Differential Geometry
MANUSCRIPT

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November 15, 2025

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Chapter 1

Differential Geometry

1 Tensors and Index Notation

Flat Space

Ordinary calculus typically deals in Cartesian coordinates with one, two, or three dimensions, where it is understood that the x-, y-, and z-axes always remain mutually perpendicular without ever bending. In such *flat space*, the rules of geometry (and thereby trigonometry, vectors, etc.) are taken everywhere for granted.

Curved Space

It is useful, however, to modify the rules of calculus and geometry so as to embed on a *curved surface* or *curved space*, which we loosely define as any situation where the rules of Euclid won't work. This is a geometric disaster, let alone what happens to trigonometry, vectors, the laws of physics, etc.

The framework for adapting to curved space comes from differential geometry, and in particular, Riemannian geometry.

1.1 Index Notation

A vector $\boldsymbol{A}$ needs one index to register its components, and usually no worry is assigned to how the index appears. As a list, $\boldsymbol{A}$ is equivalent to

$$\boldsymbol{A} = \left\langle A_x, A_y, A_z \right\rangle = \langle A^x, A^y, A^z \rangle$$

with the distinction between A_x, A^x being a stylistic choice. The first thing we need to do is dispense with such sloppiness.

Type

To advance on a cleaner index notation, let us adopt the burden that vertical placement of the index *does* matter to a vector, and to quantify this we use the word *type*.

The type of any object can be denoted (m, n) for whole numbers $m > 0$, $n > 0$. When reported as a single number, the term 'type' represents the sum $m + n$. Any type-0 tensor is simply a scalar.

Contravariant and Covariant

Using the type convention, we define a type $(1, 0)$ vector with N components as

$$x^\mu = \left(x^1, x^2, x^3, \ldots, x^N\right) \ .$$

Somewhat like a column vector, x^μ is called *contravariant*.

For a vector of type $(0, 1)$ with N components, we write

$$x_\mu = (x_1, x_2, x_3, \ldots, x_N) \ ,$$

which is somewhat like a row vector, and is called *covariant*, also known as a *dual vector*.

1.2 Tensors

Going beyond vectors, it's conceivable to work with mathematical objects that have *any* number of indices, called *tensors*, of which vectors and matrices are special cases. There are only two type-1 tensors, which are simply the vectors x^μ, x_μ written above.

For a type-2 tensor A, there are three flavors:

$$\begin{aligned} \text{type } (2,0) &= A^{\mu\nu} \\ \text{type } (1,1) &= A^\mu_\nu \\ \text{type } (0,2) &= \Lambda_{\mu\nu} \end{aligned}$$

Each tensor $A^{\mu\nu}$, Λ^μ_ν, $A_{\mu\nu}$ represents a two-dimensional 'block' of terms.

1.3 Linear System

From linear algebra, recall that at linear system $A\boldsymbol{x} = \boldsymbol{y}$ may be understood as a matrix A acting on a column vector $\boldsymbol{x}$ resulting in a new column vector $\boldsymbol{y}$. In detail, one may encounter

$$\sum_{k=1}^{N} A_{jk} x_k = y_j \ .$$

The aim is to update the above to tensor language. Taking $\boldsymbol{x}$, $\boldsymbol{y}$ as column vectors, each is replaced by x^k, y^j, respectively. This leaves the puzzle of what to do with the indices on A, as each of the following may look appealing:

$$\sum_{k=1}^{N} A^{jk} x^k \stackrel{?}{=} y^j$$
$$\sum_{k=1}^{N} A^{j}_{k} x^k \stackrel{?}{=} y^j$$
$$\sum_{k=1}^{N} A_{jk} x^k \stackrel{?}{=} y^j$$

Type Continuity

In the above, the correct option is singled out by type arguments. On the right side of each, the final type of y^j is $(1, 0)$. Only the middle case satisfies this on the left, thus we write the linear system as

$$\sum_{k=1}^{N} A^{j}_{k} x^k = y^j \ .$$

Einstein Summation Convention

By writing the linear system $A\boldsymbol{x} = \boldsymbol{y}$ in index notation, observe on the left that the k-index appears twice, once in the up-position and once in the down-position. It just happens that k is the variable we're summing over and the summation symbol can be omitted according to the *Einstein sum(mation) convention*:

$$A^{j}_{k} x^k = y^j$$

The linear system can be re-analyzed so the resulting vector y_j is type $(0, 1)$. The beauty of index notation is we can write down the answer immediately:

$$A^{k}_{j} x_k = y_j$$

In this case, the repeated index k vanishes from the left via the Einstein sum convention.

1.4 Tensor Product

Any pair of tensors A, B can be 'multiplied' in several ways. To keep things entirely general, we will assume each tensor of arbitrary type, i.e.

$$A = A^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q}$$
$$B = B^{\mu_1 \mu_2 \ldots \mu_r}_{\nu_1 \nu_2 \ldots \nu_s}$$

That is, tensor A is of type (p, q), and tensor B is of type (r, s).

Outer product

The most explosive tensor product is the *outer product*, which results in a tensor of rank $(p + r, q + s)$:

$$A \otimes B = A^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q} B^{\mu_1 \mu_2 \ldots \mu_r}_{\nu_1 \nu_2 \ldots \nu_s}$$

For a special case, the product of a $(p, 0)$ tensor and a $(0, s)$ tensor constitute a tensor of type (p, s):

$$A \otimes B = A^{\alpha_1 \alpha_2 \ldots \alpha_p} B_{\nu_1 \nu_2 \ldots \nu_s}$$

Supposing we also have $p = s = 1$, the above reduces to the product of a column vector and a row vector:

$$A \otimes B = A^{\alpha} B_{\nu}$$

If A, B have n components each, the above is equivalent to:

$$\begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{bmatrix} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$$
$$= \begin{bmatrix} A_1 B_1 & A_1 B_2 & \ldots & A_1 B_n \\ A_2 B_1 & A_2 B_2 & \ldots & A_2 B_n \\ \cdots & \cdots & \cdots & \cdots \\ A_n B_1 & A_n B_2 & \ldots & A_n B_n \end{bmatrix}$$

Contraction

Start with the structure of the outer product and let any one of the contravariant indices in A equal any one of the covariant indices in B, or vice-versa. The resulting object is called a *contraction* between A and B. Supposing the shared index is k, one could have:

$$C = A^{\alpha_1 \alpha_2 \ldots k \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q} B^{\mu_1 \mu_2 \ldots \mu_r}_{\nu_1 \nu_2 \ldots k \ldots \nu_s}$$

Using the Einstein sum convention, the contraction C is a tensor of rank $p + q + r + s - 2$. That is, the total contravariant rank is reduce by one, and the total covariant rank is reduced by one.

Problem 1

For a type-2 tensor A, which of Λ^{a}_{b}, $\Lambda^{\nu\nu}$, $\Lambda^{\mu\mu}$, Λ^{b}_{b}, Λ^{μ}_{ν}, Λ^{12} is most like the matrix trace of A? Answer: A^{b}_{b}

Inner Product

If contraction is performed recursively on all indices, the rank of the resulting quantity eventually reaches zero, leaving just a scalar $w = A \cdot B$. This is called the *inner product* of the two tensors:

$$w = A \cdot B = A^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\mu_1 \mu_2 \ldots \mu_r} B^{\mu_1 \mu_2 \ldots \mu_r}_{\alpha_1 \alpha_2 \ldots \alpha_p}$$

Norm

The *norm* of a type-1 tensor is defined as the self inner product:

$$\|A\|^2 = A^\mu A_\mu$$

The same notion is true for tensors of any type. The type-2 case involves the double sum:

$$\|B\|^2 = B^{\mu\nu} B_{\mu\nu}$$

1.5 Formal Tensor Definition

To formally qualify as a tensor, the components of any indexed object A must scale linearly with respect to a generalized coordinate transformation. For this, we let $q^{\mu'}$ represent generalized coordinates in a transformed system. An object of type (m, n) is a tensor only if:

$$A^{\mu'_1 \cdots \mu'_n}_{\nu'_1 \cdots \nu'_m} = \frac{\partial q^{\mu'_1}}{\partial q^{\mu_1}} \cdots \frac{\partial q^{\mu'_n}}{\partial q^{\mu_n}} \frac{\partial q^{\nu_1}}{\partial q^{\nu'_1}} \cdots \frac{\partial q^{\nu_m}}{\partial q^{\nu'_m}} A^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$$

For a special case, the type-1 vectors V^μ and V_μ transform as:

$$V^{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu$$
$$V_{\mu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} V_\mu$$

The above manifests three ways for type-2 tensors

$$A^{\mu'\nu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} \frac{\partial q^{\nu'}}{\partial q^\nu} A^{\mu\nu}$$
$$A^{\mu'}_{\nu'} = \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} A^\mu_\nu$$
$$A_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} \frac{\partial q^\nu}{\partial q^{\nu'}} A_{\mu\nu} \,,$$

and so on for those of higher type. For every 'up' index on the left, tack on a term like $\partial q^{\mu'}/\partial q^\mu$ to the right. Similarly for every 'down' index on the left, tack on a term like $\partial q^\mu / \partial q^{\mu'}$ to the right.

Cartesian and Polar Coordinates

Recall that the Cartesian plane is mapped by ordered pairs (x, y). Also, a polar coordinate system (r, θ) covers the same plane via

$$x = r\cos(\theta)$$
$$y = r\sin(\theta) \ .$$

A type $(0, 1)$ tensor V with Cartesian components V_x, V_y can be expressed in polar coordinates by associating:

$$q^\mu = x$$
$$q^\nu = y$$
$$q^{\mu'} = r$$
$$q^{\nu'} = \theta$$

Then, using the definition of a tensor we have:

$$V_r = \frac{\partial q^\mu}{\partial q^r} V_\mu = \frac{\partial x}{\partial r} V_x + \frac{\partial y}{\partial r} V_y$$
$$V_\theta = \frac{\partial q^\mu}{\partial q^\theta} V_\mu = \frac{\partial x}{\partial \theta} V_x + \frac{\partial y}{\partial \theta} V_y$$

Or, one can solve for the Cartesian components via

$$V_x = \frac{\partial q^\mu}{\partial x} V_\mu = \frac{\partial r}{\partial x} V_r + \frac{\partial \theta}{\partial x} V_\theta$$
$$V_y = \frac{\partial q^\mu}{\partial y} V_\mu = \frac{\partial r}{\partial y} V_r + \frac{\partial \theta}{\partial y} V_\theta \,,$$

which requires inverting the formulas for r, θ:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

From these, we gain:

$$\partial r/\partial x = x/r$$
$$\partial r/\partial y = y/r$$
$$\partial \theta/\partial x = -y/r^2$$
$$\partial \theta/\partial y = x/r^2$$

The story is similar when V is a type $(1, 0)$ tensor. For this, the tensor transformation law gives

$$V^x = \frac{\partial q^x}{\partial q^\mu} V^\mu = \frac{\partial x}{\partial r} V^r + \frac{\partial x}{\partial \theta} V^\theta$$
$$V^y = \frac{\partial q^y}{\partial q^\mu} V^\mu = \frac{\partial y}{\partial r} V^r + \frac{\partial y}{\partial \theta} V^\theta \,,$$

and correspondingly

$$V^r = \frac{\partial q^r}{\partial q^\mu} V^\mu = \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y$$
$$V^\theta = \frac{\partial q^\theta}{\partial q^\mu} V^\mu = \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y \,.$$

Problem 2

Show that:

$$V^r = \cos(\theta)\, V^x + \sin(\theta)\, V^y$$
$$V^\theta = -\frac{\sin(\theta)}{r} V^x + \frac{\cos(\theta)}{r} V^y \,.$$

1.6 Scalar Product

Using the definitions above, let us calculate the inner product of two vectors

$$w = A_\mu B^\mu \,,$$

which is similar to the special case of the outer product above up to a swap of type.

Next, write the transformed tensors

$$B^{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} B^\mu$$
$$A_{\mu'} = \frac{\partial q^\nu}{\partial q^{\mu'}} A_\nu \,,$$

and multiply the two equations:

$$A_{\mu'} B^{\mu'} = \left(\frac{\partial q^\nu}{\partial q^{\mu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \right) A_\nu B^\mu$$

Kronecker Delta

The parenthesized quantity is only nonzero if $\beta = \mu$, represented using the Kronecker delta relation (not a tensor):

$$\delta^\nu_\mu = \frac{\partial q^\nu}{\partial q^{\mu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu \end{cases}$$

Invariance

In light of the Kronecker delta, the scalar product calculation simplifies to

$$w = A_{\mu'} B^{\mu'} = A_\mu B^\mu \,.$$

The fact that the scalar product $w = A_\mu B^\mu$ is the same in any coordinate system means w is an *invariant* quantity.

In the special case $A = B$, the scalar w is the norm of the vector:

$$\|A\|^2 = A_\mu A^\mu$$

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2 Manifold and the Metric

Let $\boldsymbol{S}$ denote the position vector on a manifold parameterized by generalized coordinates $q^1, q^2, q^3, ..., q^N$, where N is the number of dimensions on the manifold:

$$\boldsymbol{S} = \left(q^1, q^2, \ldots, q^N\right)$$

Note that the physical units of any given q^μ are not limited to length.

2.1 Affine Parameter

Each coordinate q^μ can occur as a function of a parameter λ, i.e. $q^\mu = q^\mu(\lambda)$. It turns out, for our studies, that some choices of λ are more 'natural' than others. Reaching for the most natural parameter, it makes sense to associate small changes $d\lambda$ with the change in arc length dS on the manifold itself. For this reason we decide

$$d\lambda \propto dS\,,$$

where λ is called an *affine parameter*.

A convenient affine parameter invokes the notion of constant speed c on the manifold, along with a time parameter $d\tau$ during which motion takes place. From these we make the association

$$c\, d\tau \propto dS\,.$$

2.2 Differential Line Element

Denoting the differential line element vector as $d\boldsymbol{S}$, the chain rule dictates

$$d\boldsymbol{S} = \frac{\partial \boldsymbol{S}}{\partial q^1}\, dq^1 + \frac{\partial \boldsymbol{S}}{\partial q^2}\, dq^2 + \cdots + \frac{\partial \boldsymbol{S}}{\partial q^N}\, dq^N\,.$$

Generalized Basis Vectors

The differential line element is abbreviated using the Einstein sum convention

$$d\boldsymbol{S} = \boldsymbol{a}_{(\mu)} dq^\mu$$

provided we define the *generalized basis vectors* $\boldsymbol{a}_{(\mu)}$ such that:

$$\boldsymbol{a}_{(\mu)} = \frac{\partial \boldsymbol{S}}{\partial q^\mu} = \partial_\mu \boldsymbol{S}$$

In component form, the above contains the information

$$\boldsymbol{a}_{(\mu)} = \left(\frac{\partial q^1}{\partial q^\mu}, \frac{\partial q^2}{\partial q^\mu}, \cdots, \frac{\partial q^N}{\partial q^\mu}\right)\,.$$

To pick out individual components in the basis vector, write:

$$\left(\boldsymbol{a}_{(\mu)}\right)^\nu = \frac{\partial q^\nu}{\partial q^\mu} = \delta^\mu_\nu$$

As a type $(0,1)$ tensor, the basis vector $\boldsymbol{a}_{(\mu)}$ adheres to the transformation law:

$$\boldsymbol{a}_{(\mu')} = \frac{\partial q^\mu}{\partial q^{\mu'}} \boldsymbol{a}_{(\mu)}$$

Dual Line Element

For a second approach to the problem one can imagine down-shifting the index on dq^μ, thereby up-shifting the index on $\boldsymbol{a}_{(\mu)}$. With the calculation structurally the same, store the result in a vector $d\boldsymbol{S}^T$, which is the dual vector to $d\boldsymbol{S}$, via

$$d\boldsymbol{S}^T = \frac{\partial \boldsymbol{S}}{\partial q_1}\, dq_1 + \frac{\partial \boldsymbol{S}}{\partial q_2}\, dq_2 + \cdots + \frac{\partial \boldsymbol{S}}{\partial q_N}\, dq_N\,,$$

or

$$d\boldsymbol{S}^T = \boldsymbol{a}^{(\mu)} dq_\mu\,,$$

suggesting:

$$\boldsymbol{a}^{(\mu)} = \frac{\partial \boldsymbol{S}}{\partial q_\mu} = \partial^\mu \boldsymbol{S}$$

2.3 Metric Tensor

In terms of generalized basis vectors, we can write a generalized product

$$\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)} = \frac{\partial \boldsymbol{S}}{\partial q^\mu} \cdot \frac{\partial \boldsymbol{S}}{\partial q_\nu}\,,$$

which uses the notation for the dot product for illustration. Correspondingly, we have:

$$\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)} = \frac{\partial \boldsymbol{S}}{\partial q^\mu} \cdot \frac{\partial \boldsymbol{S}}{\partial q^\nu}$$
$$\boldsymbol{a}^{(\mu)} \cdot \boldsymbol{a}^{(\nu)} = \frac{\partial \boldsymbol{S}}{\partial q_\mu} \cdot \frac{\partial \boldsymbol{S}}{\partial q_\nu}$$

Under the coordinate transformation $q^\mu \to q^{\mu'}$, the above is written

$$\boldsymbol{a}_{(\mu')} \cdot \boldsymbol{a}^{(\nu')} = \frac{\partial \boldsymbol{S}}{\partial q^{\mu'}} \cdot \frac{\partial \boldsymbol{S}}{\partial q_{\nu'}}\,,$$

and similar applies for the other two. Treating each term on the right as a type-1 tensor, we further have:

$$\boldsymbol{a}_{(\mu')} \cdot \boldsymbol{a}^{(\nu')} = \left(\frac{\partial q^\mu}{\partial q^{\mu'}} \frac{\partial q_\nu}{\partial q_{\nu'}}\right) \frac{\partial \boldsymbol{S}}{\partial q^\mu} \cdot \frac{\partial \boldsymbol{S}}{\partial q_\nu}$$
$$\boldsymbol{a}_{(\mu')} \cdot \boldsymbol{a}^{(\nu')} = \left(\frac{\partial q^\mu}{\partial q^{\mu'}} \frac{\partial q_\nu}{\partial q_{\nu'}}\right) \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)}$$

Evidently, the quantity $\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)}$ is a bona fide type $(1,1)$ tensor as a matter of definition.

Similar exercises are straightforward for the other two tensor products. In summary, we conclude:

$$\text{type } (2,0) = \boldsymbol{a}^{(\mu)} \cdot \boldsymbol{a}^{(\nu)}$$
$$\text{type } (1,1) = \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)}$$
$$\text{type } (0,2) = \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)}$$

The Metric

Any of the type-2 tensors above can be called the *metric tensor*, or 'the metric' for shorthand, designated g, for the manifold $\boldsymbol{S}$. Explicitly, each form is given by:

$$\begin{aligned} g^{\mu\nu} &= \boldsymbol{a}^{(\mu)} \cdot \boldsymbol{a}^{(\nu)} \\ g^{\nu}_{\mu} &= \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)} \\ g_{\mu\nu} &= \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)} \end{aligned}$$

The 'mixed' type $(1,1)$ metric tensor g^{ν}_{μ} is almost always synonymous with the Kronecker delta δ^{ν}_{μ}. With this distinction in mind, we'll use the form g^{ν}_{μ} in calculations for extra clarity.

Using the g notation, the transformation law for the metric is:

$$g_{\mu'\nu'} = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial q^{\nu}}{\partial q^{\nu'}} g_{\mu\nu}$$

Metric Symmetry

By construction, namely commutation of the dot product, the metric tensor $g^{\mu\nu}$ is symmetric, which means any pair of indices can be swapped and the components remain the same:

$$\begin{aligned} g^{\mu\nu} &= g^{\nu\mu} \\ g_{\mu\nu} &= g_{\nu\mu} \end{aligned}$$

Jacobian Matrix

The derivative $\partial q^{\mu} / \partial q^{\mu'}$ is the $\mu\mu'$th component of the standard Jacobian matrix. Switching momentarily to matrix notation, we can write

$$\boldsymbol{a}_{(\mu')} = J^{\mu}_{\mu'} \boldsymbol{a}_{(\mu)} \ ,$$

and correspondingly,

$$\boldsymbol{a}_{(\nu')} = J^{\nu}_{\nu'} \boldsymbol{a}_{(\nu)} \ .$$

Combining the two, we have

$$\boldsymbol{a}_{(\mu')} \cdot \boldsymbol{a}_{(\nu')} = \left(J^{\mu}_{\mu'} J^{\nu}_{\nu'} \right) \boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)} \ ,$$

or

$$g_{\mu'\nu'} = \left(J^{\mu}_{\mu'} J^{\nu}_{\nu'} \right) g_{\mu\nu} \ ,$$

equivalent to the index-free notation

$$g' = J^{T} g J \ .$$

Next we need the determinant of both sides using the general identity

$$\det \left(A B A^{T} \right) = (\det A)^2 \, (\det B) \ ,$$

which for us means

$$\det (g') = (\det J)^2 \, (\det g) \ ,$$

or

$$|\det J| = \frac{\sqrt{|g'|}}{\sqrt{|g|}} \ .$$

If the original metric is Cartesian, then $g_{\mu\nu} = \delta_{\mu\nu}$ and $g = 1$.

2.4 Gradient

Consider a scalar field $f = f(q^{\mu})$ that depends on the generalized coordinates q^{μ} that cover the manifold $\boldsymbol{S}$. If $f(q^{\mu})$ is differentiable, we define the *gradient* of $f(q^{\mu})$ as:

$$\boldsymbol{\nabla} f = \boldsymbol{a}^{(\mu)} \frac{\partial f}{\partial q^{\mu}}$$

The gradient operator $\boldsymbol{\nabla}$ only makes sense in terms of the basis vector $\boldsymbol{a}^{(\mu)}$. One can attain a vector-free version of the above by distributing $\boldsymbol{a}_{(\alpha)}$ into each side:

$$\boldsymbol{a}_{(\alpha)} \cdot \boldsymbol{\nabla} f = \boldsymbol{a}_{(\alpha)} \cdot \boldsymbol{a}^{(\mu)} \frac{\partial f}{\partial q^{\mu}}$$

Recognizing g^{μ}_{α} on the right, we further have

$$\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial q^{\mu}} \ ,$$

and the left side is given the shorthand $\partial_{\mu} f$. This suggests a clean vector-free notation

$$\partial_{\mu} f = \frac{\partial f}{\partial q^{\mu}} \ ,$$

also known as the gradient.

The gradient $\partial_{\mu} f$ is a type $(0,1)$ tensor, i.e. a covariant vector, which in this case is synonymous with a vector field. To check this, we should have by definition that

$$\partial_{\mu'} f = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \partial_{\mu} f \ ,$$

which in traditional notation unpacks to

$$\frac{\partial f}{\partial q^{\mu'}} = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial f}{\partial q^{\mu}} \ .$$

This is simply a statement of the chain rule and is automatically true, therefore $\partial_{\mu} f$ is indeed a tensor.

Covariant Gradient

The gradient $\partial_\mu f = \partial f / \partial q^\mu$ may be denoted V_μ such that

$$V_\mu = \partial_\mu f \ .$$

As a covariant $(0,1)$ tensor, V_μ satisfies the definition of a tensor when imposing the change of coordinates $q^\mu \to q^{\mu'}$:

$$V_{\mu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} V_\mu$$

Standard vector notation can be restored by tacking on the basis vectors $\boldsymbol{a}^{(\mu)}$. Note that the covariant form V_μ is a dual vector, thus we assign appropriate notation:

$$\boldsymbol{V}^T = \boldsymbol{a}^{(\mu)} V_\mu$$

The low index on V_μ reminds that the gradient is a dual vector.

Problem 3

Let V_x, V_y be the Cartesian components of the gradient of the function

$$f\,(x,y) = \ln\left(\sqrt{x^2+y^2}\right) \ .$$

Calculate the gradient in Cartesian coordinates and convert the result to polar coordinates to write V_r, V_θ. Answer:

$$\begin{aligned}
V_x &= \frac{x}{x^2+y^2} \\
V_y &= \frac{y}{x^2+y^2} \\
V_r &= \frac{\partial x}{\partial r} V_x + \frac{\partial y}{\partial r} V_y = \frac{x\cos(\theta) + y\sin(\theta)}{x^2+y^2} = \frac{1}{r} \\
V_\theta &= \frac{\partial x}{\partial \theta} V_x + \frac{\partial y}{\partial \theta} V_y = \frac{-yx + xy}{x^2+y^2} = 0
\end{aligned}$$

Contravariant Gradient

A variation on the standard gradient raises the index on the derivative subscript, to give a type $(1,0)$ contravariant tensor

$$\partial^\mu f = \frac{\partial f}{\partial q_\mu} \ .$$

As a type $(1,0)$ tensor, the gradient is represented as the vector $\boldsymbol{V}$ such that

$$\boldsymbol{V} = \boldsymbol{a}_{(\mu)} V^\mu \ .$$

Reconciliation between $\boldsymbol{V}^T$ and $\boldsymbol{V}$ is done using tensor products:

$$\begin{aligned}
\boldsymbol{V}^T \cdot \boldsymbol{V} &= \left(\boldsymbol{a}^{(\mu)} \cdot \boldsymbol{a}_{(\nu)}\right) V_\mu V^\nu \\
&= \left(g^\mu_\nu V_\mu\right) V^\nu \\
&= V_\nu V^\nu \\
&= \|V\|^2
\end{aligned}$$

2.5 Differential Interval

In terms of

$$\begin{aligned}
d\boldsymbol{S} &= \boldsymbol{a}_{(\mu)} dq^\mu \\
d\boldsymbol{S}^T &= \boldsymbol{a}^{(\mu)} dq_\mu \ ,
\end{aligned}$$

the norm of the differential line element yields the *differential interval* on the manifold:

$$dS^2 = \|dS\|^2 = \left(\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)}\right) dq^\mu dq_\nu$$

As a scalar product, the differential interval is invariant with respect to a coordinate transformation, for instance:

$$dS^2 = dq^\mu dq_\mu = dq^{\mu'} dq_{\mu'}$$

Utilizing (1,1) Metric

In terms of the type $(1,1)$ metric tensor, the differential interval is

$$dS^2 = \left(g^\nu_\mu dq^\mu\right) dq_\nu \ .$$

The parenthesized term can *only* be equal to dq^ν, thus we see g^ν_μ behaving as a Kronecker delta:

$$g^\nu_\mu dq^\mu = dq^\nu$$

Conversely, we could have written

$$dS^2 = dq^\mu \left(g^\nu_\mu dq_\nu\right) \ ,$$

where the parenthesized term can only be dq_μ, thus:

$$g^\nu_\mu dq_\nu = dq_\mu$$

For a consistency check, use the invariance of dS^2 to write

$$\begin{aligned}
dS^2 &= g^\nu_\mu dq^\mu dq_\nu \\
&= g^\nu_\mu \left(g^\mu_\alpha dq^\alpha\right)\left(g^\beta_\nu dq_\beta\right) \\
&= \left(g^\mu_\alpha g^\nu_\mu g^\beta_\nu\right) dq^\alpha dq_\beta \ .
\end{aligned}$$

The quantity dS^2 remains invariant if

$$g^\mu_\alpha g^\nu_\mu g^\beta_\nu = g^\beta_\alpha \ ,$$

availing another Kronecker delta-like identity:

$$g^\mu_\alpha g^\alpha_\nu = g^\mu_\nu$$

Lowering an Index

Another formula for the differential interval can be made from only contravariant lengths:

$$dS^2 = \left(\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)}\right) dq^\mu dq^\nu \,,$$

implying

$$dS^2 = g_{\mu\nu} dq^\mu dq^\nu \,.$$

From the above, we reason that

$$g_{\mu\nu} dq^\mu = dq_\nu$$
$$g_{\mu\nu} dq^\nu = dq_\mu$$

are mutually true. Evidently, the metric tensor $g_{\mu\nu}$ applied to a contravariant form returns a covariant form. This maneuver is called 'lowering an index'.

Raising an Index

Yet another formula for the differential interval leads to

$$dS^2 = g^{\mu\nu} dq_\mu dq_\nu \,.$$

From this, we decide

$$g^{\mu\nu} dq_\mu = dq^\nu$$
$$g^{\mu\nu} dq_\nu = dq^\mu$$

must also hold. That is, the metric tensor $g^{\mu\nu}$ applied to a covariant form returns a contravariant form. This is 'raising an index'.

Condensation

Another identity involving the metric starts with $dS^2 = dq^\alpha dq_\alpha$ and shifting the vertical position of each index on the dq-terms. For this we have

$$dS^2 = \left(g^{\mu\alpha} g_{\alpha\nu}\right) dq_\mu dq^\nu \,,$$

which brings out another identity synonymous to the Kronecker delta:

$$g^{\mu\alpha} g_{\alpha\nu} = g^\mu_\nu$$

2.6 Metric-Tensor Interactions

Contractions with Vectors

Recall that for two vectors A, B, the scalar product w is defined as

$$w = A_\mu B^\mu \,.$$

Something we should check is whether the metric tensor successfully changes the index position on the vectors A, B in addition to displacements dq. For this, we postulate:

$$A_\mu = g_{\mu\nu} A^\nu$$
$$B^\mu = g^{\mu\nu} B_\nu$$

Substituting each into the formula for w, one finds

$$w = \left(g_{\mu\alpha} g^{\mu\nu}\right) A^\alpha B_\nu = g^\nu_\alpha A^\alpha B_\nu = A^\nu B_\nu \,.$$

The original w is restored provided that

$$A_\mu B^\mu = A^\mu B_\mu \,,$$

along with:

$$A^\nu = g^\nu_\alpha A^\alpha$$
$$B_\alpha = g^\nu_\alpha B_\nu$$

This is important news, as we see that the notion of shifting, raising, and lowering indices applies to any type-1 tensor.

To wring out another pair of identities, write for any vector A:

$$A_\mu = g_{\mu\alpha} A^\alpha = \left(g_{\mu\alpha} g^\alpha_\nu\right) A^\nu$$
$$A^\mu = g^{\mu\alpha} A_\alpha = \left(g^{\mu\alpha} g^\nu_\alpha\right) A_\nu$$

The parenthesized quantities can only be $g_{\mu\nu}$, $g^{\mu\nu}$, respectively. Evidently then:

$$g_{\mu\nu} = g_{\mu\alpha} g^\alpha_\nu$$
$$g^{\mu\nu} = g^{\mu\alpha} g^\nu_\alpha$$

Contractions with Tensors

The metric can be used to calculate the contravariant gradient of a scalar field $f\,(q^\mu)$ from the standard (covariant) gradient:

$$\partial^\mu f = g^{\mu\nu} \partial_\nu f$$

The same is true in reverse:

$$\partial_\mu f = g_{\mu\nu} \partial^\nu f$$

When a tensor undergoes contraction, only the common index plays into the calculation and the other indices come for the ride. For instance, if A is a type-3 tensor, we can have any of

$$A^{\mu j}_k = g^{\mu\nu} A^j_{\nu k}$$
$$A^{\mu j}_k = g^\mu_\nu A^{\nu j}_k$$
$$A^j_{\mu k} = g_{\mu\nu} A^{\nu j}_k \,,$$

and so on.

Cartesian and Polar Coordinates

In plane polar coordinates, it's straightforward to show that the nonzero components of the metric are

$$\begin{aligned} g_{rr} &= 1 \\ g_{\theta\theta} &= r^2 \,, \end{aligned}$$

and correspondingly

$$\begin{aligned} g^{rr} &= 1 \\ g^{\theta\theta} &= 1/r^2 \,. \end{aligned}$$

The metric is especially useful for transforming a vector V between coordinate systems. Previously, we found

$$\begin{aligned} V_x &= \frac{\partial r}{\partial x} V_r + \frac{\partial \theta}{\partial x} V_\theta \\ V_y &= \frac{\partial r}{\partial y} V_r + \frac{\partial \theta}{\partial y} V_\theta \,, \end{aligned}$$

along with

$$\begin{aligned} V^r &= \frac{\partial r}{\partial x} V^x + \frac{\partial r}{\partial y} V^y \\ V^\theta &= \frac{\partial \theta}{\partial x} V^x + \frac{\partial \theta}{\partial y} V^y \,, \end{aligned}$$

both requiring the inversion of the standard parameterization

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \,. \end{aligned}$$

The r, θ inversion can be avoided by first calculating the 'easy' transformations

$$\begin{aligned} V_r &= \frac{\partial x}{\partial r} V_x + \frac{\partial y}{\partial r} V_y \\ &= \cos(\theta)\, V_x + \sin(\theta)\, V_y \\ V_\theta &= \frac{\partial x}{\partial \theta} V_x + \frac{\partial y}{\partial \theta} V_y \\ &= -r\sin(\theta)\, V_x + r\cos(\theta)\, V_y \,. \end{aligned}$$

To proceed, we note $V^x = V_x$, $V^y = V_y$, and then V^r, V^θ are calculated using the metric:

$$\begin{aligned} V^r &= g^{rr} V_r \\ V^\theta &= g^{\theta\theta} V_\theta \end{aligned}$$

Thus we find

$$\begin{aligned} V^r &= V_r = \cos(\theta)\, V^x + \sin(\theta)\, V^y \\ V^\theta &= \frac{1}{r^2} V_\theta = -\frac{\sin(\theta)}{r} V^x + \frac{\cos(\theta)}{r} V^y \,. \end{aligned}$$

2.7 Tensor Symmetry

A tensor is *symmetric* in a pair of indices if it obeys

$$A^{\mu\nu} = A^{\nu\mu} \,,$$

and is *antisymmetric* if it obeys

$$A^{\mu\nu} = -A^{\nu\mu} \,.$$

Note the above comments apply to tensors of any type (p, q). Symmetry or antisymmetry need only exist in a set of indices, not in the entire tensor.

A general tensor may be expressed in terms of symmetric and antisymmetric parts via

$$A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) + \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right) \,.$$

Introducing the condensed notation

$$\begin{aligned} A^{(\mu\nu)} &= \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) \\ A^{[\mu\nu]} &= \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right) \,, \end{aligned}$$

the above can be written

$$A^{\mu\nu} = A^{(\mu\nu)} + A^{[\mu\nu]} \,.$$

2.8 Minkowski Metric

A special case of the metric tensor g is the four-dimensional *Minkowski metric*, written η, defined by

$$\begin{aligned} \eta_{00} &= -1 \\ \eta_{11} &= \eta_{22} = \eta_{33} = 1 \\ \eta_{\mu\nu} &= 0 \text{ if } \mu \neq \nu \end{aligned}$$

Akin to a diagonal matrix, off-diagonal components of the flat metric $\eta_{\mu\mu}$ are zero. The 00th component is -1 by convention. All other components on the diagonal are 1.

The same description for η works for the $(0, 2)$ version $\eta^{\mu\nu}$. Having the same utility as g to raise and lower indices, write

$$\eta^{\mu\nu} = \left(\eta^{\mu\alpha}\eta_{\beta\nu}\right)\eta_{\alpha\beta} \,.$$

Since the off-diagonal components of η are zero, it quickly follows that $\mu = \alpha$ and $\beta = \nu$. The parenthesized term can only be $+1$, and we conclude $\eta^{\mu\nu}$ and $\eta_{\alpha\beta}$ are numerically the same.

Worked Examples

Consider two tensors $X^{\mu\nu}$ and V^{μ} given by

$$X^{\mu\nu} = \begin{bmatrix} X^{00} & X^{01} & X^{02} & X^{03} \\ X^{10} & X^{11} & X^{12} & X^{13} \\ X^{20} & X^{21} & X^{22} & X^{23} \\ X^{30} & X^{31} & X^{32} & X^{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix}$$
$$V^{\mu} = (-1, 2, 0, -2) \ ,$$

respectively. Find the quantities (i) X^{μ}_{ν}, (ii) X^{ν}_{μ}, (iii) $X^{(\mu\nu)}$, (iv) $X_{[\mu\nu]}$, (v) X^{λ}_{λ}, (vi) $V^{\mu}V_{\mu}$, (vii) $V_{\mu}X^{\mu\nu}$.

Solution (i): Begin with

$$\eta_{\alpha\nu}X^{\mu\alpha} = X^{\mu}_{\nu}$$

and let $\nu = \mu$. The above becomes $\eta_{\mu\mu}X^{\mu\mu} = X^{\mu}_{\mu}$, thus the 00-component on the diagonal flips sign (namely from -2 to 2), and the components along the rest of the diagonal stay the same.

Let j, k be nonzero indices and replace $\mu = j$, $\nu = k$ to find $X^{jk} = X^{j}_{k}$, which says the lower 3×3 body of the result X^{μ}_{ν} is the same as that of the original $X^{\mu\nu}$.

For the left-most column with $\nu = 0$, the above reduces to $\eta_{00}X^{\mu 0} = X^{\mu}_{0}$. This reaffirms that the 00-component switches sign, but also tells us the terms X^{μ}_{0} are the negations of $X^{\mu 0}$.

For the top-most row with $\mu = 0$, $\nu = k$ we get $X^{0k} = X^{0}_{k}$. This means all terms in the top row except the 00-component retain sign. All together, we conclude:

$$X^{\mu}_{\nu} = \begin{bmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix}$$

Solution (ii): Begin with

$$\eta_{\mu\alpha}X^{\alpha\nu} = X^{\nu}_{\mu}$$

and let $\nu = \mu$. The above becomes $\eta_{\mu\mu}X^{\mu\mu} = X^{\mu}_{\mu}$, thus the 00-component on the diagonal flips sign, and the components along the rest of the diagonal stay the same.

Let j, k be nonzero indices and replace $\mu = j$, $\nu = k$ to find $X^{jk} = X^{k}_{j}$, which says the lower 3×3 body of the result X^{ν}_{μ} is the same as that of the original $X^{\mu\nu}$.

For the left column with $\nu = 0$, $\mu = j$ we get $X^{j0} = X^{0}_{j}$. This means all terms in the left column don't change sign.

For the top row with $\mu = 0$, the above reduces to $\eta_{00}X^{0\nu} = X^{\nu}_{0}$. This reaffirms that the 00-component switches sign, but also tells us the terms X^{ν}_{0} are the negations of $X^{0\nu}$. All together, we conclude:

$$X^{\nu}_{\mu} = \begin{bmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix}$$

Solution (iii): The transpose of $X^{\mu\nu}$, particularly $X^{\nu\mu}$, is written:

$$X^{\nu\mu} = \begin{bmatrix} 2 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & -2 \end{bmatrix}$$

By the formula

$$A^{(\mu\nu)} = \frac{1}{2}\left(A^{\mu\nu} + A^{\nu\mu}\right) \ ,$$

we calculate:

$$X^{(\mu\nu)} = \begin{bmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{bmatrix}$$

Solution (iv): Lower the indices on $X^{\mu\nu}$ via

$$X_{\mu\nu} = \eta_{\mu\alpha}\eta_{\beta\nu}X^{\alpha\beta} \ .$$

To avoid a brute-force calculation let j, k be nonzero indices and replace $\mu = j$, $\nu = k$ to find $X_{jk} = X^{jk}$, which says the lower 3×3 body of the result $X_{\mu\nu}$ is the same as that of the original $X^{\mu\nu}$.

For the left column with $\nu = 0$, the above tells us $X_{\mu 0} = -\eta_{\mu\alpha}X^{\alpha 0}$. For $\mu = 0$ we quickly find $X_{00} = X^{00}$, so the 00-component doesn't change. Instead taking $\mu = j > 0$, we have $X_{j0} = -X^{j0}$, meaning the terms in the left column (besides 00) change sign.

For the top row with $\mu = 0$, the above tells us $X_{0\nu} = -\eta_{b\nu}X^{0\beta}$. Letting $\nu = k > 0$, we have $X_{0k} = -X^{0k}$, meaning the terms in the top row (besides 00) change sign.

Putting it all together, we find

$$X_{\mu\nu} = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix} \ ,$$

along with:

$$X_{\nu\mu} = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 3 & 0 & 1 \\ 1 & 2 & 0 & -2 \end{bmatrix}$$

By the formula

$$A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right) \ ,$$

we calculate:

$$X_{[\nu\mu]} = \begin{bmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{bmatrix}$$

Solution (v): Begin with

$$\eta_{\mu\alpha} X^{\alpha\nu} = X^{\nu}_{\mu}$$

and let $\lambda = \nu = \mu$ to write $\eta_{\lambda\alpha} X^{\alpha\lambda} = X^{\lambda}_{\lambda}$, which unpacks as:

$$\begin{aligned} X^{\lambda}_{\lambda} &= \eta_{00} X^{00} + \eta_{11} X^{11} + \eta_{22} X^{22} + \eta_{33} X^{33} \\ &= -2 + 0 + 0 - 2 = -4 \end{aligned}$$

Solution (vi): First calculate the components of V_μ using $V_\mu = g_{\mu\alpha} V^{\alpha}$ to find

$$V_\mu = (1, 2, 0, -2) \ .$$

Then by the Einstein sum convention, we find

$$V^{\mu} V_{\mu} = -1 + 4 + 0 + 4 = 7 \ .$$

Solution (vii):

$$\begin{aligned} V_\mu X^{\mu\nu} &= \begin{cases} V_0 X^{00} + V_1 X^{10} + V_2 X^{20} + V_3 X^{30} = 4 \\ V_0 X^{01} + V_1 X^{11} + V_2 X^{21} + V_3 X^{31} = -2 \\ V_0 X^{02} + V_1 X^{12} + V_2 X^{22} + V_3 X^{32} = 5 \\ V_0 X^{03} + V_1 X^{13} + V_2 X^{23} + V_3 X^{33} = 7 \ , \end{cases} \end{aligned}$$

or:

$$V_\mu X^{\mu\nu} = (4, -2, 5, 7)$$

---

3 Covariant Derivative

Let $\boldsymbol{S}$ denote the position vector on an N-dimensional manifold parameterized by generalized coordinates q^1, q^2, q^3, ..., q^N.

3.1 Two Gradients Problem

Given how easily the gradient operator turns a scalar field into a type-1 tensor, it's natural to try the gradient twice to see if a type $(0, 2)$ tensor comes out.

Starting with one gradient

$$\partial_{\nu'} f = \frac{\partial q^\nu}{\partial q^{\nu'}} \partial_\nu f \ ,$$

apply another gradient $\partial_{\mu'}$ to both sides:

$$\partial_{\mu'}\partial_{\nu'} f = \frac{\partial q^\nu}{\partial q^{\nu'}} \partial_{\mu'}\partial_\nu f + \partial_{\mu'}\left(\frac{\partial q^\nu}{\partial q^{\nu'}}\right)\partial_\nu f$$

Simplifying with the chain rule, the above reads

$$\partial_{\mu'}\partial_{\nu'} f = \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}} \partial_\mu\partial_\nu f + \partial_{\mu'}\left(\frac{\partial q^\nu}{\partial q^{\nu'}}\right)\partial_\nu f \ ,$$

which contains two terms.

Alternatively, the gradient $\partial_\nu f$ can be written V_ν and the above becomes:

$$\partial_{\mu'} V_{\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}} \partial_\mu V_\nu + \partial_{\mu'}\left(\frac{\partial q^\nu}{\partial q^{\nu'}}\right) V_\nu$$

The same pattern arises from the contrvariant version of V. Starting with the derivative $\partial_{\nu'} V^{\mu'}$, use the definition of a tensor on V to write

$$\partial_{\nu'} V^{\mu'} = \partial_{\nu'}\left(\frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu\right) \ ,$$

expanding to, using the chain rule:

$$\partial_{\nu'} V^{\mu'} = \frac{\partial q^\nu}{\partial q^{\nu'}}\frac{\partial q^{\mu'}}{\partial q^\mu}\partial_\nu V^\mu + \frac{\partial q^\nu}{\partial q^{\nu'}}\frac{\partial^2 q^{\mu'}}{\partial q^\nu \partial q^\mu} V^\mu$$

From these calculations we see that two successive gradients do *not* result in a tensor, otherwise the second term would not be present on the right. The non-tensorial terms contain second derivatives of the position q^μ, analogous to an acceleration-like term that points to the center of curvature, which may be outside the tangent space defined by the basis vectors.

Antisymmetrized Gradient

In light of the two-gradient calculation not resulting in a tensor, one could wonder if some configuration of operators gets a the job done, where the non-tensorial term can be somehow subtracted off. Pursuing this, consider a quantity

$$B_{\mu\nu} = \partial\left[_\mu V_\nu\right] = \frac{1}{2}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \ ,$$

and apply the change of coordinates $q^\mu \to q^{\mu'}$:

$$B_{\mu'\nu'} = \partial\left[_{\mu'} V_{\nu'}\right] = \frac{1}{2}\left(\partial_{\mu'} V_{\nu'} - \partial_{\nu'} V_{\mu'}\right)$$

We already know how the first term turns out. As for the second term, swap $\mu' \leftrightarrow \nu'$ and apply a negative sign:

$$-\partial_{\nu'} V_{\mu'} = -\frac{\partial q^\mu}{\partial q^{\nu'}}\frac{\partial q^\nu}{\partial q^{\mu'}}\partial_\mu V_\nu - \partial_{\nu'}\left(\frac{\partial q^\nu}{\partial q^{\mu'}}\right) V_\nu$$

Note that the second term in the above is the exact negative of the troublesome term in $\partial_{\mu'} V_{\nu'}$. Therefore the result $B_{\mu'\nu'}$ reads

$$B_{\mu'\nu'} = \frac{1}{2}\left(\frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\partial_\mu V_\nu - \frac{\partial q^\mu}{\partial q^{\nu'}}\frac{\partial q^\nu}{\partial q^{\mu'}}\partial_\mu V_\nu\right) \ .$$

In the second term, the indices μ, ν can be swapped without changing $B_{\mu'\nu'}$, so

$$\begin{aligned} B_{\mu'\nu'} &= \frac{1}{2}\left(\frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\partial_\mu V_\nu - \frac{\partial q^\nu}{\partial q^{\nu'}}\frac{\partial q^\mu}{\partial q^{\mu'}}\partial_\nu V_\mu\right) \\ &= \left(\frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\right)\frac{1}{2}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \\ &= \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}} B_{\mu\nu} \ , \end{aligned}$$

therefore $B_{\mu\nu} = \partial\left[_{\mu'} V_{\nu'}\right]$ is a type $(0, 2)$ tensor.

3.2 Connection Coefficients

For a given vector $\boldsymbol{V} = V^\mu \boldsymbol{a}_{(\mu)}$, the 'two gradients' problem means that the standard derivative of $\boldsymbol{V}$ is not a tensor. Setting up a similar calculation, write the differential of $\boldsymbol{V}$ as

$$d\boldsymbol{V} = d\left(V^\mu \boldsymbol{a}_{(\mu)}\right) = dV^\mu \boldsymbol{a}_{(\mu)} + V^\mu d\boldsymbol{a}_{(\mu)} \ ,$$

and apply the chain rule to get

$$d\boldsymbol{V} = \frac{\partial V^\mu}{\partial q^\nu} dq^\nu \boldsymbol{a}_{(\mu)} + V^\mu \frac{\partial \boldsymbol{a}_{(\mu)}}{\partial q^\nu} dq^\nu \ .$$

The problematic second term shall be coerced to remain in the $\boldsymbol{a}_{(\mu)}$ basis while remaining proportional to V^μ. For this sleight of hand we need new proportionality constants called *connection coefficients*, denoted Γ, so we take:

$$\frac{\partial \boldsymbol{a}_{(\mu)}}{\partial q^\nu} = \Gamma^\gamma_{\nu\mu} \boldsymbol{a}_{(\gamma)}$$

Transformation Properties

Imposing a change of coordinates, the above is written

$$\frac{\partial \boldsymbol{a}_{(\mu')}}{\partial q^{\nu'}} = \Gamma'^{\gamma'}_{\nu'\mu'} \boldsymbol{a}_{(\gamma')} \,.$$

Substituting what we know about the basis vectors in transformed coordinates, the above is

$$\frac{\partial}{\partial q^{\nu'}} \left(\frac{\partial q^{\alpha}}{\partial q^{\mu'}} \boldsymbol{a}_{(\alpha)} \right) = \Gamma'^{\gamma'}_{\nu'\mu'} \frac{\partial q^{\alpha}}{\partial q^{\gamma'}} \boldsymbol{a}_{(\alpha)} \,.$$

Simplifying carefully, one finds the transformation law for connection coefficients:

$$\Gamma'^{\gamma'}_{\nu'\mu'} = \frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial q^{\beta}}{\partial q^{\mu'}} \frac{\partial q^{\nu}}{\partial q^{\nu'}} \Gamma^{\alpha}_{\nu\beta} + \frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial^2 q^{\alpha}}{\partial q^{\nu'} \partial q^{\mu'}}$$

An alternative to the above can also be derived. Begin with a statement of the chain rule

$$\frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial q^{\alpha}}{\partial q^{\mu'}} = \frac{\partial q^{\gamma'}}{\partial q^{\mu'}} = \delta^{\gamma'}_{\mu'} \,,$$

and differentiate with respect to $q^{\nu'}$, giving

$$\frac{\partial}{\partial q^{\nu'}} \left(\frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \right) \frac{\partial q^{\alpha}}{\partial q^{\mu'}} + \frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial^2 q^{\alpha}}{\partial q^{\nu'} \partial q^{\mu'}} = 0 \,,$$

and use the chain rule to find

$$\frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial^2 q^{\alpha}}{\partial q^{\nu'} \partial q^{\mu'}} = - \frac{\partial q^{\nu}}{\partial q^{\nu'}} \frac{\partial^2 q^{\gamma'}}{\partial q^{\nu} \partial q^{\alpha}} \frac{\partial q^{\alpha}}{\partial q^{\mu'}} \,.$$

Notice the left side of this is present in the transformation law for connection coefficients. Eliminating the common term gives an alternative:

$$\Gamma'^{\gamma'}_{\nu'\mu'} = \frac{\partial q^{\gamma'}}{\partial q^{\alpha}} \frac{\partial q^{\beta}}{\partial q^{\mu'}} \frac{\partial q^{\nu}}{\partial q^{\nu'}} \Gamma^{\alpha}_{\nu\beta} - \frac{\partial q^{\nu}}{\partial q^{\nu'}} \frac{\partial^2 q^{\gamma'}}{\partial q^{\nu} \partial q^{\alpha}} \frac{\partial q^{\alpha}}{\partial q^{\mu'}}$$

There is a tendency for indices to become a mess during derivations. Cleaning up the above using

$$\begin{aligned} \gamma' &\to \alpha' \\ \nu' &\to \beta' \\ \mu' &\to \gamma' \,, \end{aligned}$$

while relabeling dummy indices, the same transformation law reads:

$$\Gamma'^{\alpha'}_{\beta'\gamma'} = \frac{\partial q^{\alpha'}}{\partial q^{\mu}} \frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \Gamma^{\mu}_{\nu\gamma} - \frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \frac{\partial^2 q^{\alpha'}}{\partial q^{\nu} \partial q^{\gamma}}$$

To summarize, we see that Γ is not a tensor, else the second term wouldn't be present. Nonetheless, we now see what needs to happen with Γ with a coordinate transformation.

Torsion

Define the *torsion* T as:

$$T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$$

Despite Γ not being a tensor, it turns out the torsion satisfies the definition of an asymmetric type $(1,2)$ tensor. Using the transformation law, we write:

$$\begin{aligned} T'^{\alpha'}_{\beta'\gamma'} &= \Gamma'^{\alpha'}_{\beta'\gamma'} - \Gamma'^{\alpha'}_{\gamma'\beta'} \\ &= \frac{\partial q^{\alpha'}}{\partial q^{\mu}} \frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \Gamma^{\mu}_{\nu\gamma} - \frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \frac{\partial^2 q^{\alpha'}}{\partial q^{\nu} \partial q^{\gamma}} \\ &\quad - \frac{\partial q^{\alpha'}}{\partial q^{\mu}} \frac{\partial q^{\nu}}{\partial q^{\gamma'}} \frac{\partial q^{\gamma}}{\partial q^{\beta'}} \Gamma^{\mu}_{\nu\gamma} + \frac{\partial q^{\nu}}{\partial q^{\gamma'}} \frac{\partial q^{\gamma}}{\partial q^{\beta'}} \frac{\partial^2 q^{\alpha'}}{\partial q^{\nu} \partial q^{\gamma}} \end{aligned}$$

The second-order terms cancel out from the above, leaving

$$T'^{\alpha'}_{\beta'\gamma'} = \frac{\partial q^{\alpha'}}{\partial q^{\mu}} \left(\frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \Gamma^{\mu}_{\nu\gamma} - \frac{\partial q^{\nu}}{\partial q^{\gamma'}} \frac{\partial q^{\gamma}}{\partial q^{\beta'}} \Gamma^{\mu}_{\nu\gamma} \right) \,,$$

where swapping the indices ν, γ in the last term finishes the hard work

$$T'^{\alpha'}_{\beta'\gamma'} = \frac{\partial q^{\alpha'}}{\partial q^{\mu}} \frac{\partial q^{\nu}}{\partial q^{\beta'}} \frac{\partial q^{\gamma}}{\partial q^{\gamma'}} \left(\Gamma^{\mu}_{\nu\gamma} - \Gamma^{\mu}_{\gamma\nu} \right) \,,$$

as the parenthesized pair is just $T^{\mu}_{\nu\gamma}$ again.

3.3 Covariant Derivative

In terms of connection coefficients, the differential

$$d\boldsymbol{V} = \frac{\partial V^{\mu}}{\partial q^{\nu}} dq^{\nu} \boldsymbol{a}_{(\mu)} + V^{\mu} \frac{\partial \boldsymbol{a}_{(\mu)}}{\partial q^{\nu}} dq^{\nu}$$

of the vector $\boldsymbol{V} = V^{\mu} \boldsymbol{a}_{(\mu)}$ becomes

$$d\boldsymbol{V} = \frac{\partial V^{\mu}}{\partial q^{\nu}} dq^{\nu} \boldsymbol{a}_{(\mu)} + \Gamma^{\gamma}_{\nu\mu} V^{\mu} dq^{\nu} \boldsymbol{a}_{(\gamma)} \,.$$

Swap the dummy indices μ, γ on the right and simplify to find

$$d\boldsymbol{V} = \left(\partial_{\nu} V^{\mu} + \Gamma^{\mu}_{\nu\gamma} V^{\gamma} \right) dq^{\nu} \boldsymbol{a}_{(\mu)} \,.$$

We've uncovered a new formula for 'derivative' by replacing the non-tensorial term with the one involving the connection coefficients. For this we use the term *covariant derivative*, denoted D_{ν}, as

$$D_{\nu} \boldsymbol{V} = \left(\partial_{\nu} V^{\mu} + \Gamma^{\mu}_{\nu\gamma} V^{\gamma} \right) \boldsymbol{a}_{(\mu)} \,.$$

The parenthesized quantity expresses the covariant derivative acting on the contravariant vector V^{μ} such that

$$D_{\nu} V^{\mu} = \partial_{\nu} V^{\mu} + \Gamma^{\mu}_{\nu\gamma} V^{\gamma} \,,$$

or in tighter notation

$$D_{\nu} \boldsymbol{V} = \left(D_{\nu} V^{\mu} \right) \boldsymbol{a}_{(\mu)} \,,$$

and furthermore:

$$D_{\nu} V^{\mu} = \boldsymbol{a}^{(\mu)} \cdot D_{\nu} \boldsymbol{V}$$

Linearity

By construction, the covariant derivative is a linear operator, obeying

$$D_\alpha \left(A^\mu + B^\nu\right) = D_\alpha A^\mu + D_\alpha B^\nu \,,$$

and also the product rule:

$$D_\alpha \left(A^\mu B^\nu\right) = \left(D_\alpha A^\mu\right) B^\nu + A^\nu \left(D_\alpha B^\mu\right)$$

Alternate Motivation

The same operator can be motivated starting with the dual vector $\boldsymbol{V}^T = V_\mu \boldsymbol{a}^{(\mu)}$, with subsequent steps essentially the same. As a precaution we'll use a different symbol $\tilde{\Gamma}$ for the connection coefficients:

$$D_\nu V_\mu = \partial_\nu V_\mu + \tilde{\Gamma}^\gamma_{\nu\mu} V_\gamma$$

Scalar Field Derivative

A scalar field is equivalent to a type $(0,0)$ tensor $f = A^\mu B_\mu$. Calculating the covariant derivative of f, we find:

$$\begin{aligned} D_\nu f &= D_\nu \left(A^\mu B_\mu\right) \\ &= \left(D_\nu A^\mu\right) B_\mu + A_\mu \left(D_\nu B^\mu\right) \\ &= \left(\partial_\nu A^\mu + \Gamma^\mu_{\nu\gamma} A^\gamma\right) B_\mu + A^\mu \left(\partial_\nu B_\mu + \tilde{\Gamma}^\gamma_{\nu\mu} B_\gamma\right) \\ &= \partial_\nu f + \Gamma^\mu_{\nu\gamma} A^\gamma B_\mu + \tilde{\Gamma}^\gamma_{\nu\mu} A^\mu B_\gamma \\ &= \partial_\nu f + A^\gamma B_\mu \left(\Gamma^\mu_{\nu\gamma} + \tilde{\Gamma}^\mu_{\nu\gamma}\right) \end{aligned}$$

As a matter of technicality, the derivative of a scalar field does not involve connection coefficients and only $\partial_\mu f$ survives. With this we conclude that $\tilde{\Gamma}$ is the negative of the original Γ as

$$\Gamma^\mu_{\nu\gamma} = -\tilde{\Gamma}^\mu_{\nu\gamma} \,,$$

letting us rewrite:

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\gamma_{\nu\mu} V_\gamma$$

Tensor Field Derivative

Consider a type $(2,0)$ tensor $C^{\mu\nu}$ that is the tensor product $C^{\mu\nu} = A^\mu B^\nu$. Calculating the covariant derivative of C, we find

$$\begin{aligned} D_\alpha \left(C^{\mu\nu}\right) &= D_\alpha \left(A^\mu B^\nu\right) \\ &= \left(D_\alpha A^\mu\right) B^\nu + A^\mu \left(D_\alpha B^\nu\right) \\ &= \partial_\alpha \left(A^\mu B^\nu\right) + \Gamma^\mu_{\alpha\gamma} A^\gamma B^\nu + \Gamma^\nu_{\alpha\gamma} A^\mu B^\gamma \,, \end{aligned}$$

or in summary:

$$D_\alpha C^{\mu\nu} = \partial_\alpha C^{\mu\nu} + \Gamma^\mu_{\alpha\gamma} C^{\gamma\nu} + \Gamma^\nu_{\alpha\gamma} C^{\mu\gamma}$$

By the same token, one can calculate the covariant derivative of the type $(0,2)$ tensor $C_{\mu\nu} = A_\mu B_\nu$ by a similar process, giving:

$$D_\alpha C_{\mu\nu} = \partial_\alpha C_{\mu\nu} - \Gamma^\gamma_{\alpha\mu} C_{\gamma\nu} - \Gamma^\gamma_{\alpha\nu} C_{\mu\gamma}$$

The same pattern continues for more contravariant or covariant indices. For each contravariant index, include a term proportional to $+\Gamma$. For each covariant index, include a term proportional to $-\Gamma$. Mixed cases apply both rules, for instance:

$$D_\alpha C^\mu_\nu = \partial_\alpha C^\mu_\nu + \Gamma^\mu_{\alpha\gamma} C^\gamma_\nu - \Gamma^\gamma_{\alpha\nu} C^\mu_\gamma$$

Tensor Qualification

For $D_\nu V^\mu$ to qualify as a tensor, we require

$$D_{\nu'} V^{\mu'} = \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} D_\nu V^\mu \,,$$

which is calculated by brute force:

$$\begin{aligned} D_{\nu'} V^{\mu'} &= \partial_{\nu'} V^{\mu'} + \Gamma'^{\mu'}_{\nu'\gamma'} V^{\gamma'} \\ &= \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \partial_\nu V^\mu \\ &\quad + \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial^2 q^{\mu'}}{\partial q^\nu \partial q^\gamma} V^\gamma + \Gamma'^{\mu'}_{\nu'\gamma'} \frac{\partial q^{\gamma'}}{\partial q^\gamma} V^\gamma \end{aligned}$$

Proceed by factoring $\partial q^\nu / \partial q^{\nu'}$ and $\partial q^{\mu'} / \partial q^\mu$ from each term using the chain rule. Working this out, we find

$$\begin{aligned} D_{\nu'} V^{\mu'} &= \partial_{\nu'} V^{\mu'} + \Gamma'^{\mu'}_{\nu'\gamma'} V^{\gamma'} \\ &= \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \partial_\nu V^\mu \\ &\quad + \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \left(\frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial^2 q^{\alpha'}}{\partial q^\nu \partial q^\gamma}\right) V^\gamma \\ &\quad + \frac{\partial q^\nu}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^\mu} \left(\Gamma'^{\alpha'}_{\beta'\gamma'} \frac{\partial q^{\beta'}}{\partial q^\nu} \frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial q^{\gamma'}}{\partial q^\gamma}\right) V^\gamma \,. \end{aligned}$$

In order to satisfy the definition of a tensor, the two parenthesized terms must sum to $\Gamma^\mu_{\nu\gamma}$, or:

$$\Gamma^\mu_{\nu\gamma} = \frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial^2 q^{\alpha'}}{\partial q^\nu \partial q^\gamma} + \Gamma'^{\alpha'}_{\beta'\gamma'} \frac{\partial q^{\beta'}}{\partial q^\nu} \frac{\partial q^\mu}{\partial q^{\alpha'}} \frac{\partial q^{\gamma'}}{\partial q^\gamma}$$

Rearrange to isolate the Γ with the primed indices:

$$\Gamma'^{\alpha'}_{\beta'\gamma'} = \frac{\partial q^{\alpha'}}{\partial q^\mu} \frac{\partial q^\nu}{\partial q^{\beta'}} \frac{\partial q^\gamma}{\partial q^{\gamma'}} \Gamma^\mu_{\nu\gamma} - \frac{\partial q^\nu}{\partial q^{\beta'}} \frac{\partial q^\gamma}{\partial q^{\gamma'}} \frac{\partial^2 q^{\alpha'}}{\partial q^\nu \partial q^\gamma}$$

This result exactly matches the transformation law for the connection coefficients, finishing the proof.

3.4 Levi-Civita Connection

Local Torsion

Having established the torsion tensor

$$T^{\gamma}_{\nu\mu} = \Gamma^{\gamma}_{\nu\mu} - \Gamma^{\gamma}_{\mu\nu} \,,$$

let us now choose any point P on the manifold and examine what happens in a neighborhood local to P. Presuming the manifold is differentiable, it follows that the metric evaluated at a point corresponds to a 'flat' space:

$$\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)}\Big|_P = g_{\mu\nu}|_P = \eta_{\mu\nu}$$

Moreover, second derivatives of the basis vectors are zero, thus

$$\frac{\partial \boldsymbol{a}_{(\mu)}}{\partial q^{\nu}}\bigg|_P = \Gamma^{\gamma}_{\nu\mu}\boldsymbol{a}_{(\gamma)}\bigg|_P = 0 \,,$$

which is to say the connection coefficients at P are zero to first order.

Restating the connection coefficient transformation law

$$\Gamma'^{\gamma'}_{\nu'\mu'} = \frac{\partial q^{\gamma'}}{\partial q^{\alpha}}\frac{\partial q^{\beta}}{\partial q^{\mu'}}\frac{\partial q^{\nu}}{\partial q^{\nu'}}\Gamma^{\alpha}_{\nu\beta} + \frac{\partial q^{\gamma'}}{\partial q^{\alpha}}\frac{\partial^2 q^{\alpha}}{\partial q^{\nu'}\partial q^{\mu'}} \,,$$

choose indices α, ν, β such that Γ is local to P. Then, in transformed coordinates, we find:

$$\Gamma'^{\gamma'}_{\nu'\mu'}\bigg|_P = \frac{\partial q^{\gamma'}}{\partial q^{\alpha}}\frac{\partial^2 q^{\alpha}}{\partial q^{\nu'}\partial q^{\mu'}}$$

Due to the symmetry in the partial derivatives above, there is symmetry in the lower indices on Γ:

$$\Gamma'^{\gamma'}_{\nu'\mu'}\bigg|_P = \Gamma'^{\gamma'}_{\mu'\nu'}\bigg|_P$$

This immediately means that the torsion at point P is zero.

Zero Torsion

The *Levi-Civita connection* has us presume that torsion is zero not only at point P, but also everywhere on the manifold. By this, the connection coefficients are symmetric in the two lower indices:

$$\Gamma^{\gamma}_{\nu\mu} = \Gamma^{\gamma}_{\mu\nu}$$

A more direct derivation of the above begins with

$$\frac{\partial \boldsymbol{a}_{(\mu)}}{\partial q^{\nu}} = \Gamma^{\gamma}_{\nu\mu}\boldsymbol{a}_{(\gamma)} \,,$$

and then distribute $\boldsymbol{a}^{(\gamma)}$ into both sides to get a formula for Γ:

$$\Gamma^{\gamma}_{\nu\mu} = \boldsymbol{a}^{(\gamma)} \cdot \frac{\partial}{\partial q^{\nu}}\frac{\partial}{\partial q^{\mu}}\boldsymbol{S} = \boldsymbol{a}^{(\gamma)} \cdot \partial_{\nu}\boldsymbol{a}_{(\mu)}$$

Then, swapping the indices ν, μ should leave the result unchanged, and the same symmetry in the lower indices on Γ is implied.

Problem 4

If $A_{\mu\nu}$ is an antisymmetric type $(0,2)$ tensor in the absense of torsion, show that all connection coefficients cancel out of $D\left[_{\mu}A_{\nu\sigma}\right]$. Answer:

$$\begin{aligned} D\left[_{\mu}A_{\nu\sigma}\right] &= \frac{1}{3!}\cdot\begin{cases} D_{\mu}A_{\nu\sigma} + D_{\nu}A_{\sigma\mu} + D_{\sigma}A_{\mu\nu} \\ -D_{\mu}A_{\sigma\nu} - D_{\nu}A_{\mu\sigma} - D_{\sigma}A_{\nu\mu} \end{cases} \\ &= \frac{1}{3}\left(D_{\mu}A_{\nu\sigma} + D_{\nu}A_{\sigma\mu} + D_{\sigma}A_{\mu\nu}\right) \end{aligned}$$

$$\begin{aligned} D\left[_{\mu}A_{\nu\sigma}\right] &= \partial_{\mu}A_{\nu\sigma} - \Gamma^{\gamma}_{\mu\nu}A_{\gamma\sigma} - \Gamma^{\gamma}_{\mu\sigma}A_{\nu\gamma} \\ &\quad + \partial_{\nu}A_{\sigma\mu} - \Gamma^{\gamma}_{\nu\sigma}A_{\gamma\mu} - \Gamma^{\gamma}_{\nu\mu}A_{\sigma\gamma} \\ &\quad + \partial_{\sigma}A_{\mu\nu} - \Gamma^{\gamma}_{\sigma\mu}A_{\gamma\nu} - \Gamma^{\gamma}_{\sigma\nu}A_{\mu\gamma} \\ &= \partial\left[_{\mu}A_{\nu\sigma}\right] \end{aligned}$$

Problem 5

Consider a given point p on the manifold, along with a nearby point p'. Write the first-order approximation to $\partial_{\sigma}\Gamma^{\gamma}_{\mu\nu}$. Answer:

$$\partial_{\sigma}\Gamma^{\gamma}_{\mu\nu} = \frac{\Gamma^{\gamma}_{\mu\nu}\left(q^{\mu}\left(p'\right)\right) - \Gamma^{\gamma}_{\mu\nu}\left(q^{\mu}\left(p\right)\right)}{q^{\sigma}\left(p'\right) - q^{\sigma}\left(p\right)}$$

Metric Compatibility

To remove some abstraction between the covariant derivative and the connection coefficients, the Levi-Civita connection the foundational statement that the covariant derivative of the metric is identically zero:

$$D_{\sigma}g_{\mu\nu} = 0$$

The same is true in contravariant form, i.e. take the derivative of $g^{\mu\nu}g_{\nu\beta} = \delta^{\mu}_{\beta}$ to derive:

$$D_{\sigma}g^{\mu\nu} = 0$$

The zero-derivative of the metric is called *metric compatibility*. Quantitatively, we can apply the tensor field derivative to $g_{\mu\nu}$ using the formula

$$D_{\sigma}C_{\mu\nu} = \partial_{\sigma}C_{\mu\nu} - \Gamma^{\gamma}_{\sigma\mu}C_{\gamma\nu} - \Gamma^{\gamma}_{\sigma\nu}C_{\mu\gamma}$$

to derive the metric compatibility condition:

$$0 = \partial_{\sigma}g_{\mu\nu} - \Gamma^{\gamma}_{\sigma\mu}g_{\gamma\nu} - \Gamma^{\gamma}_{\sigma\nu}g_{\mu\gamma}$$

Two more copies of the compatibility condition arise by cycling the indices σ, μ, ν:

$$\begin{aligned} 0 &= \partial_{\mu}g_{\nu\sigma} - \Gamma^{\gamma}_{\mu\nu}g_{\gamma\sigma} - \Gamma^{\gamma}_{\mu\sigma}g_{\nu\gamma} \\ 0 &= \partial_{\nu}g_{\sigma\mu} - \Gamma^{\gamma}_{\nu\sigma}g_{\gamma\mu} - \Gamma^{\gamma}_{\nu\mu}g_{\sigma\gamma} \end{aligned}$$

Next, subtract the latter two equations from the first, keeping in mind the symmetry of the metric and of the connection coefficients, to find

$$0 = \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\Gamma^\gamma_{\mu\nu} g_{\sigma\gamma} \,,$$

which has only one instance of Γ. Solving for this gives a formula for the connection coefficients in terms of the metric:

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right)$$

Problem 6

Use the metric compaibility condition to calculate all nonzero connection coefficients for the Rindler metric characterized by

$$dS^2 = -c^2 d\tau^2 \left(1 + \frac{\xi \tilde{a}}{c^2} \right)^2 + d\xi^2 \,.$$

Answer:

$$\Gamma^\xi_{\tau\tau} = \frac{\tilde{a}}{c^2} \left(1 + \frac{\xi \tilde{a}}{c^2} \right)$$
$$\Gamma^\tau_{\xi\tau} = \frac{\tilde{a}/c^2}{1 + \xi\tilde{a}/c^2}$$

Problem 7

Use the metric compaibility condition to calculate all nonzero connection coefficients for the Friedman-Robertson-Walker metric characterized by

$$dS^2 = -c^2 dt^2 + a^2(t)\, \delta_{jk} dq^j dq^k \,.$$

Answer:

$$\Gamma^t_{jk} = a \frac{da}{dt} \delta_{jk}$$
$$\Gamma^j_{tk} = \frac{1}{a} \frac{da}{dt} \delta_{jk}$$

3.5 Vector Calculus Operators

Del Operator

While introducing the gradient, it's customary to reach for the Del notation via vector calculus:

$$\boldsymbol{\nabla} f = \boldsymbol{a}^{(\mu)} \frac{\partial f}{\partial q^\mu}$$

In light of the covariant derivative however, it's helpful to reassign the $\boldsymbol{\nabla}$ symbol as:

$$\boldsymbol{\nabla} = \boldsymbol{a}^{(\mu)} D_\mu$$

This does no harm to the gradient of the scalar function f. Operating on a scalar function, ∂_ν is the same as D_ν.

Divergence

Recall that for a vector field V, the covariant derivative is given by

$$D_\nu \boldsymbol{V} = \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\gamma} V^\gamma \right) \boldsymbol{a}_{(\mu)} \,.$$

Letting $\nu = \mu$ and distributing $\boldsymbol{a}^{(\mu)}$ into each side, we calculate the *divergence* of the field:

$$\boldsymbol{a}^{(\mu)} \cdot D_\mu \boldsymbol{V} = \left(\partial_\mu V^\mu + \Gamma^\mu_{\mu\gamma} V^\gamma \right) = D_\mu V^\mu$$

In terms of the Del operator, the divergence is written:

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \boldsymbol{a}^{(\mu)} \cdot D_\mu \boldsymbol{V}$$

Laplacian

For a scalar field $f\,(q^\mu)$, the Laplacian operator $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla^2 = \Delta$ involves two derivatives and must return a type $(0,0)$ tensor. In terms of the covariant derivative, this is simply:

$$\nabla^2 f = D^\mu D_\mu f = g^{\mu\nu} D_\nu D_\mu f$$

Levi-Civita Symbol

The cross product between two vectors is generalized to N dimensions using the *Levi-Civita* symbol $\epsilon_{ijk\ldots}$. If the indices $i,\ j,\ k$ are an even permutation of the sequence $1, 2, 3$, then $\epsilon = 1$. For odd permutations, $\epsilon = -1$. For any two equal indices, $\epsilon = 0$. Note ϵ_{ijk} is *not* a tensor and ignores the up- or down-placement of indices. To generalize to higher dimensions, add more indices next to $i,\ j,\ k$.

For the standard three-dimensional cross product, one writes, for two vectors $\boldsymbol{U}$, $\boldsymbol{V}$:

$$(\boldsymbol{U} \times \boldsymbol{V})^i = \epsilon_{ijk} U^j V^k$$

Curl

In terms of the Levi-Civita symbol, the ith component of the curl of a vector $\boldsymbol{V}$ reads

$$(\boldsymbol{\nabla} \times \boldsymbol{V})^i = \epsilon_{ijk} D_j V^k \,,$$

or, for the full curl:

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \left(\epsilon_{ijk} D_j V^k \right) \boldsymbol{a}_{(i)}$$

Alternatively, one can begin with the operator

$$\boldsymbol{\nabla} = \boldsymbol{a}^{(\mu)} D_\mu \,,$$

and the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \boldsymbol{a}^{(\mu)} D_\mu \times \boldsymbol{a}_{(\nu)} V^\nu \,,$$

yielding a curious formula:

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \left(\boldsymbol{a}^{(\nu)} \times \boldsymbol{a}_{(\mu)} \right) D_\nu V^\mu$$

Jacobian

In terms of Levi-Civita-like symbols, the determinant of the metric, which is the square of the Jacobian, is given by

$$\det g = \frac{1}{n!}\epsilon^{i_1 i_2 \cdots i_n}\epsilon^{j_1 j_2 \cdots j_n} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_n j_n} .$$

For three dimensions, the formula calms down a little:

$$\det g = \frac{1}{6}\epsilon^{ijk}\epsilon^{pqr} g_{ip} g_{jq} g_{kr}$$

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4 Flat Manifolds

4.1 Cartesian Coordinates

The Cartesian coordinate system in three dimensions is characterized by the position vector

$$\boldsymbol{S} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}\,.$$

The corresponding differential line element is

$$d\boldsymbol{S} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}\,,$$

with differential interval

$$dS^2 = dx^2 + dy^2 + dz^2\,.$$

In the language of tensors, assign the generalized coordinates $q^x = x$, $q^y = y$, $q^z = z$, and the above means

$$dS^2 = dq^\mu dq_\mu = g_{\mu\nu} dq^\nu dq_\mu\,,$$

with the corresponding metric

$$g_{\mu\nu} = \eta_{\mu\nu}\,,$$

which is the Minkowski metric without the 0-component:

$$\eta_{11} = \eta_{22} = \eta_{33} = 1$$
$$\eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu$$

Cartesian coordinates correspond to a flat manifold with no surprises. The basis vectors $\boldsymbol{a}_{(\mu)}$ are fixed, and thus there are no connection coefficients Γ.

4.2 Cylindrical Coordinates

Starting with the Cartesian coordinate system, make the change of coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

with

$$\hat{\boldsymbol{r}} = \cos(\theta)\,\hat{\boldsymbol{x}} + \sin(\theta)\,\hat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{\theta}} = -\sin(\theta)\,\hat{\boldsymbol{x}} + \cos(\theta)\,\hat{\boldsymbol{y}}\,.$$

With this, the position vector in cylindrical coordinates is

$$\boldsymbol{S} = r\,\hat{\boldsymbol{r}} + z\,\hat{\boldsymbol{z}}\,.$$

Line Element and Basis

From the position vector $\boldsymbol{S}$, the differential line element reads

$$d\boldsymbol{S} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\boldsymbol{z}}\,,$$

and right away we can write the differential interval:

$$dS^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Choosing the generalized coordinates $q^r = r$, $q^\theta = \theta$, $q^z = z$, we pick out the basis vectors

$$\boldsymbol{a}_{(r)} = \hat{\boldsymbol{r}}$$
$$\boldsymbol{a}_{(\theta)} = r\,\hat{\boldsymbol{\theta}}$$
$$\boldsymbol{a}_{(z)} = \hat{\boldsymbol{z}}\,.$$

Note that the coordinates q^μ themselves aren't strictly limited to having units of length. The term $q^\theta = \theta$ is indeed dimensionless, however the basis vector $\boldsymbol{a}_{(\theta)}$ has units of length.

Using $\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}^{(\nu)} = \delta^\nu_\mu$, the contravariant basis vectors are

$$\boldsymbol{a}^{(r)} = \hat{\boldsymbol{r}}$$
$$\boldsymbol{a}^{(\theta)} = r^{-1}\,\hat{\boldsymbol{\theta}}$$
$$\boldsymbol{a}^{(z)} = \hat{\boldsymbol{z}}\,.$$

Metric

From the basis vectors $\boldsymbol{a}_{(\mu)}$, all nonzero components of the metric $g_{\mu\nu}$ are straightforwardly calculated:

$$g_{rr} = \boldsymbol{a}_{(r)} \cdot \boldsymbol{a}_{(r)} = 1$$
$$g_{\theta\theta} = \boldsymbol{a}_{(\theta)} \cdot \boldsymbol{a}_{(\theta)} = r^2$$
$$g_{zz} = \boldsymbol{a}_{(z)} \cdot \boldsymbol{a}_{(z)} = 1$$

Similar goes for $g^{\mu\nu}$:

$$g^{rr} = \boldsymbol{a}^{(r)} \cdot \boldsymbol{a}^{(r)} = 1$$
$$g^{\theta\theta} = \boldsymbol{a}^{(\theta)} \cdot \boldsymbol{a}^{(\theta)} = r^{-2}$$
$$g^{zz} = \boldsymbol{a}^{(z)} \cdot \boldsymbol{a}^{(z)} = 1$$

Alternatively, or perhaps more elegantly, the same metric can be calculated by transforming the Cartesian metric:

$$g_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} \frac{\partial q^\nu}{\partial q^{\nu'}} \eta_{\mu\nu}$$

In the above, the primed indices refer to parameters r, θ, z, whereas the unprimed indices are for x, y, and z. For example, one finds for $g_{\theta\theta}$:

$$\begin{aligned} g_{\theta\theta} &= \frac{\partial q^x}{\partial q^\theta}\frac{\partial q^x}{\partial q^\theta} + \frac{\partial q^y}{\partial q^\theta}\frac{\partial q^y}{\partial q^\theta} \\ &= \left(\sin^2(\theta) + \cos^2(\theta)\right) r^2 \\ &= r^2 \end{aligned}$$

Connection

Most of the connection coefficients for cylindrical coordinates are zero. Using either of

$$\Gamma^{\gamma}_{\mu\nu} = \boldsymbol{a}^{(\gamma)} \cdot \partial_\mu \boldsymbol{a}_{(\nu)} \,,$$

$$\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2} g^{\gamma\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \,,$$

one calculates the only nonzero Γ to be:

$$\begin{aligned} \Gamma^{\theta}_{r\theta} &= \frac{1}{r} \\ \Gamma^{r}_{\theta\theta} &= -r \end{aligned}$$

Gradient

For a scalar function $f = f(r, \theta, z)$, the gradient is:

$$\boldsymbol{\nabla} f = \boldsymbol{a}^{(\mu)} \frac{\partial f}{\partial q^\mu} = \frac{\partial f}{\partial r} \hat{\boldsymbol{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z} \hat{\boldsymbol{z}}$$

Normalized Components

When dealing with vectors $\boldsymbol{V} = V^\mu \boldsymbol{a}_{(\mu)}$, notice that the basis vectors are not unit-normalized. Often it's convenient to work in such a system, in which case $\boldsymbol{V}$ is written

$$\boldsymbol{V} = v^\mu \hat{\boldsymbol{a}}_{(\mu)} \,,$$

where the contravariant components V^μ are rescaled to v^μ according to

$$\begin{aligned} v^r &= V^r \\ v^\theta &= r V^\theta \\ v^z &= V^z \,. \end{aligned}$$

Divergence

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{V} &= \boldsymbol{a}^{(\mu)} \cdot D_\mu \boldsymbol{V} = D_\mu V^\mu \\ &= \partial_r V^r + \left(\partial_\theta V^\theta + \Gamma^{\theta}_{\theta r} V^r \right) + \partial_z V^z \\ &= \frac{1}{r} \partial_r \left(r V^r \right) + \partial_\theta V^\theta + \partial_z V^z \\ &= \frac{1}{r} \partial_r \left(r v^r \right) + \frac{1}{r} \partial_\theta v^\theta + \partial_z v^z \end{aligned}$$

Laplacian

For a scalar function $f = f(r, \theta, z)$, the Laplacian is:

$$\begin{aligned} \nabla^2 f &= g^{\mu\nu} D_\nu D_\mu f = g^{\mu\nu} D_\nu \partial_\mu f \\ &= g^{\mu\nu} \left(\partial_{\nu\mu} f - \Gamma^{\gamma}_{\nu\mu} \partial_\gamma f \right) \\ &= g^{rr} \partial_{rr} f + g^{\theta\theta} \left(\partial_{\theta\theta} f + r \partial_r f \right) + g^{zz} \partial_{zz} f \\ &= \frac{1}{r} \partial_r \left(r \partial_r f \right) + \frac{1}{r^2} \partial_{\theta\theta} f + \partial_{zz} f \end{aligned}$$

Curl

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{V} &= \left(\boldsymbol{a}^{(\nu)} \times \boldsymbol{a}_{(\mu)} \right) D_\nu V^\mu \\ &= \left(\boldsymbol{a}^{(\nu)} \times \boldsymbol{a}_{(\mu)} \right) \left(\partial_\nu V^\mu + \Gamma^{\mu}_{\nu\gamma} V^\gamma \right) \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\nabla} \times \boldsymbol{V})^z &= \frac{1}{r} \left(\partial_r \left(r^2 V^\theta \right) - \partial_\theta V^r \right) \\ &= \frac{1}{r} \left(\partial_r \left(r v^\theta \right) - \partial_\theta v^r \right) \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\nabla} \times \boldsymbol{V})^r &= \frac{1}{r} \partial_\theta V^z - r \partial_z V^\theta \\ &= \frac{1}{r} \partial_\theta v^z - \partial_z v^\theta \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\nabla} \times \boldsymbol{V})^\theta &= \partial_z V^r - \partial_r V^z \\ &= \partial_z v^r - \partial_r v^z \end{aligned}$$

4.3 Spherical Coordinates

Starting again from

$$\boldsymbol{S} = x \, \hat{\boldsymbol{x}} + y \, \hat{\boldsymbol{y}} + z \, \hat{\boldsymbol{z}} \,,$$

we go from Cartesian coordinates to spherical coordinates with the substitution

$$\begin{aligned} x &= r \sin(\theta) \cos(\phi) \\ y &= r \sin(\theta) \sin(\phi) \\ z &= r \cos(\theta) \end{aligned}$$

such that

$$\boldsymbol{S} = r \, \hat{\boldsymbol{r}} \,.$$

Normalized basis vectors are calculated via

$$\hat{\boldsymbol{r}} = \frac{\partial \boldsymbol{S} / \partial r}{|\partial \boldsymbol{S} / \partial r|}$$

and similar for θ and ϕ, resolving to:

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin(\theta) \cos(\phi) \, \hat{\boldsymbol{x}} + \sin(\theta) \sin(\phi) \, \hat{\boldsymbol{y}} + \cos(\theta) \, \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} &= \cos(\theta) \cos(\phi) \, \hat{\boldsymbol{x}} + \cos(\theta) \sin(\phi) \, \hat{\boldsymbol{y}} - \sin(\theta) \, \hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin(\phi) \, \hat{\boldsymbol{x}} + \cos(\phi) \, \hat{\boldsymbol{y}} \end{aligned}$$

Line Element and Basis

The differential line element $d\boldsymbol{S}$ in spherical coordinates is calculated by direct substitution into

$$d\boldsymbol{S} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}\,,$$

resulting in

$$d\boldsymbol{S} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin(\theta)\,d\phi\,\hat{\boldsymbol{\phi}}\,,$$

and subsequently

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\,d\phi^2\,.$$

Choosing the generalized coordinates $q^r = r$, $q^\theta = \theta$, $q^z = \phi$, we pick out the basis vectors

$$\begin{aligned}\boldsymbol{a}_{(r)} &= \hat{\boldsymbol{r}}\\ \boldsymbol{a}_{(\theta)} &= r\,\hat{\boldsymbol{\theta}}\\ \boldsymbol{a}_{(\phi)} &= r\sin(\theta)\,\hat{\boldsymbol{\phi}}\,.\end{aligned}$$

Using $\boldsymbol{a}_{(\mu)}\cdot\boldsymbol{a}^{(\nu)} = \delta^\nu_\mu$, the contravariant basis vectors are

$$\begin{aligned}\boldsymbol{a}^{(r)} &= \hat{\boldsymbol{r}}\\ \boldsymbol{a}^{(\theta)} &= \frac{1}{r}\,\hat{\boldsymbol{\theta}}\\ \boldsymbol{a}^{(\phi)} &= \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\phi}}\,.\end{aligned}$$

Metric

From the basis vectors $\boldsymbol{a}_{(\mu)}$, all nonzero components of the metric $g_{\mu\nu}$ are straightforwardly calculated:

$$\begin{aligned}g_{rr} &= \boldsymbol{a}_{(r)}\cdot\boldsymbol{a}_{(r)} = 1\\ g_{\theta\theta} &= \boldsymbol{a}_{(\theta)}\cdot\boldsymbol{a}_{(\theta)} = r^2\\ g_{\phi\phi} &= \boldsymbol{a}_{(\phi)}\cdot\boldsymbol{a}_{(\phi)} = r^2\sin^2(\theta)\end{aligned}$$

Similar goes for $g^{\mu\nu}$:

$$\begin{aligned}g^{rr} &= \boldsymbol{a}^{(r)}\cdot\boldsymbol{a}^{(r)} = 1\\ g^{\theta\theta} &= \boldsymbol{a}^{(\theta)}\cdot\boldsymbol{a}^{(\theta)} = \frac{1}{r^2}\\ g^{\phi\phi} &= \boldsymbol{a}^{(\phi)}\cdot\boldsymbol{a}^{(\phi)} = \frac{1}{r^2\sin^2(\theta)}\end{aligned}$$

Alternatively, the same metric can be calculated by transforming the Cartesian metric:

$$g_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\eta_{\mu\nu}$$

In the above, the primed indices refer to parameters r, θ, ϕ, whereas the unprimed indices are for x, y, and z.

Connection

Using either of

$$\Gamma^\gamma_{\mu\nu} = \boldsymbol{a}^{(\gamma)}\cdot\partial_\mu\boldsymbol{a}_{(\nu)}\,,$$

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)\,,$$

one calculates the nonzero Γ to be:

$$\begin{aligned}\Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}\\ \Gamma^r_{\theta\theta} &= -r\\ \Gamma^r_{\phi\phi} &= -r\sin^2(\theta)\\ \Gamma^\theta_{\phi\phi} &= -\sin(\theta)\cos(\theta)\\ \Gamma^\phi_{\phi\theta} &= \cot(\theta)\end{aligned}$$

Gradient

For a scalar function $f = f(r,\theta,\phi)$, the gradient is:

$$\boldsymbol{\nabla} f = \boldsymbol{a}^{(\mu)}\frac{\partial f}{\partial q^\mu} = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Normalized Components

Reconciliation between the non-normalized vector $\boldsymbol{V} = V^\mu\boldsymbol{a}_{(\mu)}$ and its normalized counterpart $\boldsymbol{V} = v^\mu\hat{\boldsymbol{a}}_{(\mu)}$ involves transferring the 'extra' coefficients from $\boldsymbol{a}_{(\mu)}$ to v^μ as:

$$\begin{aligned}v^r &= V^r\\ v^\theta &= rV^\theta\\ v^\phi &= r\sin(\theta)\,V^\phi\end{aligned}$$

Divergence

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{V} &= \boldsymbol{a}^{(\mu)}\cdot D_\mu\boldsymbol{V} = D_\mu V^\mu\\ &= \partial_r V^r + \partial_\theta V^\theta + \Gamma^\theta_{\theta r}V^r\\ &\quad + \partial_\phi V^\phi + \Gamma^\phi_{\phi r}V^r + \Gamma^\phi_{\phi\theta}V^\theta\\ &= \frac{\partial_r\left(r^2 v^r\right)}{r^2} + \frac{\partial_\theta\left(\sin(\theta)\,v^\theta\right)}{r\sin(\theta)} + \frac{\partial_\phi\left(v^\phi\right)}{r\sin(\theta)}\end{aligned}$$

Laplacian

For a scalar function $f = f(r,\theta,z)$, the Laplacian is:

$$\begin{aligned}\nabla^2 f &= g^{\mu\nu}D_\nu D_\mu f = g^{\mu\nu}D_\nu\partial_\mu f\\ &= g^{\mu\nu}\left(\partial_{\nu\mu}f - \Gamma^\gamma_{\nu\mu}\partial_\gamma f\right)\\ &= g^{rr}\partial_r f + g^{\theta\theta}\left(\partial_{\theta\theta}f - \Gamma^r_{\theta\theta}\partial_r f\right)\\ &\quad + g^{\phi\phi}\left(\partial_{\phi\phi} - \Gamma^r_{\phi\phi}\partial_r f - \Gamma^\theta_{\phi\phi}\partial_\theta f\right)\\ &= \frac{\partial_r\left(r^2\partial_r f\right)}{r^2} + \frac{\partial_\theta\left(\sin(\theta)\,\partial_\theta f\right)}{r^2\sin(\theta)} + \frac{\partial_{\phi\phi}f}{r^2\sin^2(\theta)}\end{aligned}$$

Curl

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{V} &= \left(\boldsymbol{a}^{(\nu)}\times\boldsymbol{a}_{(\mu)}\right)D_\nu V^\mu \\ &= \left(\boldsymbol{a}^{(\nu)}\times\boldsymbol{a}_{(\mu)}\right)\left(\partial_\nu V^\mu+\Gamma^\mu_{\nu\gamma}V^\gamma\right)\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{V})^\phi &= r\partial_r V^\theta + 2V^\theta - \frac{1}{r}\partial_\theta V^r \\ &= \frac{1}{r}\left(\partial_r\left(rv^\theta\right)-\partial_\theta v^r\right)\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{V})^r &= \sin(\theta)\,\partial_\theta V^\phi + 2\cos(\theta)\,V^\phi - \frac{\partial_\phi V^\theta}{\sin(\theta)} \\ &= \frac{1}{r\sin(\theta)}\left(\partial_\theta\left(\sin(\theta)\,v^\phi\right)-\partial_\phi v^\theta\right)\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{V})^\theta &= \frac{\partial_\phi V^r}{r\sin(\theta)} - 2\sin(\theta)\,V^\phi \\ &\quad - r\sin(\theta)\,\partial_r V^\phi \\ &= \frac{1}{r}\left(\frac{\partial_\phi v^r}{\sin(\theta)}-\partial_r\left(rv^\phi\right)\right)\end{aligned}$$

4.4 Parabolic Coordinates 2D

Consider the two-dimensional Cartesian flat space with position vector

$$\boldsymbol{S} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}$$

with the substitution

$$\begin{aligned}x &= uv \\ y &= \frac{1}{2}\left(u^2-v^2\right)\,,\end{aligned}$$

called *parabolic coordinates*.

Normalized basis vectors are calculated via

$$\begin{aligned}\hat{\boldsymbol{u}} &= \frac{\partial\boldsymbol{S}/\partial u}{|\partial\boldsymbol{S}/\partial u|} = \frac{v\,\hat{\boldsymbol{x}}+u\,\hat{\boldsymbol{y}}}{\sqrt{u^2+v^2}} \\ \hat{\boldsymbol{v}} &= \frac{\partial\boldsymbol{S}/\partial u}{|\partial\boldsymbol{S}/\partial u|} = \frac{u\,\hat{\boldsymbol{x}}-v\,\hat{\boldsymbol{y}}}{\sqrt{u^2+v^2}}\,,\end{aligned}$$

which can be inverted to say

$$\begin{aligned}\hat{\boldsymbol{x}} &= \frac{\partial\boldsymbol{S}/\partial u}{|\partial\boldsymbol{S}/\partial u|} = \frac{v\,\hat{\boldsymbol{u}}+u\,\hat{\boldsymbol{v}}}{\sqrt{u^2+v^2}} \\ \hat{\boldsymbol{y}} &= \frac{\partial\boldsymbol{S}/\partial u}{|\partial\boldsymbol{S}/\partial u|} = \frac{u\,\hat{\boldsymbol{u}}-v\,\hat{\boldsymbol{v}}}{\sqrt{u^2+v^2}}\,.\end{aligned}$$

All in terms of parabolic coordinates, the position vector becomes

$$\boldsymbol{S} = \frac{1}{2}\sqrt{u^2+v^2}\,(u\,\hat{\boldsymbol{u}}+v\,\hat{\boldsymbol{v}})\,.$$

Problem 8

In terms of u, v, what is the radius of a circle centered at the origin? Answer:

$$r = \frac{1}{2}\left(u^2+v^2\right)$$

Line Element and Basis

Prom the position vector $\boldsymbol{S}$ carefully work out the differential line element $d\boldsymbol{S}$:

$$d\boldsymbol{S} = \sqrt{u^2+v^2}\,(du\,\hat{\boldsymbol{u}}+dv\,\hat{\boldsymbol{v}})$$

From the differential line element, pick out generalized coordinates $q^u = u$, $q^v = v$, along with basis vectors $\boldsymbol{a}_{(\mu)}$ are

$$\begin{aligned}\boldsymbol{a}_{(u)} &= \sqrt{u^2+v^2}\,\hat{\boldsymbol{u}} \\ \boldsymbol{a}_{(v)} &= \sqrt{u^2+v^2}\,\hat{\boldsymbol{v}}\,,\end{aligned}$$

and correspondingly

$$\begin{aligned}\boldsymbol{a}^{(u)} &= \frac{1}{\sqrt{u^2+v^2}}\,\hat{\boldsymbol{u}} \\ \boldsymbol{a}^{(v)} &= \frac{1}{\sqrt{u^2+v^2}}\,\hat{\boldsymbol{v}}\,.\end{aligned}$$

Metric

The metric $g_{\mu\nu}$ is straightforwardly calculated via $\boldsymbol{a}_{(\mu)}\cdot\boldsymbol{a}_{(\nu)}$, however it's more interesting to start from the tensor transformation law

$$g_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\eta_{\mu\nu}\,,$$

working out to

$$g_{uu} = g_{vv} = u^2+v^2\,,$$

and correspondingly

$$g^{uu} = g^{vv} = \frac{1}{u^2+v^2}\,.$$

Connection

Using either of

$$\Gamma^\gamma_{\mu\nu} = \boldsymbol{a}^{(\gamma)}\cdot\partial_\mu\boldsymbol{a}_{(\nu)}\,,$$

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}\left(\partial_\mu g_{\nu\alpha}+\partial_\nu g_{\alpha\mu}-\partial_\alpha g_{\mu\nu}\right)\,,$$

one calculates the nonzero Γ to be:

$$\begin{aligned}\Gamma^u_{uu} &= \Gamma^v_{uv} = \frac{u}{u^2+v^2} \\ \Gamma^v_{vv} &= \Gamma^u_{vu} = \frac{v}{u^2+v^2} \\ \Gamma^u_{vv} &= \frac{-u}{u^2+v^2} \\ \Gamma^v_{uu} &= \frac{-v}{u^2+v^2}\end{aligned}$$

Gradient

For a scalar function $f = f(u, v)$, the gradient is:

$$\boldsymbol{\nabla} f = \boldsymbol{a}^{(\mu)} \frac{\partial f}{\partial q^\mu} = \frac{1}{\sqrt{u^2 + v^2}} \left(\frac{\partial f}{\partial u} \hat{\boldsymbol{u}} + \frac{\partial f}{\partial v} \hat{\boldsymbol{v}} \right)$$

Normalized Components

Reconciliation between the non-normalized vector $\boldsymbol{W} = W^\mu \boldsymbol{a}_{(\mu)}$ and its normalized counterpart $\boldsymbol{W} = w^\mu \hat{\boldsymbol{a}}_{(\mu)}$ involves transferring the 'extra' coefficients from $\boldsymbol{a}_{(\mu)}$ to w^μ as:

$$w^u = \sqrt{u^2 + v^2}\, W^u$$
$$w^v = \sqrt{u^2 + v^2}\, W^v$$

Divergence

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{W} &= \boldsymbol{a}^{(\mu)} \cdot D_\mu \boldsymbol{W} = D_\mu W^\mu \\
&= \partial_u W^u + \partial_v W^v + \frac{2uW^u}{u^2 + v^2} + \frac{2vW^v}{u^2 + v^2} \\
&= \frac{\partial_u \left(\left(u^2 + v^2 \right) W^u \right) + \partial_v \left(\left(u^2 + v^2 \right) W^v \right)}{u^2 + v^2} \\
&= \frac{\partial_u \left(\sqrt{u^2 + v^2}\, w^u \right) + \partial_v \left(\sqrt{u^2 + v^2}\, w^v \right)}{u^2 + v^2}
\end{aligned}$$

Laplacian

For a scalar function $f = f(u, v)$, the Laplacian is:

$$\begin{aligned}
\nabla^2 f &= g^{\mu\nu} D_\nu D_\mu f = g^{\mu\nu} D_\nu \partial_\mu f \\
&= g^{uu} \left(\partial_{uu} f - \Gamma^\gamma_{uu} \partial_\gamma f \right) + g^{vv} \left(\partial_{vv} f - \Gamma^\gamma_{vv} \partial_\gamma f \right) \\
&= \frac{1}{u^2 + v^2} \left(\partial_{uu} f + \partial_{vv} f \right)
\end{aligned}$$

Curl

Thankfully the curl has only one component.

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{V} &= \left(\boldsymbol{a}^{(\nu)} \times \boldsymbol{a}_{(\mu)} \right) D_\nu V^\mu \\
&= \left(\boldsymbol{a}^{(\nu)} \times \boldsymbol{a}_{(\mu)} \right) \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\gamma} V^\gamma \right) \\
&= \partial_u W^v - \partial_v W^u + \frac{2\left(uW^v - vW^u \right)}{u^2 + v^2} \\
&= \frac{\partial_u \left(\left(u^2 + v^2 \right) W^v \right) - \partial_v \left(\left(u^2 + v^2 \right) W^u \right)}{u^2 + v^2} \\
&= \frac{\partial_u \left(\sqrt{u^2 + v^2}\, w^v \right) - \partial_v \left(\sqrt{u^2 + v^2}\, w^u \right)}{u^2 + v^2}
\end{aligned}$$

4.5 Parabolic Coordinates 3D

Consider the two-dimensional Cartesian flat space with position vector

$$\boldsymbol{S} = x\, \hat{\boldsymbol{x}} + y\, \hat{\boldsymbol{y}} + z\, \hat{\boldsymbol{z}}$$

with the substitution

$$\begin{aligned}
x &= uv \cos(\phi) \\
y &= uv \sin(\phi) \\
z &= \frac{1}{2} \left(u^2 - v^2 \right) ,
\end{aligned}$$

which are parabolic coordinates in three dimensions.

As it turns out, curves with constant u and constant v form confocal paraboloids characterized by:

$$z_u = \frac{1}{2} \left(\frac{x^2 + y^2}{v^2} - v^2 \right)$$
$$z_v = \frac{1}{2} \left(u^2 - \frac{x^2 + y^2}{u^2} \right)$$

Line Element and Basis

From the position vector $\boldsymbol{S}$, define generalized coordinates $q^u = u$, $q^v = v$, $q^\phi = \phi$, work out the differential line element $d\boldsymbol{S}$ as

$$d\boldsymbol{S} = dq^u \boldsymbol{a}_{(u)} + dq^v \boldsymbol{a}_{(v)} + dq^\phi \boldsymbol{a}_{(\phi)} ,$$

where:

$$\begin{aligned}
\boldsymbol{a}_{(u)} &= v \left(\cos(\phi)\, \hat{\boldsymbol{x}} + \sin(\phi)\, \hat{\boldsymbol{y}} \right) + u\, \hat{\boldsymbol{z}} \\
\boldsymbol{a}_{(v)} &= u \left(\cos(\phi)\, \hat{\boldsymbol{x}} + \sin(\phi)\, \hat{\boldsymbol{y}} \right) - v\, \hat{\boldsymbol{z}} \\
\boldsymbol{a}_{(\phi)} &= uv \left(\sin(\phi)\, \hat{\boldsymbol{x}} + \cos(\phi)\, \hat{\boldsymbol{y}} \right)
\end{aligned}$$

Metric

The metric $g_{\mu\nu}$ is calculated via $\boldsymbol{a}_{(\mu)} \cdot \boldsymbol{a}_{(\nu)}$ or using

$$g_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} \frac{\partial q^\nu}{\partial q^{\nu'}} \eta_{\mu\nu} .$$

In either case, one finds

$$\begin{aligned}
g_{uu} &= g_{vv} = u^2 + v^2 \\
g_{\phi\phi} &= u^2 v^2 ,
\end{aligned}$$

and furthermore:

$$g^{uu} = g^{vv} = \frac{1}{u^2 + v^2}$$
$$g^{\phi\phi} = \frac{1}{u^2 v^2}$$

Connection

Connection coefficients Γ are largely unchanged from the two-dimensional case. New to the mix are:

$$\Gamma^u_{\phi\phi} = \frac{-uv^2}{u^2+v^2}$$
$$\Gamma^v_{\phi\phi} = \frac{-vu^2}{u^2+v^2}$$

Laplacian

For a scalar function $f = f(u, v, \phi)$, the Laplacian is:

$$\begin{aligned}\nabla^2 f &= g^{\mu\nu} D_\nu D_\mu f = g^{\mu\nu} D_\nu \partial_\mu f \\ &= g^{uu}\left(\partial_{uu} f - \Gamma^\gamma_{uu}\partial_\gamma f\right) + g^{vv}\left(\partial_{vv} f - \Gamma^\gamma_{vv}\partial_\gamma f\right) \\ &\quad + g^{\phi\phi}\left(\partial_{\phi\phi} f - \Gamma^\gamma_{\phi\phi}\right) \\ &= \frac{1}{u^2+v^2}\left(\frac{\partial_u\left(u\partial_u f\right)}{u} + \frac{\partial_v\left(v\partial_v f\right)}{v}\right) + \frac{\partial_{\phi\phi} f}{u^2 v^2}\end{aligned}$$

4.6 Hyperspherical Coordinates

Four Spatial Dimensions

Consider a four-dimensional Cartesian flat space characterized by the position vector:

$$\boldsymbol{S} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} + w\,\hat{\boldsymbol{w}}\,,$$

and the metric extends to four dimensions accordingly:

$$\eta_{11} = \eta_{22} = \eta_{33} = \eta_{44} = 1$$
$$\eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu$$

Hyperspherical Coordinates

Such a space can be mapped by *hyperspherical coordinates* according to the substitution:

$$\begin{aligned}x &= r\cos(\psi) \\ y &= r\sin(\psi)\cos(\theta) \\ z &= r\sin(\psi)\sin(\theta)\cos(\phi) \\ w &= r\sin(\psi)\sin(\theta)\sin(\phi)\end{aligned}$$

Metric

Jumping to the metric calculation, the easiest route is to use the transformation law

$$g_{\mu'\nu'} = \frac{\partial q^\mu}{\partial q^{\mu'}}\frac{\partial q^\nu}{\partial q^{\nu'}}\eta_{\mu\nu}\,,$$

leading to:

$$\begin{aligned}g_{rr} &= 1 \\ g_{\psi\psi} &= r^2 \\ g_{\theta\theta} &= r^2\sin^2(\psi) \\ g_{\phi\phi} &= r^2\sin^2(\psi)\sin^2(\theta)\end{aligned}$$

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