 1 \,, \end{aligned}$$

meaning

$$U = \frac{11}{6}$$
$$V = \frac{1}{2\sqrt{3}} \, ,$$

from which we get

$$z_0 = 2U = \frac{11}{3} \, .$$

Restoring the x- variable, we finally have

$$x_0 = z_0 - \frac{B}{3A} = \frac{11}{3} - \frac{5}{3} = 2$$

as a solution to the original cubic expression.

With the hard part finished, we can find two more solutions to the example on hand by solving

$$0 = z^2 + \left(\frac{11}{3}\right) z + \frac{308 \cdot 3}{27 \cdot 11} \, ,$$

having solutions

$$z_1 = -\frac{7}{3}$$
$$z_2 = -\frac{4}{3} \, .$$

Restoring the x variable, these read

$$x_1 = -4$$
$$x_2 = -3 \, .$$

Finally then, we have

$$x^3 + 5x^2 - 2x - 24 = (x - 2)(x + 4)(x + 3) \, ,$$

finishing the example.

3 Binomial Theorem

For two values x and y, consider the quantity

$$P(n) = (x + y)^n \, ,$$

where n is any of

$$n = 0, 1, 2, 3, \ldots \, .$$

The quantity $P(n)$ means *polynomial of degree n.*

Explicitly, we find, using the distributive property and the properties of exponents:

$$P(0) = (x + y)^0 = 1$$
$$P(1) = (x + y)^1 = x + y$$
$$P(2) = (x + y)^2 = x^2 + 2xy + y^2$$
$$P(3) = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

There is no upper limit to the allowed n. Jotting down a few more for the sake of establishing a pattern, one finds:

$$P(4) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$P(5) = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

3.1 Pascal's Triangle

Ignoring any x- and y-symbols and listing *only* the coefficients associated with a given $P(n)$, one discovers *Pascal's Triangle*:

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

At a glance, one can see that a given number in the body of Pascal's triangle is the sum of the pair of adjacent numbers one row above.

3.2 Binomial Coefficients

A given number in Pascal's Triangle can be indexed by the row n and the column k, where $0 \leq k \leq n$. To denote the coefficient at position n, k, one writes

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \, , \tag{1.5}$$

or equivalently:

$$\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-(k-1))}{k!} \tag{1.6}$$

Reflection Identity

Due to the symmetry in Pascal's triangle, one may swap k with $n - k$ to produce the same binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k} \tag{1.7}$$

Coefficient Sum Identity

The binomial coefficients obey a summation identity

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}, \tag{1.8}$$

proven by brute force:

$$\begin{aligned}
&\binom{n}{m} + \binom{n}{m+1} \\
&= \frac{n!}{m!\,(n-m)!} + \frac{n!}{(m+1)!\,(n-(m+1))!} \\
&= \frac{n!\,(m+1)}{(m+1)!\,(n-m)!} + \frac{n!\,(n-m)}{(m+1)!\,(n-m)!} \\
&= \frac{(n+1)!}{(m+1)!\,(n-m)!} \\
&= \binom{n+1}{m+1}
\end{aligned}$$

On Pascal's triangle, the summation identity proves that the sum of a pair of coefficients equals the one 'below'.

Row Sum Identity

Summing the coefficients across the columns of Pascal's Triangle for a given row shows an interesting pattern:

$$\begin{aligned}
1 &= 2^0 \\
1+1 &= 2^1 \\
1+2+1 &= 2^2 \\
1+3+3+1 &= 2^3 \\
1+4+6+4+1 &= 2^4
\end{aligned}$$

Evidently, the sum of coefficients across row n equals 2^n. This can be seen just as quickly by setting $x = y = 1$ and then each $P(n)$ resolves to 2^n.

Sigma Notation

To represent the above row-sum identity in a single equation for any n, we use the sigma notation for the sum as follows:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{1.9}$$

For example, for $n = 3$ one has

$$\begin{aligned}
\sum_{k=0}^{3} \binom{3}{k} &= \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \\
&= \frac{3!}{0!3!} + \frac{3!}{1!2!} + \frac{3!}{2!1!} + \frac{3!}{3!0!} \\
&= 1+3+3+1 \\
&= 2^3
\end{aligned}$$

3.3 Alternating Pascal's Triangle

We can redo the whole tour of binomial multiplication by replacing y with $-y$, giving a new polynomial of degree n:

$$\tilde{P}(n) = (x-y)^n$$

This gives rise to a version of Pascal's Triangle with alternating signs. In this case, the sum of coefficients across a given row (except the zeroth row) always tallies to zero:

$$\begin{aligned}
1 &= 2^0 \\
1-1 &= 0 \\
1-2+1 &= 0 \\
1-3+3-1 &= 0 \\
1-4+6-4+1 &= 0
\end{aligned}$$

More concisely, this is:

$$\sum_{k=0}^{n>0} \binom{n}{k} (-1)^k = 0 \tag{1.10}$$

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&-1&&&&&&& \\
&&&&&&1&&-2&&1&&&&&& \\
&&&&&1&&-3&&3&&-1&&&&& \\
&&&&1&&-4&&6&&-4&&1&&&& \\
&&&1&&-5&&10&&-10&&5&&-1&&& \\
&&1&&-6&&15&&-20&&15&&-6&&1&& \\
&1&&-7&&21&&-35&&35&&-21&&7&&-1& \\
1&&-8&&28&&-56&&70&&-56&&28&&-8&&1
\end{array}$$

3.4 Binomial Theorem

Each expansion $(x+y)^n$, $(x-y)^n$ is wrapped up nicely in the *binomial theorem*:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.11}$$

For completeness, a given binomial coefficient is equal to

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

3.5 Binomial Theorem Abuse

While the binomial theorem is clearly defined for integer n, it turns out that Equation (1.11) is robust enough to handle non-integer n. Of course, the sum can't terminate at a non-integer n, so a reasonable thing to do is let the sum run to *infinity*, meaning:

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k$$

For the above to remain valid, it must be that $(y/x)^k$ tends to zero for large k.

For a detailed example, let us attempt to calculate the numerical value of $\sqrt{3}$ by setting

$$\begin{aligned} x &= 4 \\ y &= -1 \\ n &= 1/2 \end{aligned}$$

on the left side of the above equation. Then, we should have

$$(4-1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} 4^{1/2} \left(\frac{-1}{4}\right)^k ,$$

which leaves the question of what to do with a non-integer binomial coefficient.

Proceeding boldly as Newton did, use the definition of the binomial coefficient to write

$$\binom{1/2}{k} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!} ,$$

or explicitly:

$$\begin{aligned}
\binom{1/2}{0} &= 1 \\
\binom{1/2}{1} &= \frac{1}{2} \\
\binom{1/2}{2} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \frac{-1}{8} \\
\binom{1/2}{3} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{1}{16} \\
\binom{1/2}{4} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!} = \frac{-5}{128} \\
\binom{1/2}{5} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{5!} = \frac{7}{256}
\end{aligned}$$

Putting everything together, find

$$\begin{aligned}
\frac{\sqrt{3}}{2} = \frac{(4-1)^{1/2}}{2} &= 1 + \frac{1}{2}\left(\frac{-1}{4}\right)^1 - \frac{1}{8}\left(\frac{-1}{4}\right)^2 \\
&\quad + \frac{1}{16}\left(\frac{-1}{4}\right)^3 - \frac{5}{128}\left(\frac{-1}{4}\right)^4 \\
&\quad + \frac{7}{256}\left(\frac{-1}{4}\right)^5 - \cdots ,
\end{aligned}$$

leading quite correctly to:

$$\sqrt{3} = 1.7320508075\ldots$$

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