Differential Calculus
MANUSCRIPT

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Chapter 1

Differential Calculus

1 Slope at a Point

The notion of 'rise over run', which applies so naturally to straight lines, also applies to curves. Recall that for a line $y = mx+b$, the rise over run calculation $\Delta y/\Delta x$ always yields the same number m, the slope of the line, and the whole line has just one slope.

1.1 Derivative

On a curve, there is no single value m that characterizes the slope, but we can talk about the 'local' slope near a point. To illustrate, choose any point x_0 on a curve and begin 'zooming in' so the curve appears straighter and straighter until indistinguishable from a line. The slope of that line is the slope of the function in the place we've zoomed in.

This idea of *slope at a point* is also called the *derivative*. If the function is $f(x)$, the derivative is written $f'(x)$. A synonym for $f'(x)$ is written df/dx, called *Leibniz notation*. It's slightly clearer than the f' notation, as df/dx is the ratio 'differential f over differential x', which is the infinitesimal limit of $\Delta f/\Delta x$, or similarly $\Delta y/\Delta x$.

Yet another way to denote the derivative is to slip a parenthesized 1 next to f, i.e. $f^{(1)}(x)$. All of these expressions for 'slope at a point' are interchanged freely in greater literature:

$$\text{Slope at a point} = f'(x) = \frac{d}{dx} f(x) = f^{(1)}(x)$$

Definition

The formal definition of the derivative of $f(x)$ at any point x_0 is given as a limit:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1.1}$$

Letting $h = x - x_0$, the definition can be written

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} ,$$

which is in all ways the same as the above. This form is more common to your standard Calc 101 textbook.

Differentiable Functions

The definition of the derivative inevitably involves limits, thus all of of the baggage pertaining to continuity, smoothness, etc. must becomes relevant.

When a curve is 'well-behaved', which is to say continuous and smooth, the function is *differentiable*, which means the definition of the derivative can be applied and returns useful information.

Things get woolly with the derivative at or across a discontinuity.

1.2 Elementary Derivatives

The definition of the derivative can be directly used on any differentiable function. While there are plenty of extra rules and shortcuts to make calculations easier, we'll settle down a while and calculate a volley of derivatives the hard way.

Parabola

The easiest and most illustrative nontrivial derivative is the parabola $f(x) = x^2$. From the definition, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} ,$$

simplifying down to

$$f'(x) = \lim_{h \to 0} 2x - h = 2x .$$

That is, the slope on a parabola at point x is $2x$.

The same result comes from the formula that uses x_0 instead of h:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(x + x_0)\cancel{(x - x_0)}}{\cancel{x - x_0}} = 2x_0 \end{aligned}$$

Whole Number Powers

Generalizing the parabolic case, consider the function with x raised to an arbitrary (whole) power n, $f(x) = x^n$. Using the definition, the derivative of $f(x)$ is

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} .$$

To gain on this, recall an important identity attainable from polynomial division, namely

$$x^n - a^n = (x - a) \left(\sum_{k=1}^{n} a^{k-1} x^{n-k} \right) .$$

Letting $a = x_0$ while letting $x \to x_0$ inside the sum, this reads

$$x^n - x_0^n = (x - x_0) \left(\sum_{k=1}^{n} x_0^{n-1} \right) .$$

The final sum is n copies of the quantity x_0^{n-1}, and the term $x - x_0$ can be divided off to the left side:

$$\frac{x^n - x_0^n}{x - x_0} = n x_0^{n-1}$$

This is exactly what the definition of $f'(x_0)$ is asking for:

$$f'(x_0) = n x_0^{n-1}$$

For the sake of stating the function and the derivative on the same line, the result can be written:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \tag{1.2}$$

Reciprocal

For the reciprocal function $f(x) = 1/x$ we have

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{1/x - 1/x_0}{x - x_0} \\ &= \lim_{x\to x_0} \frac{-\cancel{(x-x_0)}}{xx_0\cancel{(x-x_0)}} = \frac{-1}{x_0^2} \,. \end{aligned}$$

That is, the slope of the reciprocal function at a point x_0 is $-1/x_0^2$. In summary:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} \tag{1.3}$$

As an exercise, perhaps just a mental one, it's straightforward to show that a horizontally-shifted reciprocal function obeys:

$$\frac{d}{dx}\left(\frac{1}{x+a}\right) = \frac{-1}{(x+a)^2} \tag{1.4}$$

Inverse Square

The inverse square function $f(x)$ can be dealt with using Equation (1.2), but we'll suffer the brute force approach:

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{1/x^2 - 1/x_0^2}{x - x_0} \\ &= \lim_{x\to x_0} \frac{-(x+x_0)\cancel{(x-x_0)}}{x^2x_0^2\cancel{(x-x_0)}} = \frac{-2}{x_0^3} \end{aligned}$$

Like the previous few cases, the denominator is eliminated by algebra and the derivative of the function becomes clear:

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3} \tag{1.5}$$

To go with this is the shifted version

$$\frac{d}{dx}\left(\frac{1}{(x+a)^2}\right) = \frac{-2}{(x+a)^3} \,, \tag{1.6}$$

where a is a constant.

Square Root

The derivative of the square root $f(x) = \sqrt{x}$ is also straightforward, as:

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= \lim_{x\to x_0} \frac{\cancel{\left(\sqrt{x} - \sqrt{x_0}\right)}}{\cancel{\left(\sqrt{x} - \sqrt{x_0}\right)}\left(\sqrt{x} + \sqrt{x_0}\right)} = \frac{1}{2\sqrt{x_0}} \end{aligned}$$

In one line, this is:

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \tag{1.7}$$

1.3 Exponential Derivatives

Standard Exponential

The exponential function $f(x) = n^x$ is a bit more tricky. Following traditional setup, we have

$$f'(x_0) = \lim_{x\to x_0} \frac{n^x - n^{x_0}}{x - x_0} = n^{x_0} \lim_{x\to x_0} \frac{n^{x-x_0} - 1}{x - x_0} \,,$$

where the term n^{x_0} can be factored outside of the limit.

Substituting $h = x - x_0$, the remaining limit becomes

$$f'(x_0) = n^x \lim_{h\to 0} \frac{n^h - 1}{h} \,,$$

is decidedly equivalent to the natural log of n. In summary:

$$\frac{d}{dx}\left(n^x\right) = n^x \ln(n) \tag{1.8}$$

Natural Exponential

A special case of Equation (1.8) has $n = e$, as in Euler's e, which gets rid of the ln-term because $\ln(e) = 1$. This tells us

$$\frac{d}{dx}\left(e^x\right) = e^x \,, \tag{1.9}$$

meaning $f(x) = e^x$ is its own derivative.

By making repeated use of Equation (1.2) for handling powers, one can show easily that another way to express e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \,. \tag{1.10}$$

It's worth mentioning too that the definition for e as an infinite limit, i.e. Equation

$$e = \lim_{h\to 0}\left(1 + \frac{1}{h}\right)^h$$

can be derived from the definition of the derivative. Supposing we nothing nothing of e for a moment, consider a function $f(x) = E^x$ that is named to foreshadow the answer. The derivative of f reads

$$f'(x_0) = \lim_{x \to x_0} \frac{E^x - E^{x_0}}{x - x_0} \,,$$

and E^{x_0} can be factored out:

$$f'(x_0) = E^{x_0} \left(\lim_{x \to x_0} \frac{E^{x - x_0} - 1}{x - x_0} \right)$$

If the function $f(x) = E^x$ is to be equal to its own derivative, then the parenthesized quantity in the above must resolve to one. Isolating this, we have

$$\lim_{x \to x_0} \frac{E^{x - x_0} - 1}{x - x_0} = 1 \,,$$

and then solve for E:

$$E = \lim_{x \to x_0} \left(1 + (x - x_0)\right)^{1/(x - x_0)}$$

By substituting $h = 1/(x - x_0)$, the right side becomes identical to e. Thus $E = e$ and we're done.

Natural Exp with Squared Argument

Now comes a fun one. Consider the function $f(x) = e^{x^2}$. Starting off as usual, we have

$$f'(x_0) = \lim_{x \to x_0} \frac{e^{x^2} - e^{x_0^2}}{x - x_0}$$

Using Equation (1.10), the numerator expands out to

$$x^2 - x_0^2 + \frac{\left(x^2\right)^2 - \left(x_0^2\right)^2}{2!} + \frac{\left(x^2\right)^3 - \left(x_0^2\right)^3}{3!} + \cdots \,,$$

which is an algebraic mess, because we need to factor $x - x_0$ out of the whole expression. Going in chunks, we find:

$$\begin{aligned}
x^2 - x_0^2 &= (x - x_0)(x + x_0) \\
\left(x^2\right)^2 - \left(x_0^2\right)^2 &= (x - x_0)(x + x_0)\left(x^2 + x_0^2\right) \\
\left(x^2\right)^3 - \left(x_0^2\right)^3 &= (x - x_0) \\
&\qquad \left(x^2 + x x_0 + x_0^2\right)\left(x^3 + x_0^3\right) \\
\left(x^2\right)^4 - \left(x_0^2\right)^4 &= (x - x_0) \\
&\qquad (x + x_0)\left(x^2 + x_0^2\right)\left(x^4 + x_0^4\right) \\
\left(x^2\right)^5 - \left(x_0^2\right)^5 &= (x - x_0) \\
&\qquad \left(x^4 + x^3 x_0 + x^2 x_0^2 + x x_0^3 + x_0^4\right) \\
&\qquad \left(x^5 + x_0^5\right)
\end{aligned}$$

Assuming the pattern continues, we can say that $(x - x_0)$ can be factored out of each term in the expansion, and this resolves having to further deal with the denominator in the derivative.

What remains is to evaluate everything at $x = x_0$. Doing this carefully and spotting the pattern, we see

$$\begin{aligned}
f'(x_0) &= 2x_0 + \frac{4x_0^3}{2!} + \frac{6x_0^5}{3!} + \frac{8x_0^7}{4!} \cdots \\
&= 2x_0 \left(1 + \frac{x_0^2}{1!} + \frac{x_0^4}{2!} + \frac{x_0^6}{3!} + \cdots\right) .
\end{aligned}$$

The parenthesized series is nothing more than $e^{x_0^2}$ according to Equation (1.10). Finally, the answer:

$$\frac{d}{dx}\left(e^{x^2}\right) = 2x\, e^{x^2} \tag{1.11}$$

Exponential with Squared Argument

The previous example can be done a different way by substituting $h = x - x_0$ before jumping into the algebra. To demonstrate on something more general, consider the function $f(x) = b^{x^2}$, where b is a real number. Setting up the derivative, we have

$$f'(x) = \lim_{x \to x_0} \frac{b^{x^2} - b^{x_0^2}}{x - x_0} = \lim_{h \to 0} \frac{b^{(x_0 + h)^2} - b^{x_0^2}}{h}$$

Now we must spend a moment on the quantity $(x_0 + h)^2$. Expanding this out, we have

$$(x_0 + h)^2 = x_0^2 + 2x_0 h + h^2 \,.$$

In the limit that h is going to zero, the h^2-term pales under the others and can be ignored. With this simplification, the derivative becomes

$$f'(x) = b^{x_0^2} \lim_{h \to 0} \frac{b^{2x_0 h} - 1}{h} \,.$$

The remaining limit almost looks familiar as a natural logarithm. Make the substitution $h = 2x_0 k$ to bring it into form:

$$f'(x) = 2x_0\, b^{x_0^2} \lim_{k \to 0} \frac{b^k - 1}{k} \,.$$

The remaining limit now is unambiguously equivalent to the natural log of b. Finally, we find:

$$\frac{d}{dx}\left(b^{x^2}\right) = 2x\, b^{x^2} \ln(b) \tag{1.12}$$

The special case $b = e$ recovers Equation (1.11).

X to the X

A notorious derivative to figure out is that of $f(x) = x^x$. Setting up the calculation, we have

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{x^x - x_0^{x_0}}{x - x_0} \\ &= \lim_{x\to x_0} \frac{x^{x_0}}{x - x_0} \left(x^{x-x_0} - \left(\frac{x_0}{x}\right)^{x_0} \right) , \end{aligned}$$

and let $h = x - x_0$:

$$f(x) = \lim_{h\to 0} x^{x_0} \left(\frac{x^h - (1 - h/x)^{x_0}}{h} \right)$$

Recalling the limit-based expression for the natural logarithm, namely

$$\ln(x) = \lim_{h\to 0} \frac{x^h - 1}{h} ,$$

and the above becomes:

$$f(x) = \lim_{h\to 0} x^{x_0} \left(\ln(x) + \frac{1 - (1 - h/x)^{x_0}}{h} \right)$$

The right-hand limit is best handled in isolation. Letting

$$A = \lim_{h\to 0} \frac{1 - (1 - h/x)^{x_0}}{h} ,$$

rearrange to write

$$\lim_{h\to 0} (1 - Ah) = \lim_{h\to 0} \left(1 - \frac{h}{x}\right)^{x_0} ,$$

and raise each side to the $1/h$ power:

$$\lim_{h\to 0} (1 - Ah)^{1/h} = \lim_{h\to 0} \left(1 - \frac{h}{x}\right)^{x_0/h}$$

With one more substitution $q = 1/h$, this is

$$\lim_{q\to\infty} \left(1 - \frac{A}{q}\right)^q = \lim_{q\to\infty} \left(1 - \frac{1}{qx}\right)^{qx_0}$$

Keep in mind that $q \to \infty$ also means $x \to x_0$, and the above simplifies to:

$$e^{-A} = \left(e^{-1/x_0}\right)^{x_0} = e^{-1} ,$$

telling us finally that $A = 1$. Putting the answer together:

$$\frac{d}{dx}\left(x^x\right) = x^x \left(\ln(x) + 1\right) \tag{1.13}$$

1.4 Logarithmic Derivatives

Natural Logarithm

A keystone function is the natural logarithm, $f(x) = \ln(x)$. Setting up the derivative calculation, we have

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{\ln(x) - \ln(x_0)}{x - x_0} \\ &= \frac{1}{x_0} \lim_{x\to x_0} \frac{\ln(x/x_0)}{x/x_0 - 1} , \end{aligned}$$

suggesting a substitution $x/x_0 = k$, and the limit becomes a matter of k approaching 1:

$$f'(x_0) = \frac{1}{x_0} \lim_{k\to 1} \frac{\ln(k)}{k - 1}$$

With another substitution $j = k - 1$, this is

$$f'(x_0) = \frac{1}{x_0} \lim_{j\to 0} \frac{\ln(1 + j)}{j} ,$$

equivalent to

$$f'(x_0) = \frac{1}{x_0} \lim_{j\to 0} \ln\left((1 + j)^{1/j}\right) .$$

If it doesn't look familiar yet, let $h = 1/j$ to get

$$f'(x_0) = \frac{1}{x_0} \lim_{h\to\infty} \ln\left(\left(1 + \frac{1}{h}\right)^h\right) .$$

The remaining limit patently resolves to e, and being enclosed as the argument to the ln function, resolves to just one, after all that. In summary, we find

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \tag{1.14}$$

Shifted Natural Logarithm

The shifted natural logarithm $f(x) = \ln(1 + x)$ is handled much like the vanilla natural logarithm. Setting up the derivative calculation, we have

$$f'(x_0) = \lim_{x\to x_0} \frac{\ln(1 + x) - \ln(1 + x_0)}{x - x_0} ,$$

suggesting a substitution $z = 1 + x$. The above becomes

$$f'(x_0) = \lim_{z\to z_0} \frac{\ln(z) - \ln(z_0)}{z - z_0} ,$$

which now looks identical to the the vanilla case in the variable z. Reversing the z-substitution gives the final answer:

$$\frac{d}{dx}\left(\ln(1 + x)\right) = \frac{1}{1 + x} \tag{1.15}$$

Nonlinear Natural Logarithm

For the function $f(x) = x\ln(x)$, the derivative calculation begins as

$$f'(x_0) = \lim_{x\to x_0} \frac{x\ln(x) - x_0\ln(x_0)}{x - x_0}$$

This one is best attacked with polynomial division, which leads to

$$f'(x_0) = \lim_{x\to x_0} \left(\ln(x) + x_0\left(\frac{\ln(x) - \ln(x_0)}{x - x_0}\right)\right),$$

and now the embedded limit should ring familiar as the derivative of the vanilla natural log, or simply $1/x_0$. Simplifying the rest, we find

$$\frac{d}{dx}(x\ln(x)) = \ln(x) + 1 \tag{1.16}$$

The same technique, namely polynomial division and then substitution from an easier derivative, is what is needed to find the derivative of the harder function, $f(x) = x^2\ln(x)$. Going through the exercise (which you are encouraged to do), the result should be

$$\frac{d}{dx}\left(x^2\ln(x)\right) = 2x\ln(x) + x\,. \tag{1.17}$$

Diminished Natural Logarithm

For another tough one, let us find the derivative of $f(x) = \ln(x)/x$. Going from the definition, we first write

$$f'(x_0) = \lim_{x\to x_0} \frac{\ln(x)/x - \ln(x_0)/x_0}{x - x_0}\,.$$

Using polynomial division and simplifying, we get to the intermediate step:

$$f'(x_0) = \frac{\ln(x_0)}{x_0^2} + \lim_{x\to x_0} \frac{1}{x_0^3}\left(\frac{x_0^2\ln(x) - x^2\ln(x_0)}{x - x_0}\right)$$

The remaining limit almost looks like Equation (1.16), i.e. the derivative of $x^2\ln(x)$, but sadly isn't exact.

To proceed, write the derivative of the natural logarithm in the form

$$\lim_{x\to x_0} \frac{\ln(x) - \ln(x_0)}{x - x_0} = \frac{1}{x_0}\,,$$

and then deduce the following:

$$\lim_{x\to x_0} \frac{x_0^2\ln(x)}{x - x_0} = \lim_{x\to x_0} \frac{x_0^2\ln(x_0)}{x - x_0} + x_0$$
$$\lim_{x\to x_0} \frac{x^2\ln(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{x^2\ln(x)}{x - x_0} - \frac{x^2}{x_0}$$

Subtract the bottom equation from the top, and notice the left side can replace the parenthesized quantity in our $f'(x)$ equation. Doing so, we get:

$$f'(x_0) = \frac{\ln(x_0)}{x_0^2} + \lim_{x\to x_0} \frac{1}{x_0^3}\left(x_0 + \frac{x^2}{x_0}\right) + \lim_{x\to x_0} \frac{1}{x_0^3}\left(\frac{x_0^2\ln(x_0) - x^2\ln(x)}{x - x_0}\right)$$

The latter term in the above contains the (negative) derivative of $x^2\ln(x)$ and can be replaced by Equation (1.16). This gets rid of all singularities, and we can simplify to get the answer:

$$\frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{1 - \ln(x)}{x^2} \tag{1.18}$$

Modified Natural Logarithm

For the function $f(x) = \ln\left(1 + x^2\right)$, the derivative calculation begins as

$$f'(x_0) = \lim_{x\to x_0} \frac{\ln\left(1 + x^2\right) - \ln\left(1 + x_0^2\right)}{x - x_0}$$

Letting $h = x - x_0$ and simplifying using the rules for manipulating logarithms, the above reduces way down to

$$f'(x_0) = \lim_{h\to 0} \ln\left(\left(1 + \frac{2x_0 h + h^2}{1 + x_0^2}\right)^h\right)\,.$$

In the limit $h \to 0$, the h^2-term is negligible, and the rest, after staring for long enough, contains the definition of the natural exponential:

$$f'(x_0) = \ln\left(e^{(2x_0 h)/(1 + x_0 2)}\right)$$

Since the natural log and the natural exponential are mutually-annihilating, we get the result:

$$\frac{d}{dx}\left(\ln\left(1 + x^2\right)\right) = \frac{2x}{1 + x^2} \tag{1.19}$$

1.5 Trigonometric Derivatives

All of the elementary trigonometric functions are curves, so we're obligated now to find their derivatives.

Sine

For the sine function $f(x) = \sin(x)$, we first write

$$f'(x) = \lim_{x\to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} .$$

The difference of sines is handled by the trigonometric identity

$$\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) ,$$

and the derivative becomes

$$f'(x) = \cos(x_0) \lim_{x\to x_0} \frac{2}{x - x_0}\left(\sin\left(\frac{x - x_0}{2}\right)\right)$$

Let $h = (x - x_0)/2$ to uncover a sinc function lurking about:

$$f'(x) = \cos(x_0) \lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$$

Recall that we spent some effort deciding that the parenthesized quantity is identically one, and we're left with the answer:

$$\frac{d}{dx}(\sin(x)) = \cos(x) \tag{1.20}$$

Cosine

The steps for calculating the derivative of $f(x) = \cos(x)$ are about identical to that of the sine function, except the required trig identity is

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right) .$$

This lands us at

$$f'(x) = -\sin(x_0) \lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) ,$$

and the same sinc is also present. From this we conclude

$$\frac{d}{dx}(\cos(x)) = -\sin(x) . \tag{1.21}$$

Tangent

For the tangent $f(x) = \tan(x)$, we start with

$$f'(x) = \lim_{x\to x_0} \frac{\tan(x) - \tan(x_0)}{x - x_0} ,$$

and then the trick is to divide out $\cos^2(x)$ from the limit:

$$f'(x) = \frac{1}{(\cos(x_0))^2} \lim_{x\to x_0} \frac{\sin(x)\cos(x_0) - \sin(x_0)\cos(x)}{x - x_0}$$

The remaining limit contains a trig identity for the sum of two angles, namely $x - x_0$. This again resolves to $\operatorname{sinc}(0)$, as was the case with the previous two trig functions. The result is

$$\frac{d}{dx}(\tan(x)) = \frac{1}{(\cos(x))^2} . \tag{1.22}$$

Cotangent

For the cotangent $f(x) = \cot(x)$, we start with

$$f'(x) = \lim_{x\to x_0} \frac{\cot(x) - \cot(x_0)}{x - x_0} ,$$

and then the trick is to divide out $\sin^2(x)$ from the limit:

$$f'(x) = \frac{1}{(\sin(x_0))^2} \lim_{x\to x_0} \frac{\cos(x)\sin(x_0) - \cos(x_0)\sin(x)}{x - x_0}$$

The remaining limit contains a trig identity for the sum of two angles, namely $x_0 - x$. This resolves to $-\operatorname{sinc}(0)$, and the final result is

$$\frac{d}{dx}(\cot(x)) = \frac{-1}{(\sin(x))^2} . \tag{1.23}$$

Secant

For the secant function $f(x) = 1/\cos(x)$, we start with

$$f'(x) = \lim_{x\to x_0} \frac{1/\cos(x) - 1/\cos(x_0)}{x - x_0} ,$$

and then divide out $-\cos^2(x)$ from the limit:

$$f'(x) = \frac{-1}{(\cos(x_0))^2} \lim_{x\to x_0} \frac{\cos(x) - \cos(x_0)}{x - x_0}$$

The limit now looks like the derivative of the cosine function and can be replaced by Equation (1.21), namely $-\sin(x_0)$. Reporting the result in standard form, we find

$$\frac{d}{dx}(\sec(x)) = \tan(x)\sec(x) . \tag{1.24}$$

Cosecant

For the cosecant function $f(x) = 1/\sin(x)$, we start with

$$f'(x) = \lim_{x \to x_0} \frac{1/\sin(x) - 1/\sin(x_0)}{x - x_0} ,$$

and then divide out $-\sin^2(x)$ from the limit:

$$f'(x) = \frac{-1}{(\sin(x_0))^2} \lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0}$$

The limit now looks like the derivative of the sine function and can be replaced by Equation (1.20), namely $\cos(x_0)$. Reporting the result in standard form, we find

$$\frac{d}{dx}(\csc(x)) = -\cot(x)\csc(x) . \tag{1.25}$$

Squared Argument

For the sine function with a squared argument $f(x) = \sin(x^2)$, we first write

$$f'(x) = \lim_{x \to x_0} \frac{\sin(x^2) - \sin(x_0^2)}{x - x_0} .$$

Using the same trig identity that helped with the regular sine case, the above becomes

$$f'(x) = \lim_{x \to x_0} \left(\frac{2}{x - x_0}\right) \sin\left(\frac{(x - x_0)(x + x_0)}{2}\right) \cos\left(\frac{x^2 + x_0^2}{2}\right) .$$

Lett $h = (x - x_0)/2$ and simplify a little to get

$$f'(x) = \cos(x_0^2) \lim_{h \to 0} \frac{\sin(2x_0 h)}{h} .$$

There seems to be an extra term in the remaining sinc function that cannot be ignored. To deal with this, make a new substitution $k = 2x_0 h$, which also limits to zero as h does so. With this, we now have

$$f'(x) = \cos(x_0^2)\, 2x_0 \lim_{k \to 0} \left(\frac{\sin(k)}{k}\right) .$$

The final limit is identically one, and in conclusion,

$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2)\, 2x . \tag{1.26}$$

The intermediate steps would be nearly the same had we started with cosine instead of sine, which would result in:

$$\frac{d}{dx}(\cos(x^2)) = -\sin(x^2)\, 2x . \tag{1.27}$$

X Times Sin(X)

One more before moving on. Consider the product of x and the sine of x, i.e. $f(x) = x\sin(x)$. Setting up this derivative, we write

$$f'(x) = \lim_{x \to x_0} \frac{x\sin(x) - x_0\sin(x_0)}{x - x_0}$$

To crack this one, add and subtract the quantity $x_0\sin(x)$ from the numerator, and then repack everything to get

$$f'(x) = \lim_{x \to x_0} \frac{\sin(x_0)(x - x_0)}{x - x_0} + \lim_{x \to x_0} \frac{x_0(\sin(x) - \sin(x_0))}{x - x_0} .$$

Now, one hard limit is replaced by two easy limits. The former case has $x - x_0$ canceling, leaving just $\sin(x_0)$. The latter case has x_0 multiplied by the derivative of the sine function, which we know to be $\cos(x_0)$. In conclusion, we have

$$\frac{d}{dx}(x\sin(x)) = \sin(x) + x\cos(x) . \tag{1.28}$$

The same recipe works for the $x\cos(x)$ case. Leaving the details as an exercise, the result is

$$\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x) . \tag{1.29}$$

1.6 Small-Angle Approximation

In the limit of small angles, it's easy to show that the sine and cosine obey the aptly-named *small-angle approximation*:

$$\lim_{x \to 0} \sin(x) = x - \cancel{\frac{x^3}{3!} + \cdots} \tag{1.30}$$

$$\lim_{x \to 0} \cos(x) = 1 - \cancel{\frac{x^2}{2!} + \cdots} \tag{1.31}$$

Sine and Cosine Revisited

The derivatives for the sine and cosine can be derived in a nifty way using small angles with the angle-sum formulas from trigonometry. For the sum of two angles θ and ϕ, recall that:

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$
$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

Next, suppose that ϕ is a vanishingly small angle, i.e. a differential angle. (This makes the quantity

$\theta+\phi$ analogous to $x+dx$ but we won't change letters.) In such a limit, the above identities become

$$\begin{aligned}\lim_{\phi\to 0}\sin\left(\theta+\phi\right)&=\lim_{\phi\to 0}\left(\sin\left(\theta\right)+\phi\cos\left(\theta\right)\right)\\ \lim_{\phi\to 0}\cos\left(\theta+\phi\right)&=\lim_{\phi\to 0}\left(\cos\left(\theta\right)-\phi\sin\left(\theta\right)\right)\ .\end{aligned}$$

Solve the first equation for $\cos(\theta)$ and the second equation for $-\sin(\theta)$:

$$\begin{aligned}\cos\left(\theta\right)&=\lim_{\phi\to 0}\frac{\sin\left(\theta+\phi\right)-\sin\left(\theta\right)}{\phi}\\ -\sin\left(\theta\right)&=\lim_{\phi\to 0}\frac{\cos\left(\theta+\phi\right)-\cos\left(\theta\right)}{\phi}\end{aligned}$$

This pair of results is none other than the derivative formulas for sine and cosine. By a change of variables these exactly reproduce Equations (1.20), (1.21).

2 Techniques of Differentiation

Fortunately, not every derivative needs to be calculated directly from the definition. To motivate a few tricks and shortcuts, suppose we have two functions of x, namely $f(x)$ and $g(x)$. We require that f and g be 'well-behaved' which is to say 'differentiable'. If this is the case, each has a well-defined derivative, $f'(x)$ and $g'(x)$, respectively.

2.1 Product Rule

Suppose we define $p(x)$ as the product of $f(x)$ and $g(x)$, i.e.

$$p\left(x\right)=f\left(x\right)g\left(x\right)\ .$$

The derivative of $p(x)$ is

$$p'\left(x\right)=\lim_{x\to x_0}\frac{f\left(x\right)g\left(x\right)-f\left(x_0\right)g\left(x_0\right)}{x-x_0}\ ,$$

and the job is to cook this down to something more useful.

Proceed by subtracting and adding the quantity $f(x_0)\,g(x)$ into the limit's numerator

$$\begin{aligned}p'\left(x_0\right)=&\lim_{x\to x_0}\left(\frac{f\left(x\right)g\left(x\right)-f\left(x_0\right)g\left(x\right)}{x-x_0}\right)\\ &-\lim_{x\to x_0}\left(\frac{f\left(x_0\right)g\left(x_0\right)-f\left(x_0\right)g\left(x\right)}{x-x_0}\right)\ ,\end{aligned}$$

and simplify:

$$\begin{aligned}p'\left(x_0\right)=g\left(x_0\right)&\lim_{x\to x_0}\left(\frac{f\left(x\right)-f\left(x_0\right)}{x-x_0}\right)\\ +f\left(x_0\right)&\lim_{x\to x_0}\left(\frac{g\left(x\right)-g\left(x_0\right)}{x-x_0}\right)\end{aligned}$$

Notice the two remaining limits are the individual derivatives of $f(x)$ and $g(x)$, so the above can be written might tighter

$$p'\left(x_0\right)=f'\left(x_0\right)g\left(x_0\right)+f\left(x_0\right)g'\left(x_0\right)\ ,$$

known as the *product rule* for derivatives.

Using abbreviated Leibniz notation, the product rule reads for $f(x)$ and $g(x)$:

$$\frac{d}{dx}\left(fg\right)=\frac{df}{dx}g+f\frac{dg}{dx} \tag{1.32}$$

Or, a more economical way to write the same thing:

$$\left(fg\right)'=f'g+fg'$$

All of these notations are mixed ant matched in the greater literature, and the same liberties will be taken as we proceed.

Examples

The product rule makes quick work of a few cases explored previously.

Example 1

Let $p(x)=x^2\ln(x)$. Identifying $f(x)=x^2$ and $g(x)=\ln(x)$, we have

$$\begin{aligned}p'\left(x\right)&=f'g+fg'\\ &=\ln\left(x\right)\frac{d}{dx}\ln\left(x^2\right)+x^2\frac{d}{dx}\left(\ln\left(x\right)\right)\\ &=2x\ln\left(x\right)+x\ ,\end{aligned}$$

in agreement with Equation (1.16).

Example 2

Let $p(x)=\ln(x)/x$. Identifying $f(x)=\ln(x)$ and $g(x)=1/x$, we have

$$\begin{aligned}p'\left(x\right)&=f'g+fg'\\ &=\left(\frac{1}{x}\right)\frac{d}{dx}\ln\left(x\right)+\ln\left(x\right)\frac{d}{dx}\left(\frac{1}{x}\right)\\ &=\frac{1-\ln\left(x\right)}{x^2}\ ,\end{aligned}$$

in agreement with Equation (1.18).

Example 3

Let $p(x)=x\sin(x)$. Identifying $f(x)=x$ and $g(x)=\sin(x)$, we have

$$\begin{aligned}p'\left(x\right)&=f'g+fg'\\ &=\left(\sin\left(x\right)\right)\frac{dx}{dx}+x\frac{d}{dx}\left(\sin\left(x\right)\right)\\ &=\sin\left(x\right)+x\cos\left(x\right)\ ,\end{aligned}$$

in agreement with Equation (1.28).

Example 4

Let $p(x) = x\cos(x)$. Identifying $f(x) = x$ and $g(x) = \cos(x)$, we have

$$\begin{aligned} p'(x) &= f'g + fg' \\ &= (\cos(x))\frac{dx}{dx} + x\frac{d}{dx}(\cos(x)) \\ &= \cos(x) - x\sin(x) \end{aligned}$$

in agreement with Equation (1.29).

2.2 Quotient Rule

Suppose we define $r(x)$ as the ratio of $f(x)$ and $g(x)$, i.e.

$$r(x) = \frac{f(x)}{g(x)} .$$

The derivative of $r(x)$ is

$$r'(x) = \lim_{x\to x_0} \frac{f(x)/g(x) - f(x_0)/g(x_0)}{x - x_0} ,$$

and like before, we need to simplify.

Proceed by multiplying the numerator and denominator by $g(x)g(x_0)$, and then add and subtract the quantity $f(x_0)g(x_0)$ from the numerator. Carefully treating each limit, the result is

$$r'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2} ,$$

known as the *quotient rule* for derivatives.

In Leibniz notation, the quotient rule reads for $f(x)$ and $g(x)$:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right) \tag{1.33}$$

Or, a more economical way to write the same thing:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Examples

Like the product rule, the quotient makes quick work the right kind of problem.

Example 5

Let $r(x) = \ln(x)/x$. Identifying $f(x) = \ln(x)$ and $g(x) = x$, we have

$$\begin{aligned} r'(x) &= \frac{f'g - fg'}{g^2} \\ &= \frac{(1/x)x - \ln(x)(1)}{x^2} \\ &= \frac{1 - \ln(x)}{x^2} , \end{aligned}$$

in agreement with Equation (1.18).

Example 6

Let $r(x) = \tan(x)$. Identifying $f(x) = \sin(x)$ and $g(x) = \cos(x)$, we have

$$\begin{aligned} r'(x) &= \frac{f'g - fg'}{g^2} \\ &= \frac{(\cos(x))^2 + (\sin(x))^2}{(\cos(x))^2} \\ &= \frac{1}{(\cos(x))^2} , \end{aligned}$$

in agreement with Equation (1.22).

2.3 Chain Rule

Composite Functions

Consider the *composite* function

$$c(x) = f(g(x)) .$$

To unpack this, we have a function $g(x)$ that is a typical function of x. The function f depends on g, so abbreviating the above by $c = f(g)$ is valid in the same way we would write $y = f(x)$.

Composite functions really aren't news to us. Things like $\cos(x^2)$ and e^{4x} or any nontrivial function can all be written as composite functions.

Derivation of Chain Rule

The issue of composite functions raises a subtle point, for if we have the generic scenario $y = f(x)$, it could have been all along that x itself is a function of some other variable, say t, as in $y(t) = f(x(t))$. This has curious implications for derivatives of the functions involved.

Recall the definition of the derivative of a function $f(x)$ in the generic case:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Now, suppose we just found out that x is a function of a deeper variable t such that

$$\begin{aligned} x &= x(t) \\ x_0 &= x(t_0) . \end{aligned}$$

In other words, $f(x)$ just became a composite function $f(x(t))$.

Naturally, $x(t)$ has a derivative of its own with respect to t:

$$x'(t_0) = \lim_{t \to t_0} \frac{x(t) - x(t_0)}{t - t_0}$$

The derivative of f, though, now looks like this:

$$f'(x(t_0)) = \lim_{x(t) \to x(t_0)} \frac{f(x(t)) - f(x(t_0))}{x(t) - x(t_0)}$$

To simplify the above, we first acknowledge that all functions are ultimately dependent on t, so the comment under the `lim` symbol can be replaced simply by $t \to t_0$. Next comes the key move, which is to multiply both sides by $x'(t_0)$ to get:

$$f'(x(t_0)) \cdot x'(t_0) = \lim_{t \to t_0} \frac{f(x(t)) - f(x(t_0))}{\cancel{x(t) - x(t_0)}} \cdot \frac{\cancel{x(t) - x(t_0)}}{t - t_0}$$

The quantity $x - x_0$ conveniently cancels, and we can tidy things up to write:

$$f'(x_0) \cdot x'(t_0) = \lim_{t \to t_0} \frac{f(x(t)) - f(x(t_0))}{t - t_0}$$

The right side of the above is the derivative of $f(x(t))$ with respect to t. It would be a misnomer to shorthand the right side with an f'-like symbol, as the 'prime' notation is reserved (typically) for derivatives with respect to x. It's much wiser at this instant to switch to Leibniz notation and rewrite the above as

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} , \tag{1.34}$$

known as the *chain rule* for derivatives.

The chain rule can be stacked indefinitely, i.e. if we found out that t itself depends on a deeper variable u such that $t = t(u)$, the above swiftly becomes:

$$\frac{df}{du} = \frac{df}{dx} \cdot \frac{dx}{dt} \cdot \frac{dt}{du}$$

Notice on the right that all terms 'cancel', at least visually, except for df in the numerator and du in the denominator to match the left side. This is what's nice about the chain rule - if it looks right, it *is* right.

Elementary Examples

Example 7

Consider the function $f(x) = e^{x^2}$. Letting $u = x^2$, f becomes a composite function $f(x) = e^{u(x)}$. The derivative of f with respect to x proceeds as

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} e^u\right)\left(\frac{d}{dx} x^2\right) \\ &= e^{u(x)}\, 2x \\ &= 2x\, e^{x^2} , \end{aligned}$$

in agreement with Equation (1.11).

Example 8

Consider the function $f(x) = b^{x^2}$. Letting $u = x^2$, f becomes a composite function $f(x) = b^{u(x)}$. The derivative of f with respect to x proceeds as

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} b^u\right)\left(\frac{d}{dx} x^2\right) \\ &= b^{u(x)} \ln(b)\, 2x \\ &= 2x\, b^{x^2} \ln(b) , \end{aligned}$$

in agreement with Equation (1.12).

Example 9

Consider the function $f(x) = \ln(x^x)$. Letting $u = x^x$, f becomes a composite function $f(x) = \ln(u)$. The derivative of f with respect to x proceeds as:

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \ln(u)\right)\left(\frac{d}{dx} x^x\right) \\ &= \frac{1}{u(x)} x^x (\ln(x) + 1) \\ &= \ln(x) + 1 \end{aligned}$$

Note that Equation (1.13) was quietly used in the above, making this example rather lengthy in its totality. However, this result is attained more easily by realizing $\ln(x^x)$ is equivalent to $x \ln(x)$ which is a job for the product rule. Either way, we conclude

$$\frac{d}{dx}(\ln(x^x)) = \ln(x) + 1 . \tag{1.35}$$

Example 10

Consider the function $f(x) = \ln(1 + x)$. Letting $u = 1 + x$, f becomes a composite function $f(x) = \ln(u)$. The derivative of f with respect to

x proceeds as:

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \ln(u)\right)\left(\frac{d}{dx}(1+x)\right) \\ &= \frac{1}{u(x)} \\ &= \frac{1}{1+x} \ ,\end{aligned}$$

in agreement with Equation (1.15).

Example 11

Consider the function $f(x) = \ln\left(1+x^2\right)$. Letting $u = 1+x^2$, f becomes a composite function $f(x) = \ln(u)$. The derivative of f with respect to x proceeds as:

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \ln(u)\right)\left(\frac{d}{dx}\left(1+x^2\right)\right) \\ &= \frac{1}{u(x)} 2x \\ &= \frac{2x}{1+x^2} \ ,\end{aligned}$$

in agreement with Equation (1.19).

Example 12

Consider the function $f(x) = \sin\left(x^2\right)$. Letting $u = x^2$, f becomes a composite function $f(x) = \sin(u)$. The derivative of f with respect to x proceeds as

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \sin(u)\right)\left(\frac{d}{dx} x^2\right) \\ &= \cos(u(x))\, 2x \\ &= \cos\left(x^2\right) 2x \ ,\end{aligned}$$

in agreement with Equation (1.26).

Logarithm Trick

The chain rule allows for some interesting cheats when calculating derivatives. Suppose we have a function $f(x)$ that seems to tricky to differentiate, which is to say $f'(x)$ is not straightforwardly calculated. It may help to send $f(x)$ to the natural logarithm before calculating the derivative, and then exploit the chain rule to weasel out an answer for $f'(x)$.

Applying the chain rule in, this scenario looks like

$$\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)} f'(x) \ ,$$

or

$$\frac{df}{dx} = f(x) \frac{d}{dx}(\ln(f(x))) \ , \tag{1.36}$$

which we'll call the *logarithm trick*.

Example 13

Consider the function $f(x) = b^{x^2}$. Using the logarithm trick, we find

$$\begin{aligned}\frac{df}{dx} &= b^{x^2} \frac{d}{dx}\left(\ln\left(b^{x^2}\right)\right) \\ &= b^{x^2} \ln(b) \frac{d}{dx} x^2 \\ &= 2x\, b^{x^2} \ln(b) \ ,\end{aligned}$$

in agreement with Equation (1.12).

Example 14

Consider the function $f(x) = x^x$. Using the logarithm trick, we find

$$\begin{aligned}\frac{df}{dx} &= x^x \frac{d}{dx}(\ln(x^x)) \\ &= x^x \frac{d}{dx}(x \ln(x)) \\ &= x^x (\ln(x) + 1) \ ,\end{aligned}$$

in agreement with Equation (1.13).

Power Rule Revisited

Recall that the derivative of a function $f(x) = x^n$ is established by Equation (1.2), namely

$$\frac{d}{dx}(x^n) = nx^{n-1} \ .$$

The derivation of this is guaranteed for integer n, but we did not explicitly cover what happens for non-integer n.

Luckily, the result is the same, i.e. Equation (1.2) holds for any n. To prove this, start with

$$f(x) = x^n = e^{\ln(x^n)} = e^{n \ln(x)} \ .$$

By the chain rule, we then find

$$\frac{d}{dx}(x^n) = e^{n \ln(x)} \frac{n}{x} = nx^{n-1}$$

and we're done.

Problem 1

Consider the function

$$f(x) = 4^{x^2} .$$

Find the derivative of $f(x)$ using (i) the definition of the derivative, (ii) the product rule, (iii) the chain rule, and (iv) the logarithm trick.

Hints: (i) Use Equation (1.12). (ii) Let $f(x) = g(x) = 2^{x^2}$ and use Equation (1.32). (iii) Let $f(g) = 4^g$ and $g(x) = x^2$ and calculate df/dx. (iv) Use Equation (1.36).

2.4 Implicit Differentiation

Derivative Operator

A subtlety that has been present all along, but not explicitly mentioned, is that we may talk about the derivative d/dx as an operation on some completely unspecified object, in the same way that we can talk about the 'add' ($+$) or 'multiply' ($\times$) operations without mentioning what is being added or multiplied.

Just like we'd multiply by two or divide by π, we can take the derivative across an entire equation. For instance, starting with

$$a(x) = b(x) + c(x) ,$$

it's reasonable to write

$$\frac{d}{dx} a = \frac{d}{dx}(a + b) .$$

We can go a little further by understanding the derivative to be a *linear operator*. This means the right side of the above can be broken into two separate derivatives on $b(x)$ and $c(x)$:

$$\frac{d}{dx} a = \frac{d}{dx} b + \frac{d}{dx} c$$

The act of applying the derivative operator across a whole equation has a name called *implicit differentiation*, a tool that works beautifully alongside the chain rule.

Tangent Line to the Ellipse

An ellipse characterized by semi-major axis a and semi-minor axis b centered in the Cartesian plane is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

Solving for y gives a pair of proper functions to describe the ellipse:

$$y_\pm(x) = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

If we want the slope of a tangent line to the circle, simply calculate $y'(x) = dy/dx$ (looking only at the top half of the circle for a moment):

$$y'(x) = \pm b\left(\frac{1}{2\sqrt{1 - x^2/a^2}}\right)\left(-2x/a^2\right)$$

$$y'(x) = -\frac{b^2}{a^2}\frac{x}{y}$$

Implicit differentiation is a quicker way to calculate $y'(x)$, and it works directly on the equation of the ellipse by throwing d/dx around both sides:

$$\frac{d}{dx}\left(\frac{x^2}{a^2}\right) + \frac{d}{dx}\left(\frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

Using the standard rules for differentiation, including the chain rule on the y-term, this gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 ,$$

and solving for dy/dx gives the same result as above, and there was no square root to fiddle with:

$$\frac{dy}{dx} = \frac{-b^2}{a^2}\frac{x}{y}$$

Regardless of how we know the slope of the tangent line at a given point (x_0, y_0) on the ellipse, the tangent line itself is

$$y = \left(\frac{-b^2}{a^2}\frac{x_0}{y_0}\right)x + b ,$$

which is equivalent to

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1 .$$

The proof is an exercise for the reader.

3 Mixed Techniques

Certain derivatives can require a mixture of tricks to figure out. In the following we pick and choose from the definition of the derivative, the product and quotient rules, along with the chain rule to produce results.

3.1 Inverse Trig Derivatives

Consider the set of inverse trigonometric functions, namely:

$$\arccos(x) = \cos^{-1}(x)$$

$$\arcsin(x) = \sin^{-1}(x)$$

$$\arctan(x) = \tan^{-1}(x)$$

$$\operatorname{arcsec}(x) = \sec^{-1}(x)$$
$$\operatorname{arccsc}(x) = \csc^{-1}(x)$$
$$\operatorname{arccot}(x) = \cot^{-1}(x)$$

Arccosine

Begin with the statement

$$\cos(\arccos(x)) = x\ ,$$

and apply the d/dx operator to both sides while using the chain rule

$$\sin(\arccos(x))\frac{d}{dx}\arccos(x) = -1\ ,$$

and then solve for the quantity we are after:

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(\arccos(x))}$$

There's still a little more work to do. Imagine a right triangle of hypotenuse 1 such that the adjacent side is $x = \cos(\theta)$. Comparing this to what's already written, identify $\theta = \arccos(x)$, and from geometry we also have $\sin(\theta) = \sqrt{1-x^2}$. This is enough to get the final answer:

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} \tag{1.37}$$

Arcsine

Begin with the statement

$$\sin(\arcsin(x)) = x\ ,$$

and apply the d/dx operator to both sides while using the chain rule

$$\cos(\arcsin(x))\frac{d}{dx}\arcsin(x) = 1\ ,$$

and then solve for the quantity we are after:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$

As with the previous case, there's still a little more work to do with a right triangle of hypotenuse 1 such that the opposite side is $x = \sin(\theta)$. Comparing this to what's already written, identify $\theta = \arcsin(x)$, and from geometry we also have $\cos(\theta) = \sqrt{1-x^2}$. This is enough to get the final answer:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \tag{1.38}$$

Arctangent

Begin with the statement

$$\tan(\arctan(x)) = x\ ,$$

and apply the d/dx operator to both sides while using the chain rule

$$\frac{1}{(\cos(\arctan(x)))^2}\frac{d}{dx}\arctan(x) = 1\ ,$$

and then solve for the quantity we are after:

$$\frac{d}{dx}\arctan(x) = \cos^2(\arctan(x))$$

Eliminate the $\cos^2$-term using the trig identity $\cos^2(\theta) = 1/\left(1+\tan^2(\theta)\right)$ and the answer emerges:

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \tag{1.39}$$

Arcsecant, Arccosecant, Arccotangent

The remaining three inverse functions are handled by a similar process to the first three. Without belaboring the details, which is left for an exercise, the results should be:

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{x\sqrt{x^2-1}} \tag{1.40}$$

$$\frac{d}{dx}\operatorname{arccsc}(x) = \frac{-1}{x\sqrt{x^2-1}} \tag{1.41}$$

$$\frac{d}{dx}\operatorname{arccot}(x) = \frac{-1}{1+x^2} \tag{1.42}$$

3.2 Hyperbolic Derivatives

Sinh, Cosh, Tanh

The hyperbolic trigonometric functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

are in many ways analogous to the ordinary trigonometric functions. For instance one may take $f(x) = \sinh(x)$ and use the definition of the derivative to write

$$f'(x) = \lim_{x\to x_0}\frac{\sinh(x) - \sinh(x_0)}{x - x_0}\ ,$$

which, by a similar process the led to Equation (1.20), gives:

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \tag{1.43}$$

It's a bit easier to use implicit differentiation to chug through the derivative calculation. Demonstrating on $\cosh(x)$, we have

$$\frac{d}{dx}(\cosh(x)) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right),$$

readily simplifying to

$$\frac{d}{dx}(\cosh(x)) = \sinh(x) \tag{1.44}$$

Note that this result differs from its trigonometric cousin by lacking a negative sign.

The derivative of $\tanh(x)$ follows easily from the quotient rule:

$$\begin{aligned}\frac{d}{dx}(\tanh(x)) &= \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) \\ &= \frac{(\cosh(x))^2 - (\sinh(x))^2}{(\cosh(x))^2},\end{aligned}$$

and the numerator simplifies via the identity

$$(\cosh(x))^2 - (\sinh(x))^2 = 1 .$$

In conclusion:

$$\frac{d}{dx}(\tanh(x)) = (\operatorname{sech}(x))^2 \tag{1.45}$$

Coth, Sech, Csch

The hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant are also straightforwardly handled:

$$\frac{d}{dx}(\coth(x)) = -(\operatorname{csch}(x))^2 \tag{1.46}$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x) \tag{1.47}$$

$$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x) \tag{1.48}$$

Arccosh, Arcsinh, Arctanh

The inverse hyperbolic functions are also straightforward to handle using the ordinary trig case as an analogy:

$$\frac{d}{dx}(\operatorname{arccosh}(x)) = \frac{1}{\sqrt{x^2-1}} \tag{1.49}$$

$$\frac{d}{dx}(\operatorname{arcsinh}(x)) = \frac{1}{\sqrt{x^2+1}} \tag{1.50}$$

$$\frac{d}{dx}(\operatorname{arctanh}(x)) = \frac{1}{1-x^2} \tag{1.51}$$

Arcsech, Arcscsch, Arccoth

$$\frac{d}{dx}(\operatorname{arcsech}(x)) = \frac{-1}{x\sqrt{1-x^2}} \tag{1.52}$$

$$\frac{d}{dx}(\operatorname{arccsch}(x)) = \frac{-1}{x\sqrt{x^2+1}} \tag{1.53}$$

$$\frac{d}{dx}(\operatorname{arccoth}(x)) = \frac{1}{1-x^2} \tag{1.54}$$

4 Applied Differentiation

4.1 Tangent Line

Given a differentiable function $y = f(x)$, the derivative $f'(x_0)$ gives the slope of the tangent line that, by definition, just touches the curve at the point (x_0, y_0). The equation of the *tangent line* is simply:

$$y - y_0 = f'(x_0)(x - x_0) \tag{1.55}$$

4.2 Mean Value Theorem

The *mean value theorem* is a piece of mathematics that concretizes something that my be obvious by inspecting any function. The theorem states that, for a given arc between two points in the Cartesian plane, there is at least one point on the arc where the instantaneous slope is parallel to the secant line formed by the two points.

In detail, suppose we have two points x_a, x_b that we feed to a function $f(x)$. The mean value theorem dictates that somewhere between x_a, x_b is a third point x_m satisfying

$$f'(x_m) = \frac{f(b) - f(a)}{b-a} . \tag{1.56}$$

Notice this looks a bit like the definition of the derivative, but starkly absent from the right side is any notion of limit.

The mean value theorem is easily proved. Consider the secant line

$$y(x) = \frac{f(b) - f(a)}{b-a}(x-a) + f(a)$$

that connects two points on a curve. Next, write the vertical distance between $f(x)$ and $y(x)$ as a new function $h(x)$

$$h(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a) - f(a) ,$$

and notice that $h = 0$ at the endpoints a, b.

Proceed by applying the d/dx operator (i.e. take a derivative) across the whole equation:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

Now, if $h(x)$ is zero at the endpoints and is nonzero in between, it must be that the derivative of $h(x)$ toggles between positive and negative at one (or more) points point x_m in the interval $a < x_m < b$. This can only mean $h'(x_m) = 0$ at the transition(s), and the proof is done.

4.3 L'Hopital's Rule

When deploying tools of mathematics, there are all-too-often situations where indeterminate forms, infinities, division by zero, etc., can occur. This is supposed to be a show-stopper, however the notions of 'limit' and 'derivative' grant a new cutting edge.

Motivation

Consider the ratio L of two functions $f(x)$ and $g(x)$ evaluated at a particular point x_0 such that

$$L(x_0) = \lim_{x\to x_0} \frac{f(x)}{g(x)} ,$$

where L is known to 'blow up' at x_0, which is to say the ratio resolves to $0/0$, ∞/∞, $0 \times \infty$, or similar indeterminate form.

With the notion of derivatives there is somewhere new to go, so let's try looking at the ratio of the *slope* of each function at x_0,

$$R(x_0) = \frac{f'(x_0)}{g'(x_0)} ,$$

and expand the right side using the definition of the derivative:

$$R = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \left(\frac{\cancel{x - x_0}}{\cancel{x - x_0}} \right)$$

The $0/0$ Case

Now impose the condition $f(x_0) = 0$ and $g(x_0) = 0$, and the ratio becomes

$$R(x_0) = \lim_{x\to x_0} \frac{f(x)}{g(x)} ,$$

which is in fact identical to $L(x_0)$. This can only mean, for points x_0 that cause L to blow up,

$$L(x_0) = \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} , \tag{1.57}$$

known as *L'Hopital's rule*.

In words, L'Hopital's rule says an indeterminate ratio of functions can be calculated *anyway* by calculating the ratio of their slopes. If *that* result is indeterminate, apply L'Hopital's rule until an answer comes out. While L'Hopital's rule was established using the $0/0$ case, the result is in fact quite general.

The ∞/∞ Case

To explore another extreme, suppose we have instead that $f(x_0) \to \infty$ and $g(x_0) \to \infty$.

Flipping the problem on its head slightly, one may write

$$L(x_0) = \lim_{x\to x_0} \frac{1/g(x)}{1/f(x)} ,$$

and then attack this using the chain rule. Doing so, we get

$$L(x_0) = \frac{g'(x_0)}{f'(x_0)} \lim_{x\to x_0} \left(\frac{f(x)}{g(x)} \right)^2$$
$$L(x_0) = \frac{g'(x_0)}{f'(x_0)} \left(L(x_0)\right)^2 ,$$

and ultimately,

$$L(x_0) = \frac{f'(x_0)}{g'(x_0)} ,$$

familiar already as Equation (1.57).

The Infinite-x Case

Equation (1.57) is reinforced again by investigating the case there L blows up for $x \to \infty$. To proceed, define a variable $t = 1/x$ such that

$$L(x_0) = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)} ,$$

which transforms the problem into a $0/0$-like problem.

Running the chain rule on the right side, we further find:

$$L(x_0) = \lim_{t\to 0} \frac{f'(1/t)}{g'(1/t)} \left(\frac{\cancel{-t^2}}{\cancel{-t^2}} \right)$$
$$L(x_0) = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Examples

You are encouraged to work through each of the following. For a bonus, pick out the example that helps establish that $0^0 = 1$.

Example 1

$$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Example 2

$$\lim_{x\to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2}$$

Example 3

$$\lim_{x\to 0}\frac{e^x-1-x}{\sin^2(x)}=\frac{1}{2}$$

Example 4

$$\lim_{x\to 0^+}x\ln(x)=\lim_{x\to 0^+}\frac{\ln(x)}{1/x}=0$$

Example 5

$$\lim_{x\to 0}\frac{\ln(x)}{x^p}=0$$

4.4 Critical Points

Many mathematical functions $y=f(x)$, apart from lines and constants, exhibit features akin to 'hills' and 'valleys' in the Cartesian plane. The peak of any given hill is called a *local maximum*, unless it's the tallest hill, earning the title *global maximum*. Similar notions of 'local' and 'global' apply to valleys, i.e. *minima*.

Definition

The very peak of a hill or very bottom of a valley is called an *extreme* point, also known as *critical* point. A function $f(x)$ having critical point x_c implies that the left-sided and right-sided limits near $f(x_c)$ are equal:

$$\lim_{w\to 0}f\left(x_c-\frac{w}{2}\right)=\lim_{w\to 0}f\left(x_c+\frac{w}{2}\right) \tag{1.58}$$

Critical points are locations where the derivative of the function is zero, i.e.

$$f'(x_c)=0\,.$$

While intuitive, this notion can be established using the definition

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\,,$$

and then using the shift of variables

$$\begin{aligned}x&\to x_c+w/2\\x_0&\to x_c-w/2\end{aligned}$$

such that $x-x_0=w$, we write the symmetric form

$$f'(x_c)=\lim_{w\to 0}\frac{f(x_c+w/2)-f(x_c-w/2)}{w}\,.$$

This is perhaps a more 'natural' representation of the derivative compared to what we've been working with. Enforcing Equation (1.58) nukes the right side, leaving the result $f'(x_c)=0$ as expected.

4.5 Optimization Problems

A very handy application of the derivative applies to problems of *optimization*. This is the broad set of 'real-world' problems that can be modeled as functions $f(x)$, where finding critical points $f'(x_c)=0$ could mean maximizing profits, minimizing fuel consumption, etc.

The recipe for optimization problems is almost the same every time. From the situation on hand:

1. Identify the working variable x and construct a well-behaved function $f(x)$. that characterizes the problem.
2. Calculate $f'(x_c)=0$ to identify critical point(s).
3. Feed any x_c back into $f(x)$ to produce the optimized solution(s).

Examples

Example 6

A cylindrical can of variable radius r and variable height h has fixed volume V. Find the dimensions of the can that minimize the surface area.

From the information given we can write the volume and surface area of the can:

$$\begin{aligned}V&=\pi r^2h\\A&=2\pi rh+2\pi r^2\end{aligned}$$

While there are two variables in play, r and h, we can write the area entirely in terms of r:

$$A(r)=\frac{2V}{r}+2\pi r^2$$

The idea now is to find the critical point in $A(r)$. Do so by calculating $dA/dr=0$, i.e.

$$\frac{dA}{dr}=0=-\frac{2V}{r_c^2}+4\pi r_c\,,$$

implying $h_c=2r_c$. Evidently, the most efficient can has the height equal to the diameter.

Example 7

Prove that

$$e^\pi>\pi^e\,.$$

First use the logarithm operator to get like symbols on their own sides:

$$\begin{aligned}\pi\ln(e)&=e\ln(\pi)\\\frac{\ln(e)}{e}&>\frac{\ln(\pi)}{\pi}\end{aligned}$$

This is suggestive of the function $f(x) = \ln(x)/x$, and the question translates to whether $f(e)$ is larger or smaller than $f(\pi)$.

Proceed by calculating $df/dx = 0$:

$$0 = \frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{1-\ln(x_c)}{x_c^2}$$

From this, we have that $1 = \ln(x_c)$, satisfied by $x_c = e$, and the proof is done.

Example 8

Find the largest rectangle that fits inside a 3-4-5 right triangle where one of the rectangle's edges lies on the hypotenuse.

Place the ninety-degree corner at the origin so the hypotenuse connects $(0,3)$ to $(4,0)$. Parallel to the hypotenuse is the base of the inscribed rectangle with two corners at $(0, y_*)$, $(x_*, 0)$, having length

$$b = \sqrt{x_*^2 + y_*^2} \, ,$$

and obeying the ratio

$$\frac{y_*}{x_*} = \frac{3}{4} \, .$$

The height of the inscribed rectangle is

$$h = (4 - x_*)\sin(\theta) \, ,$$

where $\sin(\theta) = 3/5$ from geometry.

The area of the rectangle is $A = bh$, or, all in terms of one variable:

$$\begin{aligned} A(x_*) &= \sqrt{x_*^2 + y_*^2}\,(4 - x_*)\,\frac{3}{5} \\ &= x_*(4 - x_*) \end{aligned}$$

The critical point x_c is found by calculating $dA/dx_* = 0$, namely

$$\frac{dA}{dx_*} = 0 = 4 - 2x_c \, ,$$

solved by $x_c = 2$, immediately meaning $y_c = 3/2$. Calculating b from these values yields $5/2$, which is half of the length of the hypotenuse. The height comes out to $h = 6/5$, and thus the area of the rectangle is $A = 3$.

4.6 Related Rate Problems

Implicit differentiation has some utility for analyzing 'real world' problems that aren't a matter of optimization. Instead, we may be concerned with the way one rate of change related to another, a class called *related rate* problems.

Melting Ice Sheet

A circular ice sheet of radius $r(t)$ in meters and area $A(t)$ is melting at a rate of $-\alpha\, m^2/s$. How quickly is the radius decreasing?

For this situation, we begin with

$$A(t) = \pi\,(r(t))^2 \, ,$$

and use implicit differentiation with respect to time:

$$\frac{d}{dt}A(t) = -\alpha = 2\pi r(t)\,\frac{d}{dt}(r(t))$$

The time derivative of A is given as alpha, whereas the time derivative of $r(t)$ is the quantity we're solving for.

So far then, we have

$$\frac{d}{dt}(r(t)) = r'(t) = \frac{-\alpha}{2\pi r(t)} \, ,$$

or in terms of A instead of r:

$$r' = \frac{-\sqrt{\pi}\alpha}{2\sqrt{A(t)}}$$

Distance from a Rocket

A person stands distance D away from a rocket that launches straight up with speed v_0 at $t = 0$. Write an equation for the distance r from the person to the rocket as a function of time, and then determine its derivative, $r'(t)$.

Use the Pythagorean theorem to get started

$$r^2 = D^2 + v_0^2 t^2 \, ,$$

and use implicit differentiation with respect to time:

$$2r(t)\,\frac{d}{dt}(r(t)) = 0 + 2v_0^2 t$$

Isolate $r'(t)$ to finish the job:

$$r'(r) = \frac{v_0^2 t}{\sqrt{D^2 + v_0^2 t^2}}$$

5 Second Derivative

5.1 Slope of Slope

Recall that, for a differentiable function $f(x)$, the notion of 'slope at a point', i.e. the derivative, can be expressed a few ways:

$$\text{Slope at a point} = f'(x) = \frac{d}{dx}f(x) = f^{(1)}(x)$$

If $f'(x)$ is itself a differentiable function, there must exist the notion of 'slope of slope', also known as the *second derivative* of $f(x)$:

$$\text{Second Derivative} = f''(x) = \frac{d^2}{dx^2} f(x) = f^{(2)}(x)$$

Starting with the definition of the derivative, a formula for the second derivative can be straightforwardly written:

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

Carefully replacing each f'-term with the definition again, we get

$$f''(x_0) = \lim_{x \to x_0} \frac{1}{x - x_0} \left(\lim_{w \to x} \frac{f(w) - f(x)}{w - x} - \frac{f(x) - f(x_0)}{x - x_0} \right) ,$$

or, after some algebra,

$$f''(x_0) = \lim_{w \to x} \lim_{x \to x_0} \left(\frac{1}{x - x_0} \right)^2 \left(\lambda f(w) - (\lambda + 1) f(x) + f(x_0) \right) ,$$

where

$$\lambda = \frac{x - x_0}{w - x} .$$

The above contains two simultaneous limits, namely $w \to x$ and $x \to x_0$. Applying each limit together, it should make sense that the ratio λ resolve to $\lambda = 1$ provided that $x_0 < x < w$. With this, we can set $x - x_0 = h$ and $w - x = h$, and the above becomes

$$f''(x_0) = \lim_{h \to 0} \lim_{x \to x_0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} ,$$

simplifying once more to the standard formula for the second derivative:

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} \tag{1.59}$$

In practice, one does not need to directly deploy Equation (1.59) to calculate the second derivative. So long as the first derivative $f'(x)$ is on hand, simply calculate the derivative of *that* to get a hold of $f''(x)$.

5.2 Stability at Critical Point

The second derivative $f''(x)$ carries important information about the function $f(x)$. To illustrate, consider the cubic curve

$$f(x) = \left(x + \frac{1}{2} \right)^3 - 3 \left(x + \frac{1}{2} \right) - \frac{1}{2}$$

as shown in Figure 1.1.

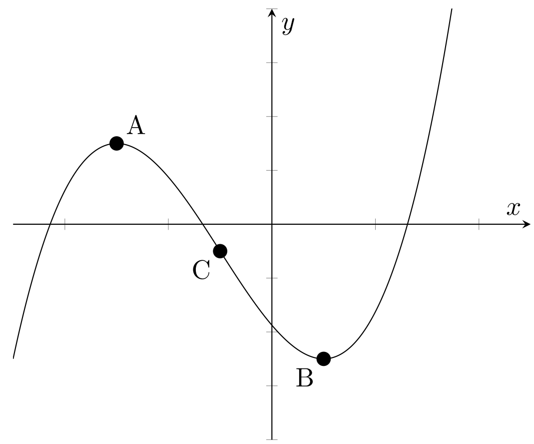

Figure 1.1: Cubic curve having two critical points and one inflection point.

Labeled in the Figure are three Cartesian points A, B, C. By a quick inspection, one sees that A and B correspond to critical points, and these can be found by the standard means of setting $f'(x) = 0$. Doing so, we first find

$$f'(x) = 3 \left(x + \frac{1}{2} \right)^2 - 3$$

for the entire curve, and $f'(x) = 0$ is solved by:

$$x_A = -3/2$$
$$x_B = 1/2$$

Concavity

While both A and B qualify as critical points, there is something clearly different about them in the sense that point A corresponds to a local maximum, and point B corresponds to a local minimum.

Introducing some new terminology, the curve $f(x)$ is *concave down* at *and near* point A, as if the only way to go is downhill. Conversely, in the 'neighborhood' of the local minimum at B, the curve is *concave up.*

In a mechanical analogy, a local maximum (such as A) is often called an *unstable equilibrium*, as if the curve $f(x)$ is embedded uniform gravity and there is a clear notion teetering on the top of a hill. A local minimum (such as B) is called a *stable equilibrium* for similar reasons.

The notion of concavity or stability begs a new question, namely how can we tell if part of a curve is concave up versus concave down without looking at the plot of the function? This is where the second derivative comes in. Starting from $f'(x)$ written above, the second derivative comes out to

$$f''(x) = 6x + 3$$

alone the whole curve.

Since we know the critical points occur at $x_A = -3/2$, $x_B = 1/2$, toss these into $f''(x)$ to learn

$$\begin{aligned} f''(-3/2) &= 6(-3/2) + 3 = -6 \\ f''(1/2) &= 6(1/2) + 3 = 6 \; . \end{aligned}$$

Evidently, the *sign* on the second derivative tells the story of concavity. When negative, the curve is concave down. When positive, the curve is concave up.

Inflection

Since the derivative operation 'knocks down' one order of x from the function, it follows that the second derivative of a cubic curve is a straight line, $y = 6x+3$ in this case. Furthermore, the second derivative has an x-intercept at $x_C = -1/2$, which is why point C is significant in Figure 1.1.

Point C is called an *inflection point*, which is where the second derivative $f''(x)$ is momentarily zero, and the concavity of the curve flips from downward to upward.

To summarize the role of the second derivative in general:

$$f''(x) = \begin{cases} < 0 & \text{Concave down} \\ = 0 & \text{Inflection} \\ > 0 & \text{Concave up} \end{cases}$$

6 Taylor's Theorem

6.1 Kinematic Motivation

In freshman kinematics, one encounters the equations of motion under uniform acceleration

$$\begin{aligned} x(t) &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ v(t) &= v_0 + a t \; , \end{aligned}$$

where $x(t)$ and $v(t)$ are the position and velocity, respectively, with their initial values written as

$$\begin{aligned} x(0) &= x_0 \\ v(0) &= v_0 \; , \end{aligned}$$

all the while acceleration a is held constant in time t.

Uniform Jerk

Extending the picture of kinematics, we consider the acceleration being allowed to vary. The simplest regime has acceleration varying *linearly* in time such that the derivative of $a(t)$ is constant called *jerk*, denoted j. In such a case, two kinematic equations are readily evident:

$$\begin{aligned} v(t) &= v_0 + a_0 t + \frac{1}{2} j t^2 \\ a(t) &= a_0 + j t \end{aligned}$$

The equation for $x(t)$ is a little more tricky though. Going from the pattern, there should be a new term proportional to jt^3, but the leading coefficient must be left as a variable

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{A} j t^3 \; ,$$

and the issue is deciding what A should be.

6.2 Time-Shift Analysis

Solving for A

To solve the riddle of the $1/A$-coefficient, consider a shift in the time variable

$$t \to t + h \; ,$$

where h is any constant. Inserting this into the above gives

$$\begin{aligned} x(t+h) = x_0 &+ v_0 (t+h) \\ &+ \frac{1}{2} a_0 (t+h)^2 + \frac{1}{A} j (t+h)^3 \; , \end{aligned}$$

and now the job is to expand all factors involving $(t+h)$. Doing so, and then combining like terms in powers of h, something interesting happens:

$$\begin{aligned} x(t+h) = & \left(x_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{1}{A} j t^3 \right) \\ & + h \left(v_0 + a_0 t + \frac{3}{A} j t^2 \right) \\ & + \frac{1}{2} h^2 \left(a_0 + \frac{6}{A} j t \right) + \frac{1}{6} h^3 (j) \end{aligned}$$

From this, we see the only way to correctly recover the identities already written is to have

$$A = 6$$

and no other choice suffices.

Time-Shifted Kinematics

Looking at the expanded $x(t+h)$ equation while knowing that $A = 6$, recall that the first parenthesized group of terms is just $x(t)$. Similarly the second group is $v(t)$, the third group, $a(t)$, and so on. Therefore we can write the same equation as

$$x(t+h) = x_t + v_t h + \frac{1}{2}a_t h^2 + \frac{1}{6}jh^3 ,$$

which is quite a beautiful result. In effect, we can pretend that t is constant, and h does the whole job of the time variable. Any point t along the path of motion can be considered as the 'initial' state.

6.3 Generalized Kinematics

Using the same procedures that led us to finding $A = 6$ in the kinematics-with-jerk analysis, it's straightforward to incorporate higher derivatives into the equations of motion. The derivative of jerk is called *snap*, denoted k. (Beyond this the derivatives aren't conventionally named.) Going through the exercise, one finds

$$x(t+h) = x_t + v_t h + \frac{a_t}{2!}h^2 + \frac{j_t}{3!}h^3 + \frac{k_t}{4!}h^4 + \cdots$$

The factorial operator is used to tightly represent the kinematic coefficients.

To handle t being considered fixed while h is the varying quantity, let us relabel t to t_p, as in 'time at some special point', and write the *effective* time variable as

$$t = t_p + h .$$

The above transforms into

$$\begin{aligned} x(t) &= x_{t_p} + v_{t_p}(t-t_p) \\ &\quad + \frac{1}{2!}a_{t_p}(t-t_p)^2 + \frac{1}{3!}j_{t_p}(t-t_p)^3 + \cdots \end{aligned}$$

Using a generalized notation to represent the velocity, acceleration, jerk, and so on, let us make the associations

$$\begin{aligned} x_{t_p} &\to x_{t_p}^{(0)} \\ v_{t_p} &\to x_{t_p}^{(1)} \\ a_{t_p} &\to x_{t_p}^{(2)} \\ j_{t_p} &\to x_{t_p}^{(3)} \\ k_{t_p} &\to x_{t_p}^{(4)} , \end{aligned}$$

and so on. On the left we've run out of 'named' items after snap, thus the general symbol $x_{t_p}^{(q)}$ is utilized to denote the qth coefficient.

Taylor Polynomial

In condensed form, $x(t)$ can be written in a most general way using summation notation

$$x(t) = x_{t_p} + \sum_{q=1}^{n} \frac{1}{q!} x_{t_p}^{(q)} (t-t_p)^q + R_n(t) , \quad (1.60)$$

known as the *Taylor polynomial*. The upper limit n can be any natural number, depending on the total number of motion coefficients in play.

The so-called 'remainder' term $R_n(t)$ contains the rest of the terms not included in the main sum. If $R_n(t)$ tends to zero for increasing n, the Taylor polynomial converges. The polynomial diverges if $R_n(t)$ fails to vanish for large n.

6.4 Taylor's Theorem

The Taylor polynomial for generalized kinematics is an extremely powerful and general result that is the center of *Taylor's theorem*. In a phrase, Taylor's theorem states that *any* n-times differentiable function can be approximated a Taylor polynomial or order n.

In terms of a function $f(x)$, near the point x_0, Taylor's theorem reads

$$p(x) = f(x_0) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}(x_0)(x-x_0)^q + R_n(x) , \quad (1.61)$$

where $f^{(q)}(x_0)$ is the qth derivative of $f(x)$ evaluated at x_0. The approximation $p(x)$ may or may not successfully approximate the entire function in its domain, but it does a great job in the neighborhood of x_0 in any case.

A less pedantic statement of Taylor's theorem omits the remainder unless it becomes necessary, and also acknowledges its approximate nature by replacing the equal sign:

$$f(x) \approx f(x_0) + \sum_{q=1}^{n} \frac{1}{q!} f^{(q)}(x_0)(x-x_0)^q$$

Proof of Taylor's Theorem

A formal proof of Taylor's theorem can begin with a new function $h_n(x)$ defined as:

$$h_n(x) = \begin{cases} (f(x) - p(x)) / ((x-x_0)^n) & x \neq a \\ 0 & x = a \end{cases}$$

It is required that $f(x)$ is an n-times differentiable function and $p(x)$ is the Taylor polynomial appearing in Equation (1.61).

The theorem is considered proven when we show that $h_n(x) = 0$ for all x near x_0. Setting this up, begin with

$$H_n = \lim_{x \to x_0} h_n(x) = \lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^n} ,$$

and recognize the right side as an indeterminate ratio.

Indeterminate ratios of this kind are handled by L'Hopital's rule, thus apply the d/dx operator to the numerator and denominator:

$$\lim_{x \to x_0} \frac{\frac{d}{dx}(f(x) - p(x))}{\frac{d}{dx}((x - x_0)^n)} = \lim_{x \to x_0} \frac{f^{(1)}(x) - p^{(1)}(x)}{n(x - x_0)^{n-1}}$$

The right side is again an indeterminate ratio, calling for another application of L'Hopital's rule:

$$H_n = \lim_{x \to x_0} \frac{f^{(2)}(x) - p^{(2)}(x)}{n(n-1)(x - x_0)^{n-2}}$$

In fact, this pattern continues $n - 1$ times, with each application of L'Hopital's rules knocking down the exponent in the denominator. Exhausting this loop, one should find:

$$H_n = \frac{1}{n!} \left(\lim_{x \to x_0} \frac{f^{(n-1)}(x) - p^{(n-1)}(x)}{x - x_0} \right)$$

The parenthesized limit is equivalent to the definition of the derivative of $f^{(n)}(x)$ evaluated at x_0. Or, use L'Hopital once more to sap the denominator entirely, and the quantity H_n evaluates to

$$H_n = \frac{1}{n!} \left(f^{(n)}(x_0) - f^{(n)}(x_0) \right) = 0$$

and the proof is done.

Order of Approximation

In certain scenarios, especially when working near the point x_0, it suffices to truncate the Taylor polynomial to a small, finite number of terms. This works only when the sum converges 'rapidly enough' so that higher powers of $x - x_0$ become negligible. To put a label to the first few approximations, we have, for a function $f(x)$:

Zeroth Order:

$$p_0(x) = f(x_0)$$

First Order:

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

Second Order:

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

6.5 Testing Taylor's Theorem

In a certain sense, Taylor's theorem contains the entire lesson of elementary calculus. Here we spend a moment recovering some already-known results.

Geometric Series

Consider the function

$$f(x) = \frac{1}{1 - x}$$

in the domain $|x| < 1$. Near any point x_0, the infinite Taylor polynomial approximation to $f(x)$ reads:

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$$

The derivatives f', f'', etc., are straightforwardly attained from $f(x)$:

$$f'(x_0) = 1/(x - x_0)^2$$
$$f''(x_0) = 2!/(x - x_0)^3$$
$$f'''(x_0) = 3!/(x - x_0)^4$$

Substituting these into $p(x)$ and performing the obvious cancellations gives:

$$p(x) = \frac{1}{1 - x_0}\left(1 + \lambda + \lambda^2 + \cdots\right) ,$$

where for brevity, λ contains the x-dependence via

$$\lambda = \frac{x - x_0}{1 - x_0} .$$

Note, of course, that the parenthesized sum containing powers of λ is a geometric series guaranteed to converge because $|x| < 1$. Realizing this, replace the infinite sum with the ratio $1/(1 - \lambda)$ as

$$p(x) = \frac{1}{1 - x_0} \frac{1}{1 - \lambda} ,$$

simplifying to

$$p(x) = \frac{1}{1 - x} = f(x) .$$

Evidently, the infinite Taylor polynomial approximation of the geometric series is no approximation at all - the result is exact.

Trigonometric Functions

Another regime where the Taylor polynomial exactly approximates the function is in trigonometry, namely the sine and cosine. Choosing the sine function to play with, we have $f(x) = \sin(x)$, and then,

$$\begin{aligned} f'(x_0) &= \cos(x_0) \\ f''(x_0) &= -\sin(x_0) \\ f'''(x_0) &= -\cos(x_0) \\ f''''(x_0) &= \sin(x_0) \end{aligned}$$

and by the fourth derivative we're back to $\sin(x)$.

The simplest case has $x_0 = 0$, which nixes all of the sine terms in the list of evaluated derivatives. The cosines all resolve to either 1 or -1, and we quickly find

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

Choosing a juicier example, let $x_0 = \pi/2$ and all the signs change:

$$\sin(x) = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots$$

On the right is the polynomial expression for the cosine offset by $-\pi/2$, and the whole thing reduces to the trig identity

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right) .$$

6.6 Recovering Differentiation Rules

Taylor's theorem also jives with the rules of differentiation.

Product and Quotient Rules

Consider two differentiable functions $f(x)$, $g(x)$. From these, construct the product $P(x) = f(x) \cdot g(x)$ along with the quotient $Q(x) = f(x)/g(x)$. A question that immediately arises from this is, what are the first-order approximations to $P(x)$, $Q(x)$?

To handle the product case, write each function $f(x)$, $g(x)$ to first-order approximation,

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ g(x) &\approx g(x_0) + g'(x_0)(x - x_0) \end{aligned}$$

with the understanding that x is near x_0.

Denoting

$$\Delta x = x - x_0 ,$$

the product $P(x)$ reads

$$\begin{aligned} P(x) \approx\ & f(x_0)\, g(x_0) \\ & + \Delta x \left(f'(x_0)\, g(x) + f(x)\, g'(x_0)\right) \\ & + \cancel{(\Delta x)^2 f'(x_0)\, g'(x_0)} \end{aligned}$$

where the term $(\Delta x)^2$ is negligible compared to the others and is dropped.

The middle term in the above is Δx multiplied by the derivative of the product $f(x)\, g(x)$ per Equation (1.32), i.e. the product rule. After simplifying, we can summarize by writing the first-order approximation to $P(x)$:

$$P_1(x) = f(x_0)\, g(x_0) + \frac{d}{dx}\left(f(x)\, g(x)\right)\bigg|_{x_0} \Delta x \quad (1.62)$$

The case for quotients is a little harder. To prepare, let us apply Taylor's theorem to $1/g(x)$ on its own. Begin using the first order approximation for $g(x)$ via

$$\frac{1}{g(x)} \approx \lim_{x \to x_0} \frac{1}{g(x_0) + g'(x_0)(x - x_0)} ,$$

and then factor out $1/g(x_0)$:

$$\frac{1}{g(x)} \approx \frac{1}{g(x_0)} \lim_{x \to x_0} \frac{1}{1 + \lambda} ,$$

where the x-dependence is wrapped up in λ:

$$\lambda = \frac{g'(x_0)}{g(x_0)}(x - x_0)$$

Like we've seen before, it suffices to proceed with $|\lambda| < 1$ for all x, and the fraction $1/(1 + \lambda)$ can be replaced with the geometric series:

$$\frac{1}{1+\lambda} = 1 - \lambda + \lambda^2 - \lambda^3 + \cdots$$

Of course, terms λ^2 and above are omitted in the first-order approximation, thus we have

$$\frac{1}{g(x)} \approx \frac{1}{g(x_0)} \lim_{x \to x_0} \left(1 - \frac{g'(x_0)}{g(x_0)}(x - x_0)\right) .$$

The first-order approximation to $Q(x)$ can be taken as the product $f_1(x)$ and $1/g_1(x)$. Doing this out while dropping the inevitable $(\Delta x)^2$ term, we find:

$$Q(x) \approx \frac{f(x_0)}{g(x_0)} + \Delta x \left(\frac{f'(x_0)\, g(x) - f(x)\, g'(x_0)}{(g(x_0))^2}\right)$$

The latter term in the above is Δx multiplied by the derivative of the quotient $f(x)/g(x)$ per Equation (1.33), i.e. the quotient rule. After simplifying, we can summarize by writing the first-order approximation to $Q(x)$:

$$Q_1(x) = \frac{f(x_0)}{g(x_0)} + \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\bigg|_{x_0} \Delta x \qquad (1.63)$$

Chain Rule

Consider the composite function

$$C(x) = f(g(x)) \ .$$

To a first-order approximation the functions f and g obey

$$\begin{aligned} f(g) &\approx f(g_0) + f'(g_0)(g - g_0) \\ g(x) &\approx g(x_0) + g'(x_0)(x - x_0) \ , \end{aligned}$$

where $g_0 = g(x_0)$.

With these, the composite function reads

$$f(g(x)) \approx f(g(x_0)) + f'(g(x_0))\, g'(x_0)(x - x_0) \ ,$$

or more succinctly:

$$C_1(x) = f(g(x_0)) + f'(g(x_0))\, g'(x_0)\, \Delta x \tag{1.64}$$

Second Derivative

The formula for the second derivative can also be wriggled from the Taylor polynomial. First write the standard approximation of $f(x)$:

$$\begin{aligned} f(x) \approx f(x_0) &+ f'(x_0)(x - x_0) \\ &+ \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots \end{aligned}$$

The same function can be approximated from a different base point, namely $x \to 2x_0 - x$. Writing this out, we have

$$\begin{aligned} f(2x_0 - x) \approx f(x_0) &+ f'(x_0)(x_0 - x) \\ &+ \frac{f''(x_0)}{2!}(x_0 - x)^2 \\ &+ \frac{f'''(x_0)}{3!}(x_0 - x)^3 + \cdots \ , \end{aligned}$$

which has the effect of reversing the sign on Δx on the odd-powered terms.

Next take the sum of the two equations to make all odd-powered terms cancel, and then and reshuffle a little to write

$$\frac{f(x) + f(2x_0 - x) - 2f(x_0)}{(x - x_0)^2} \approx f''(x_0) + \cancel{\frac{2}{2!} f''''(x_0)\, \Delta x^2} \ ,$$

where terms containing powers of Δx^2 and above are negligible in a first-order approximation.

In the infinitesimal limit $x \to x_0$, the above reduces to the exact formula for the second derivative, Equation (1.59). Specifically, let $x - x_0 = h$ and let h go to zero.

6.7 Binomial Expansion

Consider the function

$$f(x) = (x + a)^r \ ,$$

where a is a constant and r is an arbitrary exponent. In preparation for Taylor's theorem, crank out the first few derivatives of $f(x)$ and spot the pattern:

$$\begin{aligned} f^{(1)}(x_0) &= r(x_0 + a)^{r-1} \\ f^{(2)}(x_0) &= r(r-1)(x_0 + a)^{r-2} \\ f^{(3)}(x_0) &= r(r-1)(r-2)(x_0 + a)^{r-3} \\ f^{(q)}(x_0) &= \frac{r!}{(r-q)!}(x_0 + a)^{r-q} \end{aligned}$$

As a sum, the approximation for $f(x)$ then reads

$$\begin{aligned} f(x) \approx{}& (x_0 + a)^r \\ &+ \sum_{q=1}^{n} \frac{r!}{q!\,(r-q)!}(x_0 + a)^{r-q}(x - x_0)^q \ . \end{aligned}$$

Next, impose the condition

$$x \approx x_0 = 0 \ ,$$

which causes increasing powers of Δx^q tend to zero quickly. The above becomes

$$f(x) \approx a^r + a^r \sum_{q=1}^{n} \frac{r!}{q!\,(r-q)!}\left(\frac{x}{a}\right)^q \ ,$$

and the condition $x \approx 0$ is represented by $(x/a)^q$ tending to zero for increasing q.

Binomial Coefficients

The pattern of factorials in the above has a special name called the *binomial coefficients*, which follow a special notation:

$$\binom{r}{q} = \frac{r!}{q!\,(r-q)!} \tag{1.65}$$

In terms of binomial coefficients, the sum representing $f(x)$ is written

$$(x + a)^r \approx a^r \sum_{q=0}^{n} \binom{r}{q}\left(\frac{x}{a}\right)^q \tag{1.66}$$

valid for 'small' x. This is called the *binomial expansion* formula. The above can be written in open form for more practical use:

$$\begin{aligned} (x + a)^r \approx a^r + r a^{r-1} x &+ \frac{r(r-1)}{2!} a^{r-2} x^2 \\ &+ \frac{r(r-1)(r-2)}{3!} a^{r-3} x^3 + \cdots \end{aligned}$$

Examples

Example 1

Expand $\sqrt{1+x}$ for small x.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \tag{1.67}$$

Example 2

Expand $1/\sqrt{1+x}$ for small x.

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots \tag{1.68}$$

6.8 Generalized Taylor Expansion

Taylor's theorem can be used to approximate any differentiable function in addition to polynomials.

Shifted Natural Logarithm

Consider the shifted natural logarithm

$$f(x) = \ln(1+x) \; .$$

At a point x_0, the derivatives of $f(x)$ are:

$$\begin{aligned}
f^{(1)}(x_0) &= 1/(1+x_0) \\
f^{(2)}(x_0) &= -1/(1+x_0)^2 \\
f^{(3)}(x_0) &= 2/(1+x_0)^3 \\
f^{(4)}(x_0) &= -3 \cdot 2/(1+x_0)^4 \\
f^{(q)}(x_0) &= (-1)^{q-1}(q-1)!/(1+x_0)^q
\end{aligned}$$

Then, the approximation for $f(x)$ near x_0 reads

$$f(x) \approx \ln(1+x_0) + \sum_{q=1}^{n} \frac{(-1)^{q-1}}{q} \frac{(x-x_0)^q}{(1+x_0)^q} \; .$$

This result boils down to a quaint infinite series for x near zero:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1.69}$$

Arctangent Near Zero

Using Taylor's theorem involves derivative calculations that can get increasingly messy without an obvious pattern showing.

To demonstrate, let's run through the exercise using $f(x) = \arctan(x)$, where we have:

$$\begin{aligned}
f^{(1)}(x_0) &= \frac{1}{1+x_0^2} \\
f^{(2)}(x_0) &= \frac{-2}{(1+x_0^2)^2} \\
f^{(3)}(x_0) &= \frac{6x_0^2-2}{(1+x_0^2)^3} \\
f^{(4)}(x_0) &= \frac{-24x_0(x_0^2-1)}{(1+x_0^2)^4} \\
f^{(5)}(x_0) &= \frac{24(5x_0^4-10x_0^2+1)}{(1+x_0^2)^5}
\end{aligned}$$

Clearly the derivatives are not exhibiting a clear pattern. To reign in the work we're doing, let $x_0 = 0$ and simplify to end up with an infinite series approximation for the $\arctan(x)$ near $x = 0$:

$$\arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{1.70}$$

The last term $x^7/7$ wasn't directly calculated, but tacked on due to the prevailing pattern in the coefficients. This move isn't safe unless you're sure the pattern really is there.

Arctangent Near One

Taking another easy case, the arctangent near $x = 1$ can be found in the same way as above, resulting in:

$$\begin{aligned}
\arctan(x)_{x\approx 1} = & \frac{\pi}{4} + \frac{x-1}{2} \\
& - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} \\
& - \frac{(x-1)^5}{40} + \frac{(x-1)^6}{48} - \cdots
\end{aligned} \tag{1.71}$$

Arctangent near Two

For the sake of completeness, the arctangent near $x = 2$ case works out to be:

$$\begin{aligned}
\arctan(x)_{x\approx 2} = & \arctan(2) + \frac{x-2}{5} \\
& - \frac{2(x-2)^2}{25} + \frac{11(x-2)^3}{375} - \cdots
\end{aligned} \tag{1.72}$$

Arctangent of Two

It's worth pausing a moment on the quantity $\arctan(2)$, which is required to evaluate Equation (1.72).

A situation with $\arctan(2)$ could arise from a right triangle with adjacent side 1, opposite side 2, and hypotenuse $\sqrt{5}$. No rational multiple of π radians or degrees corresponds to the interior angles of such a triangle. Moreover, Equation (1.71) cannot be used as $x = 2$ is outside the valid domain of approximation.

To crack this problem, consider some argument z and list the trig identity

$$\cot\left(\frac{\pi}{2} - z\right) = \tan(z) \ .$$

Then substitute $z = \arctan(x)$ to get

$$\cot\left(\frac{\pi}{2} - \arctan(x)\right) = \tan(\arctan(x)) = x \ ,$$

and simplifying further:

$$\frac{\pi}{2} - \arctan(x) = \text{arccot}(x)$$

To deal with $\text{arccot}(x)$, recall also from trigonometry that

$$\text{arccot}(x) = \arctan\left(\frac{1}{x}\right) \ ,$$

thus we land at a powerful identity:

$$\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right) \tag{1.73}$$

This puts us in position to finally calculate $\arctan(2)$ via

$$\arctan(2) = \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) \ .$$

Either of Equations (1.70), (1.71) is sufficient to calculate $\arctan(1/2)$.

Tangent Near Zero

The tangent function is a bit ugly for having vertical asymptotes at integer multiples of $\pm\pi/2$, $\pm 3\pi/2$, etc. The function is otherwise handled in typical fashion, first by listing off the first few derivatives of $f(x_0) = \tan(x_0)$:

$$\begin{aligned}
f^{(1)}(x_0) &= \sec^2(x_0) \\
f^{(2)}(x_0) &= 2\sec^2(x_0)\tan(x_0) \\
f^{(3)}(x_0) &= 2\sec^2(x_0)\left(\sec^2(x_0) + 2\tan^2(x_0)\right) \\
f^{(4)}(x_0) &= 16\sec^4(x_0)\tan(x_0) \\
&\quad + 8\sec^2(x_0)\tan^3(x_0) \\
f^{(5)}(x_0) &= 88\sec^4(x_0)\tan^2(x_0) + 16\sec^6(x_0) \\
&\quad + 16\tan^4(x_0)\sec^2(x_0)
\end{aligned}$$

The above simplifies differently depending on which x_0 is chosen. Going with $x_0 = 0$ first, acknowledge that

$$\begin{aligned}
\sec(0) &= 1 \\
\tan(0) &= 0 \ ,
\end{aligned}$$

and quickly find:

$$\begin{aligned}
f^{(1)}(0) &= 1 & f^{(2)}(0) &= 0 \\
f^{(3)}(0) &= 2 & f^{(4)}(0) &= 0 \\
f^{(5)}(0) &= 16
\end{aligned}$$

Plugging this information into Taylor's theorem yields a useful approximation to the tangent function:

$$\tan(x) \approx x + \frac{x^3}{3} + \frac{2x^5}{15} + O\left(x^7\right) \tag{1.74}$$

The symbol $O\left(x^7\right)$ signifies that the next nonzero term in the approximation is of order 7, and then the terms get smaller after that. In this particular case, it happens that

$$O\left(x^7\right) = \frac{17x^7}{315} \ ,$$

which you are welcome to verify.

Tangent Near Pi/4

Shifting the base point to $x_0 = \pi/4$, we can recycle all of the work in calculating the derivatives of $\tan(x)$ and re-evaluate using

$$\begin{aligned}
\sec(\pi/4) &= \sqrt{2} \\
\tan(\pi/4) &= 1 \ ,
\end{aligned}$$

which gives:

$$\begin{aligned}
f^{(1)}(\pi/4) &= 2 & f^{(2)}(\pi/4) &= 4 \\
f^{(3)}(\pi/4) &= 16 & f^{(4)}(\pi/4) &= 80 \\
f^{(5)}(\pi/4) &= 512
\end{aligned}$$

Not forgetting the shift by x_0 units, the approximation for the tangent near $x = \pi/4$ reads

$$\begin{aligned}
\tan(x)_{x\approx\pi/4} &= 1 + 2\left(x - \frac{\pi}{4}\right) \\
&\quad + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 \\
&\quad + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5 \\
&\quad + O\left(x - \frac{\pi}{4}\right)^6
\end{aligned} \tag{1.75}$$

Cotangent Near Pi/2

The same routine can be applied to the cotangent. For $f(x) = \cot(x)$, find:

$$\begin{aligned}
f^{(1)}(x_0) &= -\csc^2(x_0) \\
f^{(2)}(x_0) &= 2\csc^2(x_0)\cot(x_0) \\
f^{(3)}(x_0) &= -2\csc^2(x_0)\left(\csc^2(x_0) + 2\cot^2(x_0)\right) \\
f^{(4)}(x_0) &= 16\csc^4(x_0)\cot(x_0) \\
&\quad + 8\csc^2(x_0)\cot^3(x_0) \\
f^{(5)}(x_0) &= -88\csc^4(x_0)\cot^2(x_0) - 16\csc^6(x_0) \\
&\quad - 16\cot^4(x_0)\csc^2(x_0)
\end{aligned}$$

Setting $x_0 = \pi/2$ first, note that

$$\begin{aligned}
\csc(\pi/2) &= 1 \\
\cot(\pi/2) &= 0\,,
\end{aligned}$$

and quickly find:

$$\begin{aligned}
f^{(1)}(0) &= -1 & f^{(2)}(0) &= 0 \\
f^{(3)}(0) &= -2 & f^{(4)}(0) &= 0 \\
f^{(5)}(0) &= -16
\end{aligned}$$

Evidently, the expansion for the cotangent near $x = \pi/2$ is somewhat like the tangent near $x = 0$ with the signs reversed. For conciseness, let $z = x - \pi/2$ and find

$$\cot(z) \approx -z - \frac{z^3}{3} - \frac{2z^5}{15} - O\left(z^7\right)\,. \tag{1.76}$$

6.9 Expansion Near Asymptotes

Tangent near Pi/2

Returning to the problem of the tangent function, we know $\tan(x)$ has a hopeless singularity at $x = \pi/2$ tending to $+\infty$ on the left and $-\infty$ on the right. With this, how can derivatives evaluated *at* $\pi/2$, which are surely divergent, mean anything?

It seems that Taylor would have nothing to say about expansion near an asymptote, but there is a trick. Since the tangent and cotangent are mutually reciprocal, then it should make sense to approximate the ratio $1/\cot(x)$ near $x = \pi/2$ and get the answer we want.

Letting $z = x - \pi/2$, this means we start with

$$\tan(z)_{z\approx 0} = \frac{1}{\cot(z)_{z\approx 0}} = \frac{-1}{z + z^3/3 + 2z^5/15}\,,$$

where any terms of order 7 or higher are ignored as negligible. Carrying out the polynomial division leads to an infinite sum:

$$\tan(z) \approx -\frac{1}{z} + \frac{z}{3} + \frac{z^3}{45} + O\left(z^5\right) \tag{1.77}$$

Notice the result is two orders lower than the quantity we started with, thus any terms of order 5 or greater can't be trusted. More important are the low-order terms, and we see $-1/z$ being the dominant one. This captures the divergent behavior of the tangent near its first asymptote and the trailing terms improve accuracy.

Cotangent Near Zero

The cotangent function behaves asymptotically near $x = 0$, thus the same trick is needed to explore this case. That is, take the approximation for $\tan(x)$ near $x = 0$ and perform long division. Leaving the details for an exercise, the result is:

$$\cot(x) \approx \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - O\left(x^5\right) \tag{1.78}$$

6.10 Kinematics with Air Damping

When studying kinematics, one comes to understand that it all starts with a uniform gravitational field in vacuum, which on earth near sea level means

$$a = -g = -9.8m/s^2\,.$$

From this, we know the velocity will be a linear function in time, i.e.

$$v(x) = v_0 + at\,,$$

and the position is a quadratic:

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

Of course, this is the most baseline picture of kinematics in the sense that there are no jerk-like terms or higher derivatives. Starkly absent too are real-world effects that would alter the idealized image of projectile motion, particularly the presence of the atmosphere as a resisting fluid.

Using a simplified model for air damping, we can imagine a new component to the acceleration that tries to slow down an object by an amount proportional to its speed. To capture this, we let the acceleration vary in time via

$$a(t) = -g - bv(t)\,,$$

where b is a *linear damping* coefficient and $v(t)$ is the velocity.

Acknowledging that $a(t)$ is the second derivative of $x(t)$ and $v(t)$ is the first derivative, the above is more clearly stated in Leibniz notation as

$$\frac{d^2x}{dt^2} = -g - b\frac{dx}{dt} \,,$$

which is an honest-to-goodness *differential equation*. The signature of a differential equation is that the function $x(t)$ is tied up in some kind of relationship with its own derivative(s), and solving for x can't be done algebraically. A less intimidating version of the same equation can be written strictly in terms of $v(t)$:

$$\frac{dv}{dt} = -g - bv \tag{1.79}$$

Frobenius Method

There is a brilliant trick attributed to Ferdinand Georg Frobenius (1849-1917) for solving equations like the one above. Suppose $v(t)$ takes the form of an infinite, Taylor-like polynomial with unknown coefficients:

$$v(t) = v_0 + A_1 t + A_2 t^2 + A_3 t^3 + \cdots$$

Without knowing much else about $v(t)$, we can still compute the derivative:

$$\frac{dv}{dt} = A_1 + 2A_2 t + 3A_3 t^2 + 4A_4 t^3 + \cdots$$

Now, plug both of these into $-g = bv + dv/dt$:

$$\begin{aligned} -g = bv_0 &+ bA_1 t + bA_2 t^2 + bA_3 t^3 + \cdots \\ &+ A_1 + 2A_2 t + 3A_3 t^2 + 4A_4 t^3 + \cdots \end{aligned}$$

This seems to be a greater mess than we started with until *matching coefficients*, which means the coefficients on matching powers of t must balance. This means

$$\begin{aligned} -g &= bv_0 + A_1 \\ 0 &= bA_1 + 2A_2 \\ 0 &= bA_2 + 3A_3 \\ 0 &= bA_3 + 4A_4 \,, \end{aligned}$$

and the pattern continues forever.

Velocity

Astonishingly, notice that all of the coefficients can all be related back to the first few, and $v(t)$ can now be written:

$$v(t) = v_0 + A_1 t - \frac{bA_1}{2!}t^2 + \frac{b^2 A_1}{3!}t^3 - \frac{b^3 A_1}{4!}t^4 + \cdots$$

The polynomial on the right looks tantalizingly close to an exponential, which it indeed is. Proceeding carefully, we next have

$$v(t) = v_0 + \frac{A_1}{b}\left(1 - e^{-bt}\right) \,,$$

simplifying once more to

$$v(t) = \frac{-g}{b} + \left(v_0 + \frac{g}{b}\right)e^{-bt} \,. \tag{1.80}$$

The final unknown v_0 is the initial velocity $v(0)$.

If an object is left in freefall in atmosphere for a long time, it's likely to achieve a state called *terminal velocity* where the force of gravity balances the force of air damping. To see this, let t run to infinity in the above, and we find

$$v_{\text{terminal}} = \lim_{t\to\infty} v(t) = \frac{-g}{b} \,.$$

Position

It just happens that the Frobenius method works for attaining $x(t)$. Postulating

$$x(t) = x_0 + B_1 t + B_2 t^2 + B_3 t^3 + \cdots \,,$$

one finds, after plugging into

$$-g = \frac{d^2x}{dt^2} + b\frac{dx}{dt} \,,$$

that

$$x(t) = x_0 - \frac{gt}{b} - \frac{2B_2}{b^2}\left(1 - e^{-bt}\right) \,,$$

where x_0 is the initial position $x(0)$.

Since the derivative of $x(t)$ is identically $v(t)$, we can relate the coefficient B_2 to the initial velocity via

$$B_2 = -\frac{1}{2}\left(g + bv_0\right) \,,$$

and the position equation takes the form:

$$x(t) = x_0 - \frac{gt}{b} + \frac{1}{b}\left(v_0 + \frac{g}{b}\right)\left(1 - e^{-bt}\right) \tag{1.81}$$

Small-b Limit

In the case that the damping constant b is small, the velocity and position equations ought to restore to their ideal form, or at least approximately. Doing the $v(t)$-case first, Equation (1.80) in the small-b limit reads

$$v(t) \approx \frac{-g}{b} + \left(v_0 + \frac{g}{b}\right)(1 - bt) \,,$$

reducing readily to

$$v(t) \approx v_0 - (g + v_0 b)\, t$$

with no factor of b in the denominator. The $v_0 b$ term plays essentially no role in the numerator and the form $v \approx v_0 - gt$ is recovered.

The position equation ought to churn out something similar. Starting from Equation (1.81) and expanding the exponential to second order gives:

$$x(t) \approx x_0 - \frac{gt}{b} + \frac{1}{b}\left(v_0 + \frac{g}{b}\right)\left(bt - \frac{1}{2}b^2t^2\right) ,$$

which simplifies to

$$x(t) \approx x_0 + v_0 t - \frac{1}{2}(g + v_0 b)\, t^2$$

as expected.

7 Numerical Methods

Transcendental Equations

Fairly often, solving a problem by analytical means is not a straightforward task, and a whole class of creatures called *transcendental equations* have no analytical solution at all. For these, the best thing we can do is approximate the answer.

For instance, try solving for x in the equation

$$x = \cos(x) ,$$

but don't try too long. While it is possible to manipulate a transcendental equation, there is no satisfactory way to isolate x. One may transform the above into either of

$$\arccos(x) = x$$
$$x = \cos(\cos(\cos(\cdots x \cdots))) ,$$

but each of these are also transcendental. To actually solve the problem on hand, you're better off plotting $y = x$ with $y = \cos(x)$ and hunting for the intersection of the two.

7.1 Newton's Method

A fascinating trick called *Newton's Method* can be used for solving certain problems, including transcendental equations, by numerical estimation.

Borrowing from the example above, consider a function

$$g(x) = x - \cos(x) ,$$

which has solutions $g(x_*) = 0$ for some (or several or many) special x_*. This setup, of course, works for any scenario where $g(x)$ is a differentiable function, and we'll proceed as if working in general.

Expand $g(x)$ to a first-order approximation

$$g_1(x) = g(x_0) + g'(x_0)(x - x_0) ,$$

where x_0 is some value in the domain of $g(x)$, called an *initial guess* that is presumably not equal to x_*. The variable x represents any point near x_0.

Now, we already know $g(x) = 0$ is hard to deal with, but $g_1(x)$ is *easy* to deal with. Imposing the condition $g_1(x) = 0$ causes x to take on a new value x_1 away from x_0 and presumably closer to x_* as:

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

The reason x_1 is an 'improvement' over the initial guess x_0, i.e. closer to x_*, can be seen geometrically in Figure 1.2. In the Cartesian plane, $g_1(x)$ is the tangent line to the function at x_0. If x_0 is reasonably close to x_* to begin with, we're dealing with a 'zoomed in' picture of $g(x)$ where things behave linearly *anyway*, supposing the function is well-behaved.

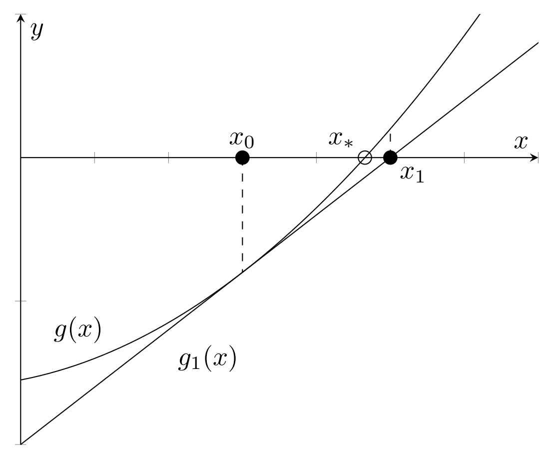

Figure 1.2: Newton's method.

With the improved guess x_1 attained, the process can be repeated to generate x_2, which forms the initial guess for x_3, and so on until you get tired. The process is captured in a single *recursive* formula

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} , \tag{1.82}$$

which, just to remind, attempts to solve $g(x_*) = 0$.

X Equals Cos(X)

Finishing the example that got us here, namely

$$g(x) = x - \cos(x)$$

implies

$$g'(x) = 1 + \sin(x) ,$$

thus we write

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)} .$$

Choosing a reasonable initial guess such as $x_0 = 0.25$, the evolution of x_n proceeds as:

n	x_n
0	0.25
1	0.8263268718020449
2	0.7406169010184902
3	0.7390856504615118
4	0.7390851332152197
5	0.7390851332151607
6	0.7390851332151607

By the sixth iteration, the approximation for x_* seems to have converged to a number whose precision outruns that of the numerical system used. In conclusion, we find $x_* = \cos(x_*)$ is solved by

$$x_* \approx 0.7390851332151607\ldots .$$

Cube Roots

Newton's method need not work only on transcendental equations, as things like cube roots are just as straightforward to churn out as well. The nice part is, you only need a standard four-function calculator to do so. For example, take

$$g(x) = x^3 - 29 ,$$

solved by the cube root of 29. Setting up the proper recursive formula, we have

$$x_{n+1} = x_n - \frac{x^3 - 29}{3x^2}$$

With an initial guess of $x_0 = 3$, the evolution of x_n proceeds as:

n	x_n
0	3
1	3.0740740740740740
2	3.0723178299991580
3	3.0723168256861757
4	3.0723168256858470
5	3.0723168256858475
6	3.072316825685847...

Stopping at six iterations, the result seems to be converging near $x_* \approx 3.072\ldots$, or

$$(29)^{1/3} \approx 3.072316825685847 .$$

Digits of Pi

On a scientific calculator set to radians, type

$$3.14 - \tan(3.14)$$

to get an approximate output

$$\pi \approx 3.14159265\ldots .$$

The reason this works is left as an exercise for the reader.

Second-Order Newton's Method

It's possible to improve the convergence time of Newton's method by including the second order term via

$$g_2(x_0 + h) = g(x_0) + g'(x_0)\,h + \frac{g''(x_0)}{2!}h^2 ,$$

where $h = x - x_0$.

Playing a similar game as the first-order case, the original curve is approximated using a parabola instead of a line. Solutions are attained by setting $g_2(x_0 + h) = 0$ and isolating h with the quadratic formula:

$$h = \frac{-g'(x_0)}{g''(x_0)} \pm \frac{g'(x_0)}{g''(x_0)}\sqrt{1 - \frac{2g(x_0)\,g''(x_0)}{(g'(x_0))^2}}$$

To deal with the square root term, we turn to another order-two approximation in the form of Equation (1.67). Churning through the algebra gives, in abbreviated notation,

$$h = \frac{-g'}{g''} \pm \left(\frac{g'}{g''} - \frac{g}{g'} - \frac{g^2 g''}{2(g')^3}\right) .$$

The second-order result needs to reduce to the first-order result in the small g''-limit, thus we choose the positive root in the solution for h. In final form, h reads

$$h = \frac{-g}{g'}\left(1 + \frac{g g''}{2(g')^2}\right) .$$

Restoring the iterative notation and writing the above as a recursive formula yields a useful improvement to Newton's method:

$$x_{n+1} = x_n - \frac{-g(x_n)}{g'(x_n)}\left(1 + \frac{g(x_n)\,g''(x_n)}{2(g'(x_n))^2}\right) \tag{1.83}$$

7.2 Babylonian Method

A procedure less powerful but slightly more straightforward than Newton's method is something that works on roots alone, credited to the ancient Babylonians.

Square Root

Suppose that we need to estimate the square root of some number N. Proceed by assuming N to be comprised of some lesser number $Q < N$, along with a smaller contribution $x \ll Q$ such that $Q + x = N$, or also

$$Q^2 + 2Qx + x^2 = N^2 \ .$$

If x is 'small enough', then the term x^2 is negligible, allowing the first-order equation in x to be written:

$$x \approx \frac{N^2 - Q^2}{2Q}$$

The formula $Q + x = N$ is replaced with

$$Q + \frac{N^2 - Q^2}{2Q} \approx N \ .$$

Now, if the left side always evaluates to approximately N, it does so especially well for $Q \approx N$, and it should be true that whatever number we get on the left can become the next Q. In other words, we have a recursive formula

$$Q_{n+1} = Q_n + \frac{N^2 - Q_n^2}{2Q_n} \ , \tag{1.84}$$

or more simply,

$$Q_{n+1} = \frac{Q_n}{2} + \frac{N^2}{2Q_n}$$

Cube Root

The Babylonian method for cube roots starts the same as the square root case. This time though, we write the third-power expansion of $Q + x$:

$$Q^3 + 3Q^2x + 3x^2Q + x^3 = N^3 \ ,$$

and then take the x^2- and x^3 terms to be negligible. This means x is approximately

$$x \approx \frac{N^3 - Q^3}{3Q^2} \ ,$$

the recursive formula settles to

$$Q_{n+1} = \frac{2}{3}Q_n + \frac{N^3}{3Q_n^2} \ .$$

Kth Root

One may pursue the generalized Babylonian method for the kth root of the number N. Leaving the details as an exercise, the recursive formula is

$$Q_{n+1} = \left(1 - \frac{1}{k}\right) Q_n + \frac{N^k}{kQ_n^{k-1}} \ .$$

Perhaps not surprisingly, this result is recovering what Newton's method would have said about the same problem. The above can also be written

$$Q_{n+1} = Q_n - \frac{Q_n^k - N^k}{kQ_n^{k-1}} \ ,$$

which is indeed Newton's method applied to

$$g\left(x\right) = x^k - N^k \ ,$$

solved by $x_* = N$.

7.3 Euler's Method

Numerical methods need not be limited to estimating individual numbers, as estimating entire curves is also fair game.

Revisiting the scenario of kinematics with air damping, the situation is governed by the differential equation

$$\frac{dv}{dt} = -g - bv \ ,$$

where $v\left(t\right)$ is the velocity of a falling body, g is the local gravity constant, and b is the linear damping coefficient. By the Frobenius method we were able to jot down exact solutions to this problem, namely

$$\begin{aligned} v\left(t\right) &= \frac{-g}{b} + \left(v_0 + \frac{g}{b}\right) e^{-bt} \\ x\left(t\right) &= x_0 - \frac{gt}{b} + \frac{1}{b}\left(v_0 + \frac{g}{b}\right)\left(1 - e^{-bt}\right) \ , \end{aligned}$$

where after supplying the initial values v_0, x_0, the motion is completely determined.

The haunting question now is, what if we could not easily get hold of solutions for $v\left(t\right)$, $x\left(t\right)$? It seems that things fell into place by pure luck in a sense, and if the differential equation had been more complicated, maybe solutions would be be hopelessly tangled up.

A technique called *Euler's method* allows for approximating the path of motion directly from the differential equation. Assuming v_0, x_0 as given, the idea is, much like Newton's method, to calculate the updated information v_1, x_1 using linear approximations.

Forward Euler's Method

Starting with an easy case, consider the frictionless constant-acceleration scenario characterized by

$$\begin{aligned} dv/dt &= -g \\ dx/dt &= v(t) \ . \end{aligned}$$

Expanding each of these to first order, we have

$$\begin{aligned} v(t) &\approx v(t_0) - g\Delta t \\ x(t) &\approx x(t_0) + v(t_0)\,\Delta t \ . \end{aligned}$$

To turn these into something useful, we first understand that the quantity Δt is meant to be a 'small enough' number such that $g\Delta t$ and $v(t_0)\,\Delta t$ are small compared to $v(t_0)$, $x(t_0)$. (This is the essence of the first-order approximation.)

Also, note that the right side of each equation contains all 'known' information, and the left side contains the 'updated' versions of v, x. Much like Newton's method, this setup invites a recursive representation given by

$$v_{n+1} = v_n - g\Delta t \tag{1.85}$$

$$x_{n+1} = x_n + v_n\Delta t \tag{1.86}$$

This setup in particular is called the *forward* Euler's method. Supplying v_0, x_0 as initial values, the above can be used to estimate the subsequent motion as a set of points. The number of points generated has to do with the size of Δt and the total time interval being considered.

Backward Euler's Method

Recalling the definition of the derivative, namely

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \ ,$$

note that the definition remains intact by reversing the sign on h:

$$f'(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \ .$$

Moreover, no error is made if we simply shift variables $x \to x + h$, so we can also write

$$\lim_{h\to 0} f'(x+h) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \ ,$$

which essentially recovers the definition.

Recasting the above as a first-order approximation, we get, after rearranging,

$$f(x+h) \approx f(x) + hf'(x+h) \ .$$

Interestingly, the f'-term uses the updated version of x, namely $x + h$ as its argument. This configuration leads to the *backward* Euler'd method, and is an implicit formula in the sense that some extra work needs to be done to isolate $f(x+h)$ in terms of initial quantities.

For the problem on hand, the backward Euler's method is represented recursively via

$$v_{n+1} = v_n - g\Delta t \tag{1.87}$$

$$x_{n+1} = x_n + v_{n+1}\Delta t \ , \tag{1.88}$$

which is subtlety different from Equations (1.85), (1.86). Note that in a more general case, the g-term would instead be a_{n+1}.

Energy Considerations

In the absence of friction, freefall kinematics qualifies as an energy-conserving system. At any point during motion, the kinetic energy is given by

$$T(v) = \frac{1}{2}mv^2 \ ,$$

where m is the mass of the object that is falling. Meanwhile, freefall near sea level implies the potential energy is

$$U(x) = mgx \ ,$$

where g is the familiar gravity constant. For this situation, conservation of energy means

$$E = T(v) + U(x)$$

is constant.

If conservation of energy is to hold, then the recursive formulas for the forward and backward Euler's method ought to reflect this. At a given step n, the energy is

$$E_n = \frac{1}{2}mv_n^2 + mgx_n \ .$$

At the next step $n+1$ the same energy ought to read

$$E_{n+1} = \frac{1}{2}mv_{n+1}^2 + mgx_{n+1} \ .$$

All is fair until we want to substitute v_{n+1} and x_{n+1} into E_{n+1}. That is, nothing says to only use the forward method represented by Equations (1.85), (1.86), or for that matter, nothing forbids the pair of Equations (1.87), (1.88). Which pair is correct? At this point we're obligated to try both, and doing each case carefully, we find:

$$\text{forward: } E_{n+1} = E_n + \frac{1}{2}mg^2\Delta t^2$$

$$\text{backward: } E_{n+1} = E_n - \frac{1}{2}mg^2\Delta t^2$$

Evidently, the energy is conserved to zeroth order and first order by each method. There is, however, a pesky second-order term lingering in each result. This will surely introduce artificiality in the results.

It should be noted that error can be minimized when Δt is very small, but this still leaves the question of, can we do better?

Mixed Euler's Method

Given how the forward and backward Euler's method produce equal and opposite errors in the total energy, one has to wonder if some mixture of the methods will be better than either alone. Trying the average of the two, we write the *mixed* Euler's method for this problem:

$$v_{n+1} = v_n - g\Delta t \tag{1.89}$$

$$x_{n+1} = x_n + \frac{1}{2}\left(v_n + v_{n+1}\right)\Delta t \tag{1.90}$$

As it turns out, this pair of equations does in fact satisfy $E_{n+1} = E_n$ which you are encouraged to verify.

To see each of Euler's methods in action against a concrete problem, consider the one-dimensional motion of a body that begins at $x_0 = 10\,m$ at $t = 0\,s$ with initial upward speed of $v_0 = 10\,m/s$. For a time step we'll use $\Delta t = 0.1\,s$ over a total of 20 iterations. Writing the appropriate C program and producing graphs with *gnuplot*, we generate the outputs shown in Figure 1.3.

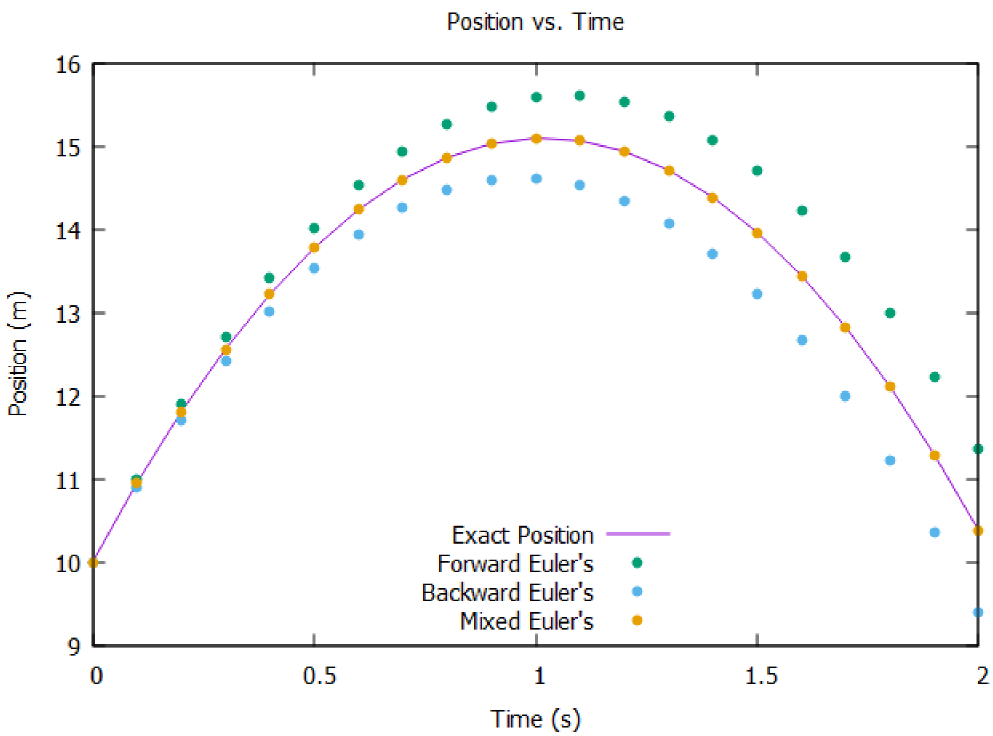

Figure 1.3: Various Euler's method approximations of ideal freefall motion compared to exact solution.

As shown in the Figure, the exact solution is traced by a solid line with the three approximations, namely forward, backward, mixed, appearing as unconnected colored dots. The forward method sails consistently over the exact solution, while the backward method sails under. Perhaps not surprisingly, the mixed method approximation stays perfectly with the exact solution.

Air Damping Problem

Returning to the problem of kinematics with air damping, governed by

$$\frac{dv}{dt} = -g - bv\,,$$

we can immediately dispense with any hope of conserving energy, as the effect of friction eats away at the kinetic component without replenishing the potential. Nonetheless, we may still approximate solutions with variations of Euler's method.

For the air damping problem, a set of 'forward' equations are (as always) easy to write explicitly:

$$v_{n+1} = v_n - g\Delta t - bv_n\Delta t$$
$$x_{n+1} = x_n + v_n\Delta t$$

That is, the above is analogous to Equations (1.85), (1.86) and only differ by the presence of the b-term.

As for a 'backward' set of equations, replace downstream n on the right with $n+1$ to get

$$v_{n+1} = v_n - g\Delta t - bv_{n+1}\Delta t$$
$$x_{n+1} = x_n + v_{n+1}\Delta t \,,$$

analogous to Equations (1.87), (1.88). Solving for v_{n+1} and x_{n+1} explicitly, we find these to mean

$$v_{n+1} = \frac{v_n - g\Delta t}{1 + b\Delta t}$$
$$x_{n+1} = x_n + \left(\frac{v_n - g\Delta t}{1 + b\Delta t}\right)\Delta t \,.$$

Finally, we can pursue a set of 'mixed' equations by imposing the average on the downstream terms as

$$v_{n+1} = v_n - g\Delta t - \frac{b}{2}\left(v_n + v_{n+1}\right)\Delta t$$
$$x_{n+1} = x_n + \left(v_n + v_{n+1}\right)\Delta t \,,$$

which are analogous to Equations (1.89), (1.90). After some effort, these can be expressed entirely with $n+1$ on the left and n on the right:

$$v_{n+1} = \frac{v_n\left(1 - b\Delta t/2\right) - g\Delta t}{1 + b\Delta t/2}$$
$$x_{n+1} = x_n + \left(\frac{v_n - g\Delta t/2}{1 + b\Delta t/2}\right)\Delta t$$

We're now in position to plot each of the three approximations along with the exact solution for $x\left(t\right)$ for the air damping problem. Using the same initial conditions as previous while introducing a damping coefficient of $b = 3\,s^{-1}$, we generate the output shown in Figure 1.4.

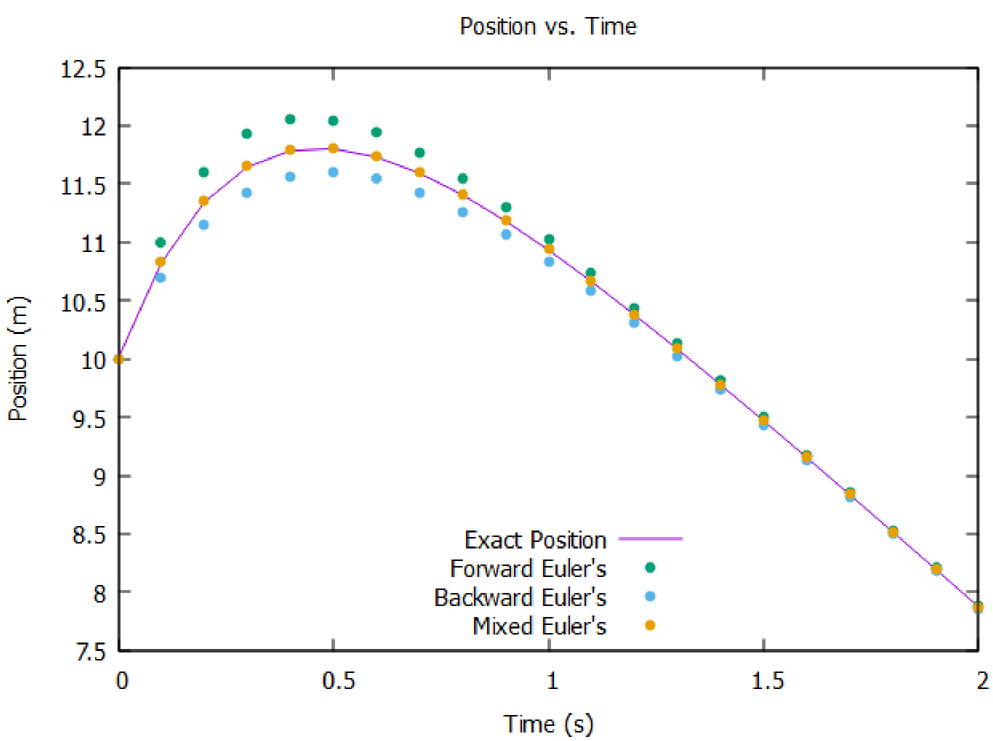

Figure 1.4: Various Euler's method approximations of damped freefall motion compared to exact solution.

In the Figure, note that all approximations agree with the exact solution in the large-t limit, corresponding to the falling body reaching terminal velocity. The slope of the asymptotic line implied ought to be $-g/b$, or roughly $-3.3\,m/s$ in the plot.

Comparing the overall performance of each approximation, as seen in the ideal case, the forward method is a little too generous in its output, and the backward method is a little too thrifty. Astonishingly though, the mixed method is spot on with the exact solution.

8 Antiderivative

Reflecting on the notion of the derivative $f'(x)$ as it relates to the original function $f(x)$, there is a sense that the derivative operator is a one-way arrow from one 'space' of functions to another. For any given $f(x)$, we can more-or-less confidently calculate $f'(x)$ using the techniques gained above.

The derivative calculation begs an interesting question though, namely, can we start with $f'(x)$ and infer what the original $f(x)$ could have been? This idea is like running the derivative operator backwards, and is aptly named the *antiderivative*.

8.1 Motivation

A natural way to frame the antiderivative question starts with definition of the derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} ,$$

and then set the left side to a function that is given, call it $g(x)$:

$$g(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Next, consider any sequence of manipulations, symbolized by Q, that is applied to both sides of the above. By 'manipulations', we mean adding zero, multiplying by one, and so on. Symbolically, this would mean

$$Q(g(x_0)) = Q\left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right) .$$

If the sequence Q is chosen properly, the quantity $Q(g(x_0))$ on the left is some new function of x_0, which, and this is the key - should match the form of a known derivative. That is, we should be able to recognize $Q(g(x_0))$ as the derivative of some previously-cataloged function $r(x)$. This lets us replace the right side of the above:

$$Q(g(x_0)) = \frac{d}{dx}(r(x))\bigg|_{x_0}$$

Finally, isolate $g(x_0)$ algebraically via

$$g(x_0) = Q^{-1}\left(\frac{d}{dx}(r(x))\bigg|_{x_0}\right) ,$$

where Q^{-1} reverses the manipulations represented by Q.

8.2 Exemplary Cases

Natural Logarithm

Going for an interesting example, it turns out that the natural logarithm $\ln(x)$ didn't turn up as the result of any derivative calculation previously done. Letting $f'(x) = \ln(x)$, we can puzzle out $f(x)$ starting with:

$$\begin{aligned} \ln(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ \ln(x_0) + 1 &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + 1 \end{aligned}$$

By adding 1 to each side, the left is suddenly recognizable from Equation (1.35), which reads

$$\frac{d}{dx}(\ln(x^x)) = \ln(x) + 1 .$$

Knowing this, replace the right side of our working equation:

$$\begin{aligned} \ln(x_0) + 1 &= \frac{d}{dx}(\ln(x^x))\bigg|_{x_0} \\ \ln(x_0) + 1 &= \lim_{x \to x_0} \frac{\ln(x^x) - \ln(x_0^{x_0})}{x - x_0} \end{aligned}$$

Now subtract 1 from each side, thereby applying Q^{-1}, and simplify:

$$\ln(x_0) = \lim_{x \to x_0} \frac{(x\ln(x) - x) - (x_0 \ln(x_0) - x_0)}{x - x_0}$$

The right side is none other than the derivative of $x\ln(x) - x$, and we're done:

$$\ln(x) = \frac{d}{dx}(x\ln(x) - x) \tag{1.91}$$

X Times Cos(X)

It's a bit more practical to work in Leibniz notation if we have a good handle of how to isolate the desired derivative.

To illustrate, suppose we want to know which function has a slope of $f'(x) = x\cos(x)$. Reaching for a table of derivatives, recall Equation (1.28), namely

$$\frac{d}{dx}(x\sin(x)) = \sin(x) + x\cos(x) ,$$

which contains the answer as the rightmost term. To proceed, note that the sine term can be replaced by the negative derivative of the cosine:

$$\frac{d}{dx}(x\sin(x)) = \frac{d}{dx}(-\cos(x)) + x\cos(x)$$

Since the derivative operator is a linear one, we can cram all derivative terms on the same side to write the final answer:

$$x\cos(x) = \frac{d}{dx}\left(x\sin(x) + \cos(x)\right) \tag{1.92}$$

By identical reasoning, one can work out the case $f'(x) = x\sin(x)$ using Equation (1.29). Leaving the details as an exercise, the result is

$$x\sin(x) = \frac{d}{dx}\left(\sin(x) - x\cos(x)\right) . \tag{1.93}$$

8.3 Powers and Roots

The rule governing powers and roots is covered by Equation (1.2), namely

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} ,$$

which reads cleanly in both directions, i.e. it's ready for derivative and antiderivatives.

In light of the chain rule, we can replace x with a function $f(x)$ to have:

$$\frac{d}{dx}\left((f(x))^n\right) = \frac{n}{(f(x))^{n-1}}\frac{d}{dx}\left(f(x)\right)$$

Despite the above result being general, it's still a bit messy and not worth memorizing. One exception, though, is the special case $n = 1/2$:

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$

As you train your eye solve antiderivative problems, it helps to know that the ratio $f'/\sqrt{f}$ can be dealt with using the above.

To illustrate, consider the case

$$g'_{\pm}(x) = \frac{\pm x}{\sqrt{1 \pm x^2}} ,$$

which has $f(x) = 1 \pm x^2$ and $f'(x) = \pm 2x$. Immediately from this, we can write:

$$\frac{\pm x}{\sqrt{1 \pm x^2}} = \frac{d}{dx}\sqrt{1 \pm x^2} \tag{1.94}$$

Reciprocal

One exception to the usual pattern for powers and roots is the reciprocal function $f'(x) = 1/x$. This antiderivative is handled by Equation (1.14) going backwards, namely:

$$\frac{1}{x} = \frac{d}{dx}\left(\ln(x)\right)$$

8.4 Logarithmic Antiderivatives

Diminished Natural Logarithm

Much of the struggle in calculating antiderivatives is deciding which functions to try. For instance, suppose we have the diminished natural logarithm represented by

$$f'(x) = \frac{\ln(x)}{x} .$$

After some fiddling with the chain rule, one eventually stumbles upon

$$\frac{d}{dx}\left((\ln(x))^2\right) = \frac{2}{x}\ln(x) . \tag{1.95}$$

Recognizing the original problem embedded on the right, we have the answer:

$$\frac{\ln(x)}{x} = \frac{1}{2}\frac{d}{dx}\left((\ln(x))^2\right) . \tag{1.96}$$

Shifted Natural Logarithm

Consider the case $f'(x) = \ln(x+1)$. For this, use the product rule to establish

$$\frac{d}{dx}\left((x+1)\ln(x+1)\right) = \ln(x+1) + 1 .$$

In order to isolate $x\ln(x)$, everything else must be part of the same derivative, and this can be done by replacing the 1-term via

$$1 = \frac{dx}{dx} .$$

Substituting this into the above and simplifying gives the result we're after:

$$\ln(x+1) = \frac{d}{dx}\left((x+1)\ln(x+1) - x\right) \tag{1.97}$$

Nonlinear Natural Logarithm

Consider the case $f'(x) = x\ln(x)$. For this, use the product rule to establish

$$\frac{d}{dx}\left(x^2\ln(x)\right) = 2x\ln(x) + x .$$

Like the previous case, in order to isolate $x\ln(x)$, everything else must be part of the same derivative, and this can be done by replacing the x-term via

$$x = \frac{1}{2}\frac{d}{dx}\left(x^2\right) .$$

Substituting this into the above and simplifying gives the result we're after:

$$x\ln(x) = \frac{1}{2}\frac{d}{dx}\left(x^2\left(\ln(x) - \frac{1}{2}\right)\right) \tag{1.98}$$

Modified Natural Logarithm

Finding the antiderivative of the modified natural logarithm $f'(x) = \ln\left(1+x^2\right)$ is a challenge. To begin we'll write something that contains $1+x^2$ and hope for the best, particularly:

$$\frac{d}{dx}\left(x\ln\left(1+x^2\right)\right) = \ln\left(1+x^2\right) + \frac{2x^2}{1+x^2}$$

The rightmost term can be split apart by algebra

$$\frac{2x^2}{1+x^2} = 2\left(1-\frac{1}{1+x^2}\right) ,$$

and now the problem reduces to writing the parenthesized quantity as a derivative. Luckily, we know exactly how to do this:

$$\frac{2x^2}{1+x^2} = 2\left(\frac{dx}{dx} - \frac{d}{dx}\arctan(x)\right)$$

Putting all derivative terms on the same side yields the answer:

$$\ln\left(1+x^2\right) = \frac{d}{dx}\left(x\left(\ln\left(1+x^2\right)-2\right)\right) + 2\frac{d}{dx}\left(\arctan(x)\right) \tag{1.99}$$

8.5 Exponential Antiderivatives

Exponential Times X

To handle the case $f'(x) = xe^x$, use the product rule on the same quantity

$$\frac{d}{dx}\left(xe^x\right) = e^x + xe^x ,$$

and notice the right-most term contains the answer we want. Exploiting the fact that e^x is its own derivative, we can write everything else as a total derivative to have the answer:

$$xe^x = \frac{d}{dx}\left(e^x\left(x-1\right)\right) \tag{1.100}$$

Exponential Times X*X

To handle the case $f'(x) = x^2e^x$, use the product rule on the same quantity

$$\frac{d}{dx}\left(x^2e^x\right) = 2xe^x + x^2e^x ,$$

and notice the right-most term contains the answer we want. The middle term would be show-stopper if it weren't for Equation (1.100), which allows the rest to be written as a total derivative:

$$x^2e^x = \frac{d}{dx}\left(e^x\left(x^2-2x+2\right)\right) \tag{1.101}$$

Exponential Times Cos(X)

The set of problems

$$f_1'(x) = e^x\cos(x)$$
$$f_2'(x) = e^x\sin(x)$$

can be handled simultaneously. First, write two results easily attainable by the product rule:

$$\frac{d}{dx}\left(e^x\sin(x)\right) = e^x\sin(x) + e^x\cos(x)$$
$$\frac{d}{dx}\left(e^x\cos(x)\right) = e^x\cos(x) - e^x\sin(x)$$

Next, take the sum and the difference of the two above equations and exploit the linearity of the derivative operator to get both results at once:

$$e^x\cos(x) = \frac{1}{2}\frac{d}{dx}\left(e^x\cos(x) + e^x\sin(x)\right) \tag{1.102}$$
$$e^x\sin(x) = \frac{1}{2}\frac{d}{dx}\left(e^x\sin(x) - e^x\cos(x)\right) \tag{1.103}$$

8.6 Trigonometric Antiderivatives

Tangent and Cotangent

The case for $f'(x) = \tan(x)$ is a bit tricky. Hunting for any derivative calculation that has $\tan(x)$ as part of the answer, Equation (1.24) comes to mind, namely

$$\frac{d}{dx}\left(\sec(x)\right) = \tan(x)\sec(x) .$$

Proceed by letting $u = \sec(x)$ and separate variables:

$$\frac{1}{u}\frac{du}{dx} = \tan(x)$$

By the chain rule, or equivalently by the 'logarithm trick' represented by Equation (1.36), the left side is equivalent to the derivative of the natural log of u:

$$\frac{d}{dx}\left(\ln(u)\right) = \tan(x)$$

Reversing the u-substitution, the final answer is

$$\tan(x) = \frac{d}{dx}\left(-\ln\left(\cos(x)\right)\right) . \tag{1.104}$$

By a similar line of reasoning, the cotangent version can also be done, with the details left as an exercise:

$$\cot(x) = \frac{d}{dx}\left(\ln\left(\sin(x)\right)\right) . \tag{1.105}$$

Secant and Cosecant

The case of $f'(x) = \sec(x)$ can be attacked with partial fractions. Following the algebra, we find

$$\begin{aligned}\frac{1}{\cos(x)} &= \frac{\cos(x)}{\cos^2(x)} = \frac{\cos(x)}{1-\sin^2(x)} \\ &= \frac{1}{2}\left(\frac{\cos(x)}{1-\sin(x)} + \frac{\cos(x)}{1+\sin(x)}\right) .\end{aligned}$$

Now, spotting this takes some getting used to, but the above can be rewritten using the logarithm trick

$$\begin{aligned}\frac{1}{\cos(x)} &= -\frac{1}{2}\frac{d}{dx}\left(\ln\left(1-\sin(x)\right)\right) \\ &\quad + \frac{1}{2}\frac{d}{dx}\left(\ln\left(1+\sin(x)\right)\right) ,\end{aligned}$$

which can be simplified and the problem is solved:

$$\sec(x) = \frac{1}{2}\frac{d}{dx}\left(\ln\left(\frac{1+\sin(x)}{1-\sin(x)}\right)\right) \tag{1.106}$$

With a little more algebra, the above can be simplified to

$$\sec(x) = \frac{d}{dx}\left(\ln\left(\sec(x)+\tan(x)\right)\right) .$$

Leaving the details for an exercise, a similar exercise leads to the cosecant version

$$\csc(x) = -\frac{1}{2}\frac{d}{dx}\left(\ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right)\right) , \tag{1.107}$$

or

$$\csc(x) = -\frac{d}{dx}\left(\ln\left(\csc(x)+\cot(x)\right)\right) .$$

Cos(X) Squared

The set of problems

$$\begin{aligned}f_1'(x) &= \cos^2(x) \\ f_2'(x) &= \sin^2(x)\end{aligned}$$

can be handled simultaneously. First, note from the product rule that

$$\frac{d}{dx}\left(\sin(x)\cos(x)\right) = \cos^2(x) - \sin^2(x) ,$$

which is equivalent to both of:

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)\cos(x)\right) &= 1 - 2\sin^2(x) \\ \frac{d}{dx}\left(\sin(x)\cos(x)\right) &= 2\cos^2(x) - 1\end{aligned}$$

Finally, note that the factor 1 is equivalent to dx/dx, which allows the left side (in each) to be written as a total derivative, leading to

$$\sin^2(x) = \frac{1}{2}\frac{d}{dx}\left(x - \sin(x)\cos(x)\right) \tag{1.108}$$

$$\cos^2(x) = \frac{1}{2}\frac{d}{dx}\left(x + \sin(x)\cos(x)\right) \tag{1.109}$$

Cos(X) Times Sin(X)

The case $f'(x) = \cos(x)\sin(x)$ has two unique answers. Handling both possibilities in one blow, use the chain rule to write

$$\frac{d}{dx}\left(\sin^2(x)\right) = -\frac{d}{dx}\left(\cos^2(x)\right) = 2\cos(x)\sin(x) ,$$

and the result is isolated:

$$\cos(x)\sin(x) = \frac{1}{2}\frac{d}{dx}\left(\left(\sin(x)\right)^2\right) \tag{1.110}$$

$$\cos(x)\sin(x) = \frac{-1}{2}\frac{d}{dx}\left(\left(\cos(x)\right)^2\right) \tag{1.111}$$

8.7 Inverse Trig Antiderivatives

Arccosine and Arcsine

The roulette of inverse trigonometric functions can also be tackled. Going for $f'(x) = \arccos(x)$ first, consider the following application of the product rule:

$$\frac{d}{dx}\left(x\arccos(x)\right) = \arccos(x) + x\frac{d}{dx}\left(\arccos(x)\right)$$

The derivative of $\arccos(x)$ can be replaced by Equation (1.37), namely

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} ,$$

and the above becomes

$$\frac{d}{dx}\left(x\arccos(x)\right) = \arccos(x) - \frac{x}{\sqrt{1-x^2}} .$$

The square root term is itself the derivative of a function obeying Equation (1.94), or

$$\frac{-x}{\sqrt{1\pm x^2}} = \frac{d}{dx}\sqrt{1-x^2} .$$

Condensing derivatives on one side with $\arccos(x)$ on the other gives the answer:

$$\arccos(x) = \frac{d}{dx}\left(x\arccos(x) - \sqrt{1-x^2}\right) \tag{1.112}$$

By similar reasoning, the $\arcsin(x)$ case works out as

$$\arcsin(x) = \frac{d}{dx}\left(x\arcsin(x) + \sqrt{1-x^2}\right) . \tag{1.113}$$

Arctangent and Arccotangent

To handle $f'(x) = \arctan(x)$, consider the following application of the product rule:

$$\frac{d}{dx}(x \arctan(x)) = \arctan(x) + x\frac{d}{dx}(\arctan(x))$$

The derivative of $\arctan(x)$ can be replaced by Equation (1.39), namely

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} ,$$

and the above becomes

$$\frac{d}{dx}(x \arctan(x)) = \arctan(x) + \frac{x}{1+x^2} .$$

The rightmost term needs to be replaced with the derivative of something. It turns out that Equation (1.15) does the job, namely

$$\frac{d}{dx}\left(\ln\left(1+x^2\right)\right) = \frac{2x}{1+x^2} .$$

Condensing derivatives on one side with arctan on the other gives the answer:

$$\arctan(x) = \frac{d}{dx}\left(x \arctan(x) - \frac{\ln\left(1+x^2\right)}{2}\right) \tag{1.114}$$

By similar reasoning, the $\operatorname{arccot}(x)$ case works out as

$$\operatorname{arccot}(x) = \frac{d}{dx}\left(x \operatorname{arccot}(x) + \frac{\ln\left(1+x^2\right)}{2}\right) . \tag{1.115}$$

8.8 Trigonometric Substitution

Arcsecant and Arccosecant

The last two inverse trig functions, namely the arcsecant and the arc-cosecant, don't follow as easily as the others. To handle $f'(x) = \operatorname{arcsec}(x)$, consider the application of the product rule

$$\frac{d}{dx}(x \operatorname{arcsec}(x)) = \operatorname{arcsec}(x) + x\frac{d}{dx}(\operatorname{arcsec}(x)) ,$$

which, by Equation (1.40), is equivalent to

$$\frac{d}{dx}(x \operatorname{arcsec}(x)) = \operatorname{arcsec}(x) + \frac{1}{\sqrt{x^2-1}} .$$

As usual, the rightmost term needs to be the derivative of something else. Innocent as this looks, a different technique called *trigonometric substitution* must be used. For the example on hand, introduce a new variable θ such that

$$x = \sec(\theta) .$$

Then, standard trig identities tell us

$$\tan^2(\theta) = \sec^2(\theta) - 1 = x^2 - 1 ,$$

or

$$\tan(\theta(x)) = \sqrt{x^2-1} .$$

Meanwhile, we can take the θ-derivative of x to write

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sec(\theta)) = \sec(\theta)\tan(\theta) = x\sqrt{x^2-1} ,$$

and by the chain rule, this means

$$\frac{d\theta}{dx} = \frac{1}{x\sqrt{x^2-1}} .$$

Now, if the term $1/\sqrt{x^2-1}$ is to be the derivative of some unknown function $q(x)$, we have, by the chain rule,

$$\frac{dq}{dx} = \frac{dq}{d\theta}\frac{d\theta}{dx} = \frac{dq}{d\theta}\frac{1}{x\sqrt{x^2-1}} ,$$

which can only mean

$$1 = \frac{dq}{d\theta}\frac{1}{x} .$$

Since x is already known as $\sec(\theta)$, the question has come to looking for a function whose derivative is $\sec(\theta)$. For this we may refer to Equation (1.106) using θ as the variable:

$$\sec(\theta) = \frac{d}{d\theta}(\ln(\sec(\theta) + \tan(\theta))) ,$$

or

$$q(\theta) = \ln(\sec(\theta) + \tan(\theta)) .$$

Switching variables back to x, the above reads

$$q(x) = \ln\left(x + \sqrt{x^2-1}\right) .$$

This result alone is worth noting,

$$\frac{1}{\sqrt{x^2-1}} = \frac{d}{dx}\left(\ln\left(x + \sqrt{x^2-1}\right)\right) , \tag{1.116}$$

and more importanly, we can write the final answer to the $\operatorname{arcsec}(x)$ antiderivative:

$$\operatorname{arcsec}(x) = \frac{d}{dx}\left(x \operatorname{arcsec}(x) - \ln\left(x + \sqrt{x^2-1}\right)\right) \tag{1.117}$$

A similar exercise gives the $\operatorname{arccsc}(x)$ version:

$$\operatorname{arccsc}(x) = \frac{d}{dx}\left(x \operatorname{arccsc}(x) + \ln\left(x + \sqrt{x^2-1}\right)\right) \tag{1.118}$$

8.9 Hyperbolic Cases

The nontrivial hyperbolic trig antiderivatives are mostly analogous to their ordinary trig counterparts:

$$\tanh(x) = \frac{d}{dx}\left(\ln\left(\cosh(x)\right)\right) \tag{1.119}$$

$$\coth(x) = \frac{d}{dx}\left(\ln\left(\sinh(x)\right)\right) \tag{1.120}$$

The hyperbolic secant and cosecant are a little less obvious, but turn out to be:

$$\operatorname{sech}(x) = \frac{d}{dx}\left(2\arctan\left(\tanh\left(\frac{x}{2}\right)\right)\right) \tag{1.121}$$

$$\operatorname{csch}(x) = \frac{d}{dx}\left(\ln\left(\tanh\left(\frac{x}{2}\right)\right)\right) \tag{1.122}$$

The details are left for an exercise.

Arccosh and Arcsinh

Luckily, some of the inverse hyperbolic trig derivatives are analogous to calculations previously done. For the cases arccosh, arcsinh, one straightforwardly finds

$$\operatorname{arccosh}(x) = \frac{d}{dx}\left(x\operatorname{arccosh}(x) - \sqrt{x^2-1}\right) \tag{1.123}$$

$$\operatorname{arcsinh}(x) = \frac{d}{dx}\left(x\operatorname{arcsinh}(x) - \sqrt{x^2+1}\right) . \tag{1.124}$$

Arctanh and Arccoth

The arccosh, arcsinh functions have antiderivatives that are remarkably similar to their trig counterparts:

$$\operatorname{arctanh}(x) = \frac{d}{dx}\left(x\operatorname{arctanh}(x) + \frac{\ln\left(1-x^2\right)}{2}\right) \tag{1.125}$$

$$\operatorname{arccoth}(x) = \frac{d}{dx}\left(x\operatorname{arccoth}(x) + \frac{\ln\left(1-x^2\right)}{2}\right) \tag{1.126}$$

Arcsech and Arccsch

The arcsech, arccsch functions can be aced with a trick. Handling both at the same time, introduce two unknown functions $f(x)$ and $g(x)$ to write the following total derivatives

$$\operatorname{arcsech}(x) = \frac{d}{dx}\left(f(x) + x\operatorname{arcsech}(x)\right)$$
$$\operatorname{arccsch}(x) = \frac{d}{dx}\left(g(x) + x\operatorname{arccsch}(x)\right) ,$$

and now the whole problem is about finding out what $f(x)$ and $g(x)$ must be.

Applying the product rule across the right side, we find

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$
$$g'(x) = \frac{1}{\sqrt{1+x^2}} ,$$

telling us, according to Equations (1.38), (1.50), that

$$f(x) = \arcsin(x)$$
$$g(x) = \operatorname{arcsinh}(x) ,$$

and thus:

$$\operatorname{arcsech}(x) = \frac{d}{dx}\left(x\operatorname{arcsech}(x) + \arcsin(x)\right) \tag{1.127}$$

$$\operatorname{arccsch}(x) = \frac{d}{dx}\left(x\operatorname{arccsch}(x) + \operatorname{arcsinh}(x)\right) \tag{1.128}$$

9 Simple Harmonic Oscillator

The *simple harmonic oscillator* is a mathematical model used for approximating many real-world systems. Common simple harmonic oscillators are (i) a mass attached to a spring moving in a frictionless environment, (ii) a hanging pendulum making small deflections from equilibrium, or most generally, (iii) small displacements in any system featuring a local minimum in the potential energy.

Problem Setup

To get going, consider a body of mass m tethered to the point $x = 0$ subject to the linear restoring force

$$F = -kx$$

corresponding to a potential energy

$$U_{spring}(x) = \frac{1}{2}kx^2 .$$

By Newton's second law

$$m\frac{d^2}{dt^2}x(t) = -\frac{d}{dx}U(x) ,$$

we can assemble an equation governing the so-called harmonic motion of the oscillator:

$$\frac{d^2}{dt^2}x(t) = -\frac{k}{m}x(t) \tag{1.129}$$

Finding the Solution

The task now is to 'solve' the above equation, which means to find the correct $x(t)$ that satisfies it. With $x(t)$ in hand, we will know the position of the body as a function of time.

We seek $x(t)$ as a function whose second derivative is equal to the negative of itself multiplied by a constant. Right away, two trigonometric functions come to mind:

$$\frac{d^2}{dt^2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{-k}{m}\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$
$$\frac{d^2}{dt^2}\sin\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{-k}{m}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

Angular Frequency

Both the cosine and the sine seem to satisfy Equation (1.129), so let's keep track of both for a moment. The quantity $\sqrt{k/m}$ is called the *angular frequency* and is designated the symbol ω (Greek *omega*):

$$\omega = \sqrt{\frac{k}{m}}$$

So far we can sketch out two possible solutions:

$$x_1(t) \propto \cos(\omega t)$$
$$x_2(t) \propto \sin(\omega t)$$

Scaling Constants

Notice now that scaling each of these by an unknown constant to make $A\cos(\omega t)$, $B\sin(\omega t)$ would leave the oscillator equation unchanged, yet the presence of scaling constants would clearly affect the final solution. The updated solutions are now

$$x_1(t) = A\cos(\omega t)$$
$$x_2(t) = B\sin(\omega t)$$

fur two undetermined coefficients A, B.

General Solution

With the preparatory work done, we can write a general solution for the problem:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \ ,$$

and from $x(t)$ we can take a time derivative to get the velocity:

$$v(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

For a sanity check, we should be able to take the time derivative of $v(t)$ to recover Equation (1.129). Doing so, we find

$$\begin{aligned}\frac{d^2}{dt^2}x(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\\ &= -\omega^2\left(A\cos(\omega t) + B\sin(\omega t)\right)\\ &= -\omega^2 x(t)\\ &= \frac{-k}{m}x(t)\end{aligned}$$

as expected.

Initial Conditions

To refine the solution to the harmonic oscillator equation, suppose at $t = 0$ the body is known to be at position x_0 with initial velocity v_0. Such *initial conditions* can be worked into the solution by setting $t = 0$ in the x- and v-equations

$$x_0 = x(0) = A\cos(0) + 0$$
$$v_0 = v(0) = 0 + B\omega\cos(0)$$

to discern:

$$A = x_0$$
$$B = v_0/\omega$$

With this, the updated position and velocity read:

$$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$
$$v(t) = -x_0\omega\sin(\omega t) + v_0\cos(\omega t)$$

Magnitude and Phase

While the above is a workable solution to the simple harmonic oscillator, everything can be made tighter by introducing a *magnitude* coefficient R, along with a *phase* coefficient ϕ such that:

$$x_0 = R\cos(\phi)$$
$$\frac{v_0}{\omega} = -R\sin(\phi)$$

The magnitude and phase have a trigonometric relationship to the initial conditions, namely

$$R = \sqrt{x_0^2 + v_0^2/\omega^2}$$
$$\phi = \arctan\left(\frac{-v_0}{\omega x_0}\right) .$$

In terms of R and ϕ, the solution $x(t)$ reads

$$x(t) = R\cos(\phi)\cos(\omega t) - R\sin(\phi)\sin(\omega t) \ ,$$

which, using a trigonometric angle-sum formula, simplifies to:

$$x(t) = R\cos(\omega t + \phi)$$