Conic Sections
MANUSCRIPT

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Chapter 1

Conic Sections

1 Ellipse

1.1 Definition

In the Cartesian plane, consider a point labeled *focus* that is distance p from a vertical line labled *directrix*. Now, let us seek the set of points $\{s\} = \{(x, y)\}$ that satisfy the following rule: the distance R to the focus divided by the (purely horizontal) distance Q to the directrix equals a constant $e < 1$. In algebraic terms, this means

$$\frac{R}{Q} = e < 1 \,. \tag{1.1}$$

Sketched in Figure 1.1 are some of the points that obey such a rule.

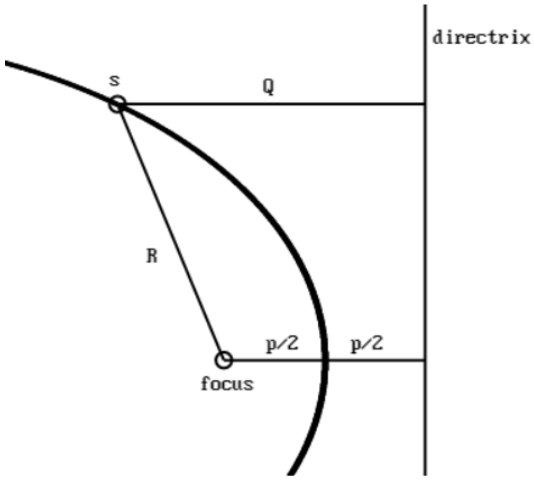

Figure 1.1: Points obeying $R/Q = e < 1$ as measured from a directrix and a focus separated by distance p. The origin is at the focus.

To determine the proper shape defined by the rule, begin with Equation (1.1) and discern from inspection that R, Q can be written:

$$R = \sqrt{x^2 + y^2} \tag{1.2}$$

$$Q = p - x \tag{1.3}$$

Inserting the above into Equation (1.1) and completing the square in x, one finds

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{1.4}$$

describing an *ellipse* centered at $x = -c$. The *semimajor axis* a, *semiminor axis* b, and offset c relate to e, p by:

$$a = \frac{ep}{1-e^2} \tag{1.5}$$

$$b = \frac{ep}{\sqrt{1-e^2}} \tag{1.6}$$

$$c = ae \tag{1.7}$$

The ellipse has two *vertex* points at $(-c \pm a, 0)$, and two *covertex* points at $(-c, \pm b)$.

Problem 1

Derive Equation (1.4) simultaneously with Equations (1.5), (1.6), (1.7).

1.2 Eccentricity

The constant e is called the *eccentricity* of the ellipse, and characterizes the proportions of the semimajor and minor axes. The special case $e = 0$ reduces the ellipse to a circle of radius $r = a = b$.

1.3 Internal Relations

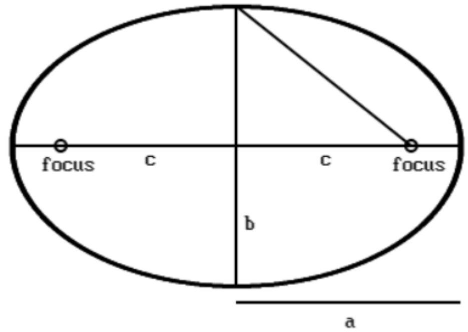

Figure 1.2: Ellipse displaying internal relations beteween a, b, c.

Having established that the set of points obeying $R/Q = e < 1$ forms an ellipse, we label the semimajor and semiminor axes, along with the center-to-focus distance as shown in Figure 1.2. Note that the minor axis b is *never* greater than the major axis a.

Problem 2

Derive the internal relations:

$$a^2 - b^2 = c^2 \tag{1.8}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{1.9}$$

Problem 3

Determine the length of the line segment connecting the upper vertex (height b from the center) to the focus (horizontal distance c from the center).

Semilatus Rectum

Problem 4

The *semilatus rectum* is the vertical distance from the focus to the ellipse. Prove this is equal to b^2/a.

1.4 Symmetry

Reflected Origin

We decided by writing Equation (1.2) that the origin is placed at the focus of the ellipse, which is to say the origin is not at the ellipse's geometric center. Due the vertical symmetry of our construction, there also exists a complimentary focus with its own directrix in the mirror image of the ellipse as shown in Figure 1.3. Should we wish to choose to rebuild using the 'left' focus as the origin, the resulting equation is complimentary to Equation (1.4), with the sign on c reversing:

$$\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=1$$

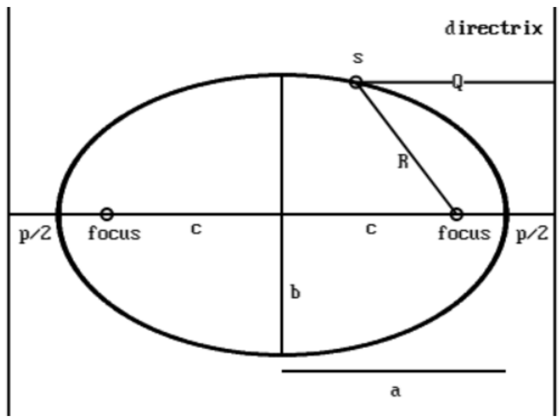

Figure 1.3: Vertical symmetry of the ellipse implies another directrix and focus.

1.5 Translations

Centered Origin

Placing the origin at the geometric center, the most symmetric equation of the ellipse reads

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,. \tag{1.10}$$

Having no offset term, the focii are located symmetrically at $x=\pm c$.

Shifted Origin

An ellipse centered at the point (x_0, y_0) is represented by

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1\,. \tag{1.11}$$

Problem 5

For the ellipse

$$5x^2+y^2-20x=0\,,$$

find the major and minor axes, the center, the eccentricity, the vertex points, the covertex points, and the focii. Answer: major $=2\sqrt{5}$, minor $=2$, center $=(2,0)$, $e=2/\sqrt{5}$, vertices $=\left(2,\pm2\sqrt{5}\right)$, covertices $=(2\pm2,0)$, focii $=(2,\pm4)$

Problem 6

For the ellipse

$$x^2+2y^2+4y-6=0\,,$$

find the major and minor axes, the center, the eccentricity, the vertex points, the covertex points, and the focii. Answer: major $=2\sqrt{2}$, minor $=2$, center $=(0,-1)$, $e=1/\sqrt{2}$, vertices $=\left(\pm2\sqrt{2},-1\right)$, covertices $=(0,-1\pm2)$, focii $=(\pm2,-1)$

Problem 7

Find the equation of the ellipse with vertices at $(3,1)$ and $(-1,1)$ and eccentricity $e=2/3$. Answer: $(x-1)^2/4+9\,(y-1)^2/20=1$

1.6 Polar Representation

In polar coordinates, a point (x,y) in the Cartesian plane is represented by

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta)\ , \end{aligned}$$

where r is the distance to the origin and θ is the angular parameter. These can be inverted to solve for r, θ with respect to x, y:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

The definition (1.1) combined with Equations (1.2), (1.3) lends naturally to polar coordinates:

$$e=\frac{R}{Q}=\frac{r}{p-x}=\frac{r}{p-r\cos(\theta)}$$

Solving for $r(\theta)$, one finds

$$r=\frac{pe}{1+e\cos(\theta)}\,. \tag{1.12}$$

Equation (1.12) traces an ellipse in the plane from an origin placed at the 'right' focus ($x=c$). To trace the ellipse from the 'left' focus ($x=-c$), we change the sign on the cosine term:

$$r=\frac{pe}{1-e\cos(\theta)}$$

Problem 8

Show that the polar representation (focus on the left)

$$r=\frac{b^2/a}{1-e\cos(\theta)}$$

is equivalent to the Cartesian version centered at $(ae,0)$:

$$\frac{(x-ae)^2}{a^2}+\frac{y^2}{b^2}=1$$

1.7 Parametric Representation

Consider an ellipse centered at the origin described by Equation (1.10):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

By comparing the above to the fundamental identity of triginometry, namely

$$\cos(\phi)^2 + \sin(\phi)^2 = 1$$

for any angle ϕ, we cannot help but make the association

$$x = a\cos(\phi) \tag{1.13}$$
$$y = b\sin(\phi) \; . \tag{1.14}$$

Equations (1.13), (1.14) constitute a *parametric representation* of the ellipse.

Problem 9
Check that Equations (1.13), (1.14) combine to recover Equation (1.10).

1.8 Interior Identities

Sum of Radii

Consider a point (x, y) on an ellipse centered on the origin, and let

$$r_1 = \sqrt{(x+c)^2 + y^2}$$
$$r_2 = \sqrt{(x-c)^2 + y^2}$$

be the distance from (x, y) to each respective focus as shown in Figure 1.4. By brute force, we can show that the sum of r_1 and r_2 is a *constant*. Proceed by writing

$$A = r_1 + r_2 \; ,$$

and square both sides to get

$$A^2 = 2\left(x^2 + y^2 + c^2\right) + 2\sqrt{\left(x^2e^2 + a^2 + 2cx\right)\left(x^2e^2 + a^2 - 2cx\right)} \; ,$$

simplifying further to

$$A^2 = 2\left(x^2e^2 + a^2\right) + 2\left(a^2 - x^2e^2\right) \; .$$

Performing the final cancelation, we find $A^2 = 4a^2$, or

$$r_1 + r_2 = 2a \; . \tag{1.15}$$

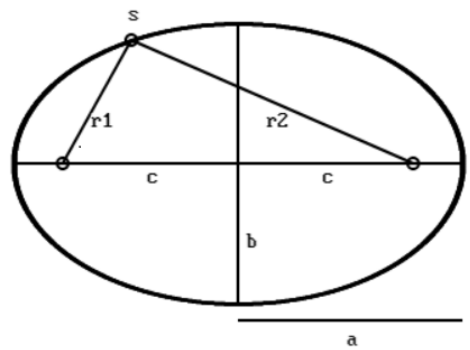

Figure 1.4: Line segments $r_{1,2}$ connect each focus to the point s on the ellipse. The sum $r_1 + r_2$ always equals the constant $2a$.

Drawing an Ellipse

The interior length identity (1.15) teaches how to draw an ellipse in the plane. Fix two pins a distance $2c$ apart, and then place a closed loop of string around the pins. Use a pen to pull the string tight, and trace around the pins while maintaining tension. The resultant shape is an ellipse with a pin at each focus.

Problem 10
Derive Equation (1.15).

Problem 11
An ellipse with eccentricity $e = 0.5$ is traced in the plane using two pins and a string. In terms of the semimajor axis a, how far apart are the pins and how long is the string?

Difference of Radii

We learned from Equation (1.15) that the sum of the interior radii $r_1 + r_2$ in the ellipse always yields the constant $2a$. Natuarally one wonders if the difference of radii $r_2 - r_1$ simplifies in any nice way. Recycling most of the work done previously, we quickly find

$$r_1 - r_2 = 2xe \; . \tag{1.16}$$

Problem 12
Derive Equation (1.16).

Decoupled Identities

Having Equations (1.15) and (1.16) in hand, we can isolate each of $r_{1,2}$ to yield a pair of tight formulas representing the ellipse:

$$r_1 = a + xe \tag{1.17}$$
$$r_2 = a - xe \tag{1.18}$$

1.9 Tangent Line to the Ellipse

At a point $s = (x, y)$ on an ellipse, there exists a *tangent line AB* that represents the instantaneous slope m_s of the ellipse as shown in Figure 1.6. The value of m_s is straightforwardly attained by implicit differentiation[1] of (1.10), which comes out to

$$m_s = \frac{-b^2}{a^2}\frac{x}{y} \,. \tag{1.19}$$

Problem 13

At a point $(\tilde{x}, \tilde{y})$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that the tangent line is

$$\frac{x\tilde{x}}{a^2} + \frac{y\tilde{y}}{b^2} = 1 \,.$$

Derivation of Slope

If 'implicit differentiation' sounds foreign, we must calculate m_s the hard way by asking about the slope m_c of a chord connecting two points $s_1 = (x_1, y_1)$, $s_2 = (x_2, y_2)$ on the ellipse as sketched in Figure 1.5. Jotting down the rise-over-run formula for the slope, we have

$$m_c = \frac{y_2 - y_1}{x_2 - x_1} \,,$$

and by placing the origin at the center of the ellipse, we use Equation (1.10) to eliminate y_1, y_2 such that

$$m_c = \frac{b\sqrt{1 - x_2^2/a^2} - b\sqrt{1 - x_1^2/a^2}}{x_2 - x_1} \,.$$

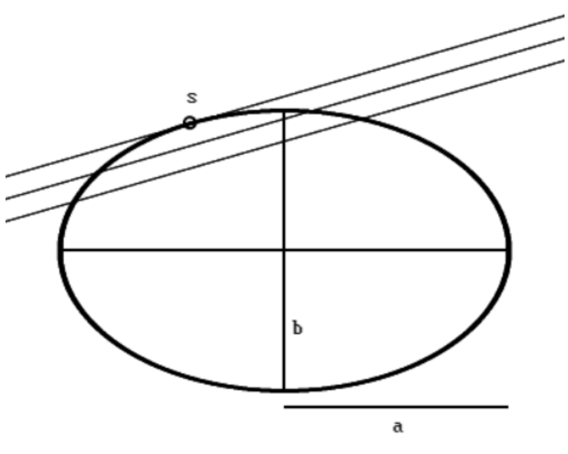

Figure 1.5: A chord cutting through the ellipse is constructed on the ellipse's edge so the endpoints share a common location s. The extension of the chord at s is the tangent line to the ellipse having slope m_s. The origin is at the center.

[1] A trick from calculus.

Our next move is to suppose that the points s_1, s_2 are very close to each other such that

$$h = x_2 - x_1$$

is a small number. In such a case, the chord connecting s_1 to s_2 becomes shorter and moves closer to the edge of the ellipse. In the limit that s_1 is approximately equal to s_2, the chord has essentially zero length, but has a slope equal to that of the tangent line to the ellipse at $s_{1,2}$. To summarize, the slope is written in 'limit' notation:

$$m_s = \lim_{h \text{ small}} \frac{b\sqrt{1 - (x_1 + h)^2/a^2} - b\sqrt{1 - x_1^2/a^2}}{h}$$

To simplify this, we exploit the 'smallness' of h by realizing that h^2 is *so* small that it can be ignored altogether such that

$$(x_1 + h)^2 \approx x_1^2 + 2x_1 h + \cancel{h^2} \,.$$

The argument inside the square root shall be handled by the approximation

$$\lim_{z \text{ small}} \sqrt{1 + z} \approx 1 + \frac{z}{2} - \frac{z^2}{8} + \cdots \,, \tag{1.20}$$

where any terms of power 2 or higher can be ignored. For our problem, this means

$$b\sqrt{1 - \frac{x_1^2 + 2x_1 h}{a^2}} \approx y_1 - \frac{x_1 b^2 h}{y_1 a^2} \,,$$

allowing the formula for m_s to simplify with all factors of h canceling out:

$$m_s = \lim_{h \text{ small}} \frac{1}{h}\left(y_1 - \frac{x_1 b^2 h}{y_1 a^2} - y_1\right) = -\frac{b^2 x_1}{a^2 y_1}$$

Note that the 1-subscript becomes redundant, as the points s_1 and s_2 become the same point s (with no subscript). Finally then, we recover Equation (1.19) for the slope of the ellipse at point s.

1.10 Reflection Property

Consider an ellipse centered on the origin as shown in Figure 1.6, with respective focii labeled $f_{1,2}$. The radii extending to a point $s = (x, y)$ on the ellipse are labeled $r_{1,2}$, and the tangent line AB is indicated. The *reflection property* of the ellipse states that *a ray emerging from one focus will reflect from the ellipse* *to the other focus.* To prove this, let us define:

$$\begin{aligned}\text{Angle } Asf_1 &= \theta \\ \text{Angle } Bsf_2 &= \phi \\ \text{Slope of } AB &= m_s \\ \text{Slope of } r_1 &= m_1 \\ \text{Slope of } r_2 &= m_2\end{aligned}$$

The slopes m_1, m_2 are straightforward to write by inspsection of Figure 1.6:

$$\begin{aligned}m_1 &= \frac{y}{x+c} \\ m_2 &= \frac{y}{x-c}\end{aligned}$$

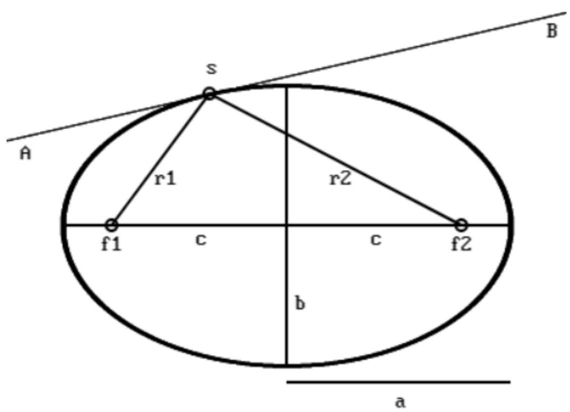

Figure 1.6: A ray emerging from one focus will reflect from the ellipse to the other focus.

Next, we'll need to use the angle-sum identity for tangent, namely

$$\tan(\alpha+\beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)} , \tag{1.21}$$

and observe again from the Figure that

$$\begin{aligned}\tan(\theta) &= \frac{m_1 - m_s}{1 + m_1 m_s} \\ \tan(\phi) &= \frac{m_2 - m_s}{1 + m_2 m_s} .\end{aligned}$$

Simplifying each expression delivers

$$|\tan(\theta)| = |\tan(\phi)| = \frac{b^2}{cy} , \tag{1.22}$$

telling us that $\theta = \phi$ and the proof is done.

<u>Problem 14</u>

Prove Equation (1.22).

1.11 Normal Line to the Ellipse

Consider a *normal line* q that is perpendicular to the tangent line at point $s = (x, y)$ on the ellipse as shown in Figure 1.7. The slope of the normal line is defined as the negative reciprocal of the tangent's slope, namely $-1/m_s$ given by (1.19). The normal line q can thus be written

$$y_q = -x_q/m_s + b_q ,$$

with $b_q = y + x/m_s$. Such a line is more conveniently expressed as

$$y_q = y + (x - x_q)/m_s . \tag{1.23}$$

The normal line intersects the x-axis at the point $x_q = q_0$, which we determine by setting $y_q = 0$:

$$\begin{aligned}q_0 &= x + m_s y = x + \left(\frac{-b^2}{a^2}\frac{x}{y}\right) y \\ q_0 &= x\left(1 - \frac{b^2}{a^2}\right) = xe^2 \end{aligned} \tag{1.24}$$

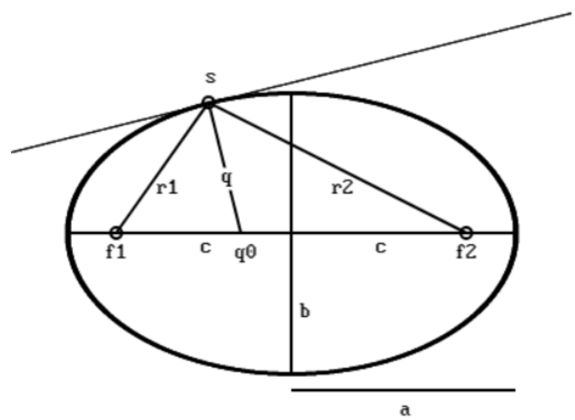

Figure 1.7: A point s on the ellipse implies a normal line q that intersects the x-axis at $x = q_0$. The origin is at the focus.

<u>Problem 15</u>

Determine where the normal line intersects the y-axis.

<u>Problem 16</u>

At a point $(\tilde{x}, \tilde{y})$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that the normal line is

$$\frac{x}{\tilde{x}} - \frac{y}{\tilde{y}}\left(1 - e^2\right) = e^2 .$$

1.12 Vector Analysis (Optional)

Three Vectors

Refering again to Figure 1.7, let us construct three vectors from straight lines:

$$\begin{aligned}\text{Line } f_1 s &= \vec{r}_1\\ \text{Line } f_2 s &= \vec{r}_2\\ \text{Line } q_0 s &= \vec{q}\end{aligned}$$

Introducing the unit vector $\hat{x}$ that points horizontally to the right, we can jot down three ways to get from the origin to point s:

$$\vec{s} = -c\hat{x} + \vec{r}_1 \tag{1.25}$$

$$\vec{s} = c\hat{x} + \vec{r}_2 \tag{1.26}$$

$$\vec{s} = q_0\hat{x} + \vec{q} \tag{1.27}$$

Solving (1.25), (1.26) for $\vec{r}_1$, $\vec{r}_2$ respectively, we can divide each by its own magnitude to write unit vectors:

$$\hat{r}_1 = \frac{\vec{s} + c\hat{x}}{r_1}$$

$$\hat{r}_2 = \frac{\vec{s} - c\hat{x}}{r_2}$$

Recovering Reflection Property

In terms of the vectors $\vec{r}_1$, $\vec{r}_1$, $\vec{q}$, the reflection property of the ellipse can be proposed by writing

$$\hat{r}_1 \cdot \vec{q} = \hat{r}_2 \cdot \vec{q} \,, \tag{1.28}$$

which is to claim that the angle formed bewtween either $\hat{r}_j$ and $\vec{q}$ is the same. Proceeding carefully, one can simplify the above down to

$$\frac{b^2}{a}\cancel{\left(\frac{a + xe}{r_1}\right)} = \frac{b^2}{a}\cancel{\left(\frac{a - xe}{r_2}\right)} \,, \tag{1.29}$$

and the claim is proven. The parenthesized terms cancel due to Equations (1.17), (1.18). Evidently, we discover that the dot product between either unit vector $\hat{r}_{1,2}$ and the normal vector $\vec{q}$ equals a constant:

$$\hat{r}_{1,2} \cdot \vec{q} = \frac{b^2}{a} \tag{1.30}$$

<u>Problem 17</u>

Derive Equation (1.29) from (1.28).

Recovering Interior Length Identites

Next, we make use of the reflection property of the ellipse to realize that the sum of $\hat{r}_1$ and $\hat{r}_2$ must be parallel to (i.e. proportional to) the vector $\vec{q}$. Following this lead, we first take the sum

$$\hat{r}_1 + \hat{r}_2 = \vec{s}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + c\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\hat{x} \,,$$

simplifying down to

$$\hat{r}_1 + \hat{r}_2 = \left(\frac{r_1 + r_2}{r_1 r_2}\right)\left(\vec{s} + \frac{e}{2}\left(r_2 - r_1\right)\hat{x}\right) \tag{1.31}$$

Note that the interior length identity $r_1 + r_2 = 2a$ was used along the way. Comparing the above to Equation (1.27), we must have

$$q_0 = \frac{e}{2}\left(r_1 - r_2\right) \,. \tag{1.32}$$

<u>Problem 18</u>

Derive Equations (1.31), (1.32) and then recover Equation (1.16).

2 Hyperbola

2.1 Definition

In the Cartesian plane, consider a point labeled *focus* that is distance p from a vertical line labled *directrix*. Now, let us seek the set of points $\{s\} = \{(x, y)\}$ that satisfy the following rule: the distance R to the focus divided by the (purely horizontal) distance Q to the directrix equals a constant $e > 1$. In algebraic terms, this means

$$\frac{R}{Q} = e > 1 \,. \tag{1.33}$$

Sketched in Figure 1.8 are some of the points that obey such a rule.

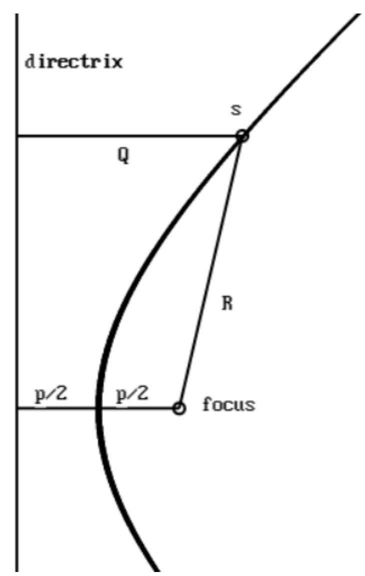

Figure 1.8: Points obeying $R/Q = e > 1$ as measured from a directrix and a focus separated by distance p. The origin is at the focus.

To determine the proper shape defined by the rule, begin with (1.33) and discern from inspection that R, Q can be written:

$$R = \sqrt{x^2 + y^2} \tag{1.34}$$

$$Q = p + x \tag{1.35}$$

Inserting the above into (1.33) and completing the square in x, one finds

$$\frac{(x+c)^2}{a^2} - \frac{y^2}{b^2} = 1 \,, \tag{1.36}$$

describing a *hyperbola*. The constants a, b, c relate to e, p by:

$$a = \frac{ep}{e^2 - 1} \tag{1.37}$$

$$b = \frac{ep}{\sqrt{e^2 - 1}} \tag{1.38}$$

$$c = ae \tag{1.39}$$

The hyperbola has two vertex points occurring at $x = \pm a - c$.

2.2 Asymptotes

The hyperbola is not a closed curve like a circle or ellipse, and it's worthwhile to inquire about the hyperbola far from the origin. Supposing we let x and y be very large, especially such that $x \gg c$, Equation (1.36) roughly reads

$$\frac{x^2}{a^2} \approx \frac{y^2}{b^2} \,,$$

implying a pair of straight lines having slope

$$m_\pm = \pm \frac{b}{a} \tag{1.40}$$

that are *asymptopes* to the hyperbola.

To find an exact equation for each asymptote, begin with the equation of the hyperbola (1.36) and let $y = 0$ to determine the value x^* at which the curve touches the x-axis. Doing so, we find

$$x^* = \begin{cases} a - c \\ -a - c \end{cases} .$$

Each x-intercept is negative, i.e. to the left of the focus, however the curve sketched in Figure 1.8 has a focus at $a - c$ and 'opens up' to the right. Evidently, a second x-intercept occurs at $-a - c$, implying a mirror image hyperbola opening up to the left.

The line of vertical symmetry between each copy of the hyperbola is defined by the average of each x^*-value, or

$$x^*_{\text{ave}} = \frac{(a-c) + (-a-c)}{2} = -c$$

By horizontal symmetry, we argue that each asymptote crosses the x-axis at x^*_{ave}. This is enough to determine the equation of each asymptote, coming out to

$$y = \pm \frac{b}{a} (x + c) \;. \tag{1.41}$$

Our findings are summarized in Figure 1.9.

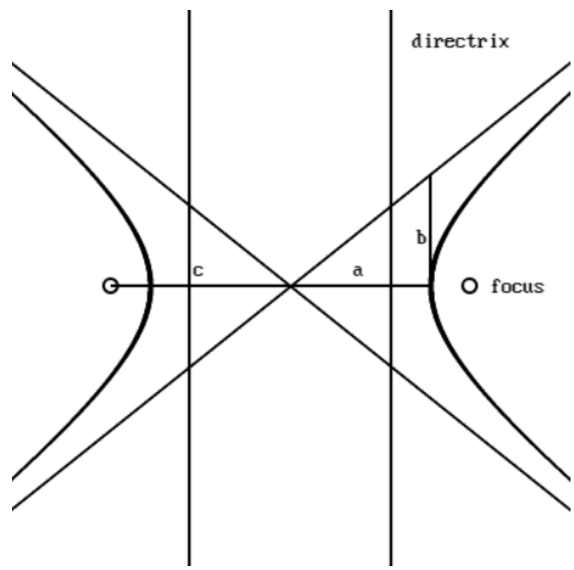

Figure 1.9: Vertical symmetry of the hyperbola implies another directrix and focus.

2.3 Internal Relations

Problem 19

Derive the internal relations:

$$a^2 + b^2 = c^2 \tag{1.42}$$

$$e = \sqrt{\frac{b^2}{a^2} + 1} \tag{1.43}$$

Problem 20

Show that a is the distance from the vertex of the hyperbola to the intersection of the asymptotes. Show that b is the vertical distance from the focus to the asymptote. Sbow that c is the distance from the focus to the intersection of the asymptotes.

2.4 Symmetry

Reflected Origin

We decided by wtiring Equation (1.34) that the origin is placed at the right focus of the hyperbola, which is to say the origin is not at the hyperbola's geometric center. Due the vertical symmetry of our construction, there also exists a complimentary focus with its own directrix in the mirror image of the hyperbola as shown in Figure 1.9. Should we wish to choose to rebuild using the 'left' focus as the origin, the resulting equation is complimentary to (1.36), with the sign on c reversing:

$$\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$$

2.5 Translations

Centered Origin

Placing the origin at the geometric center, the most symmetric equation of the hyperbola reads

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \; . \tag{1.44}$$

Having no offset term, the focii are located symmetrically at $x = \pm c$.

Shifted Origin

A hyperbola centered at the point (x_0, y_0) is represented by

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1 \; . \tag{1.45}$$

Problem 21

For the hyperbola

$$16y^2 - 9x^2 = 144 \; ,$$

find the major and minor axes, the center, the eccentricity, the vertex points, the asymptotes, and the focii. Answer: major $= 4$, minor $= 3$, center $= (0,0)$, $e = 5/4$, vertices $= (0, \pm 3)$, $y = \pm 3x/4$, focii $= (0, \pm 5)$

Problem 22

For the hyperbola

$$12x^2 - 32y^2 - 12x + 96y + 27 = 0 \; ,$$

find the major and minor axes, the center, the eccentricity, the vertex points, the asymptotes, and the focii. Answer: major $= \sqrt{3}$, minor $= 2\sqrt{2}$, center $= (1/2, 3/2)$, $e = \sqrt{11/3}$, vertices $= \left(1/2, 3/2 \pm \sqrt{3}\right)$, $y = \pm\sqrt{3/8}x \mp \sqrt{3/32} + 3/2$, focii $= \left(1/2, 3/2 \pm \sqrt{11}\right)$

Problem 23

Find the equation of the hyperbola with vertices at $(0, \pm 2)$ with asymptotes $y = \pm x/2$. Answer: $y^2/4 - x^2/16 = 1$

Problem 24

Find the equation of the hyperbola with focus points $(7,0)$ and $(-1,0)$ passes through $\left(6, \sqrt{15}\right)$. Answer: $(x-3)^2/4 - y^2/12 = 1$

2.6 Polar Representation

In polar coordinates, recall that a point (x, y) in the Cartesian plane is represented by

$$x = r\cos(\theta)$$
$$y = r\sin(\theta) \; ,$$

where r is the distance to the origin and θ is the angular parameter. These can be inverted to solve for r, θ with respect to x, y:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

The definition (1.33) combined with (1.34), (1.35) lends naturally to polar coordinates:

$$e = \frac{R}{Q} = \frac{r}{p+x} = \frac{r}{p + r\cos(\theta)}$$

Solving for $r(\theta)$, one finds

$$r = \frac{pe}{1 - e\cos(\theta)} \; . \tag{1.46}$$

Equation (1.46) traces a hyperbola in the plane from an origin placed at the 'right' focus ($x = c$). To trace the hyperbola from the 'left' focus ($x = -c$), we change the sign on the cosine term:

$$r = \frac{pe}{1 + e\cos(\theta)}$$

2.7 Parametric Representation

Consider a hyperbola centered at the origin described by Equation (1.44):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

By comparing the above to the fundamental identity of hyperbolic trigonometry, namely

$$\cosh(\phi)^2 - \sinh(\phi)^2 = 1$$

for any value of ϕ, we cannot help but make the association

$$x = a\cosh(\phi) \tag{1.47}$$
$$y = b\sinh(\phi)\ . \tag{1.48}$$

Equations (1.47), (1.48) constitute a parametric representation of the hyperbola.

<u>Problem 25</u>

Check that Equations (1.47), (1.48) combine to recover Equation (1.44).

2.8 Interior Identities

Difference of Radii

Consider a point (x, y) on a hyperbola centered on the origin, and let

$$r_1 = \sqrt{(x+c)^2 + y^2}$$
$$r_2 = \sqrt{(x-c)^2 + y^2}$$

be the distance from (x, y) to each respective focus as shown in Figure 1.10. By brute force, we can show that the difference between r_1 and r_2 is a *constant*. Proceed by writing

$$A = r_1 - r_2\ ,$$

and square both sides to get

$$A^2 = 2\left(x^2 + y^2 + c^2\right) - 2\sqrt{\left(x^2e^2 + a^2 + 2cx\right)\left(x^2e^2 + a^2 - 2cx\right)}\ ,$$

simplifying further to

$$A^2 = 2\left(x^2e^2 + a^2\right) + 2\left(a^2 - x^2e^2\right)\ .$$

Performing the final cancellation, we find $A^2 = 4a^2$, or

$$r_1 - r_2 = 2a\ . \tag{1.49}$$

<u>Problem 26</u>

Derive Equation (1.49).

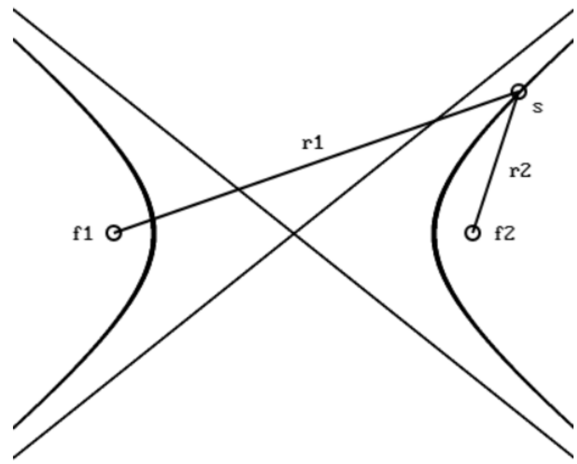

Figure 1.10: Line segments $r_{1,2}$ connect each focus to the point s on the hyperbola. The difference $r_1 - r_2$ always equals a constant $2a$.

Locating Ships

The interior length identity (1.49) teaches how to locate ships at sea by noticing that a signal emitted from any point on a hyperbola will reach each focus a fixed time apart. If the signal propagation speed (the speed of light for radar) is v, then the time interval Δt is $2a/v$. Supposing two receiver stations are separated by distance $d = 2c$ on land, we use the internal relation $a^2 + b^2 = c^2$ to write

$$b = \pm\sqrt{c^2 = a^2} = \pm\frac{1}{2}\sqrt{d^2 - v^2\Delta t^2}\ ,$$

telling us the ship is somewhere on a known hyperbola. The ship's exact location can be discerned using a third station and the intersection of two hyperbolas.

Sum of Radii

We learned from Equation (1.49) that the difference of the interior radii $r_1 - r_2$ in the hyperbola always yields the constant $2a$. Natuarally one wonders if the sum of radii $r_2 + r_1$ simplifies in any nice way. Recycling most of the work done previously, we quickly find

$$r_1 + r_2 = 2xe\ . \tag{1.50}$$

<u>Problem 27</u>

Derive Equation (1.50).

Decoupled Identities

Having Equations (1.49) and (1.50) in hand, we can isolate each of $r_{1,2}$ to yield a pair of tight formulas representing the hyperbola:

$$r_1 = a + xe \tag{1.51}$$
$$r_2 = -a + xe \tag{1.52}$$

2.9 Tangent Line to the Hyperbola

At a point $s = (x, y)$ on a hyperbola, there exists a tangent line AB that represents the instantaneous slope m_s of the hyperbola as shown in Figure 1.11. The value of m_s is straightforwardly attained by implicit diffentiation of Equation (1.44), which comes out to

$$m_s = \frac{b^2}{a^2}\frac{x}{y} . \tag{1.53}$$

Problem 28

At a point $(\tilde{x}, \tilde{y})$ on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that the tangent line is

$$\frac{x\tilde{x}}{a^2} - \frac{y\tilde{y}}{b^2} = 1 .$$

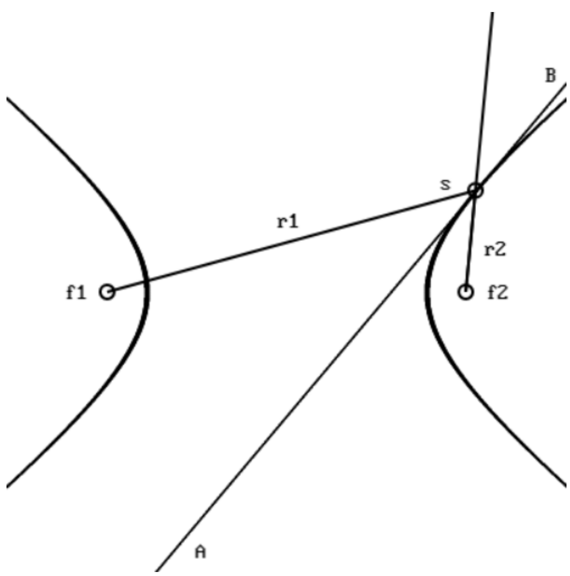

Figure 1.11: An incoming ray aimed at f_2 intersects the hyperbola at $s = (x, y)$. The reflected ray goes toward f_1. The angle formed between the incoming ray and the tangent line AB is is identical to angle Asf_2. The origin is at the focus.

2.10 Reflection Property

Consider a hyperbola centered on the origin as shown in Figure 1.11, with respective focii labeled $f_{1,2}$. The radii extending to a point $s = (x, y)$ on the hyperbola are labeled $r_{1,2}$, and the tangent line AB is indicated. The reflection property of the hyperbola states that *an external ray aimed at a focus will be reflected by the hyperbola to the other focus.* To prove this, let us define:

$$\begin{aligned} \text{Angle } Asf_1 &= \theta \\ \text{Angle } Asf_2 &= \phi \\ \text{Slope of } AB &= m_s \\ \text{Slope of } r_1 &= m_1 \\ \text{Slope of } r_2 &= m_2 \end{aligned}$$

The slopes m_1, m_2 are straightforward to write by inspsection of Figure 1.11:

$$m_1 = \frac{y}{x + c}$$
$$m_2 = \frac{y}{x - c}$$

Next, we'll need to use the angle-sum identity for tangent, namely

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)} ,$$

and observe again from the Figure that

$$\tan(\theta) = \frac{m_1 - m_s}{1 + m_1 m_s}$$
$$\tan(\phi) = \frac{m_2 - m_s}{1 + m_2 m_s} .$$

Simplifying each expression delivers

$$|\tan(\theta)| = |\tan(\phi)| = \frac{b^2}{cy} , \tag{1.54}$$

telling us that $\theta = \phi$ and the proof is done.

Problem 29

Prove Equation (1.54).

2.11 Normal Line to the Hyperbola

Consider a normal line q that is perpendicular to the tangent line at point $s = (x, y)$ on the hyperbola as shown in Figure 1.12. The slope of the normal line is defined as the negative reciprocal of the tangent's slope, namely $-1/m_s$ given by Equation (1.53). The normal line q can thus be written

$$y_q = -x_q/m_s + b_q ,$$

with $b_q = y + x/m_s$. Such a line is more conveniently expressed as

$$y_q = y + (x - x_q)/m_s . \tag{1.55}$$

The normal line intersects the x-axis at the point $x_q = q_0$, which we determine by setting $y_q = 0$:

$$q_0 = x + m_s y = x + \left(\frac{b^2}{a^2}\frac{x}{y}\right) y$$

$$q_0 = x\left(1 + \frac{b^2}{a^2}\right) = xe^2 \tag{1.56}$$

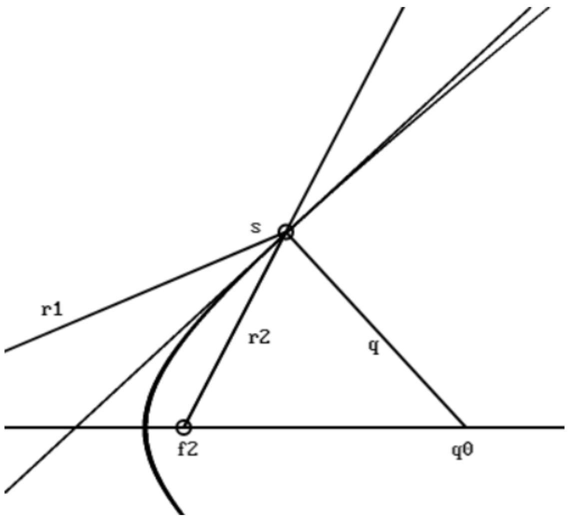

Figure 1.12: A point s on the hyperbola implies a normal line q that intersects the x-axis at $x = q_0$. The origin is at the focus.

Problem 30

Determine where the normal line intersects the y-axis.

Problem 31

At a point $(\tilde{x}, \tilde{y})$ on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that the normal line is

$$\frac{x}{\tilde{x}} + \frac{y}{\tilde{y}}\left(e^2 - 1\right) = e^2 \; .$$

3 Parabola

3.1 Definition

In the Cartesian plane, consider a point labeled *focus* that is distance p from a vertical line labled *directrix*. Now, let us seek the set of points $\{s\} = \{(x, y)\}$ that satisfy the following rule: the distance R to the focus divided by the (purely horizontal) distance Q to the directrix equals a constant $e = 1$. In algebraic terms, this means

$$\frac{R}{Q} = 1 \; . \tag{1.57}$$

Sketched in Figure 1.13 are some of the points that obey such a rule.

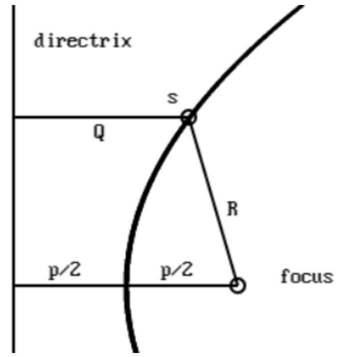

Figure 1.13: Points obeying $R/Q = 1$ as measured from a directrix and a focus separated by distance p. The origin is at the focus.

To determine the proper shape defined by the rule, begin with (1.57) and discern from inspection that R, Q can be written:

$$R = \sqrt{x^2 + y^2} \tag{1.58}$$

$$Q = p + x \tag{1.59}$$

Inserting the above into (1.57), one finds

$$y^2 = p^2 + 2px \; , \tag{1.60}$$

describing a *parabola* that ‘opens up’ to the right. The parabola has one focus point (the second one is at infinity, if you like). If we want the parabola to open up to the left, place the focus on the other side of the directrix. This has the effect of reversing the sign on p. The vertex of the parabola occurs at $x = -p/2$. The line of symmetry halving the parabola is called the *axis*.

3.2 Opening Direction

A parabola is often found in the wild opening in the up- or down-direction (as opposed to left or right). To generate the parabola that opens ‘upward’, let the directrix run horizontally and place the focus above it, effectively swapping the x- and y- variables in the equation of the parabola as shown in Figure 1.14:

$$x^2 = p^2 + 2py \tag{1.61}$$

The sign on the p-term determines the opening direction in either case.

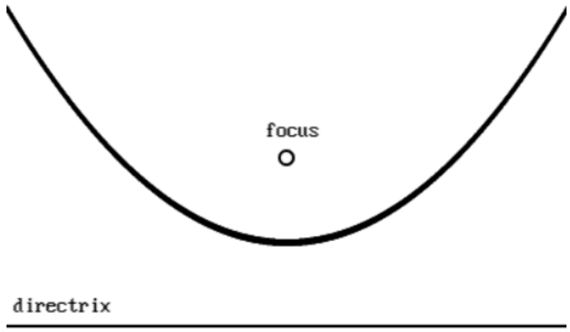

Figure 1.14: The directrix-focus construction rotated by ninety degrees produces an up- (or down-) opening parabola.

3.3 Parabolic Expressions

In Equations (1.60), (1.61), the value p^2 acts to translate the parabola vertically or horzontally. The factor of $2p$ is a scaling factor that stretches or skews the overall parabola. If we let scaling be handled by a new variable a, and let the translation vector (x_0, y_0) take care of the absolute placement of the parabola, Equation (1.60) is more generally written as

$$x = x_0 + a\,(y - y_0)^2 \ . \tag{1.62}$$

By the same token, Equation (1.61) is more generally written as

$$y = y_0 + a\,(x - x_0)^2 \ . \tag{1.63}$$

The sign on a dictates whether the parabola opens up right-left or up-down, respectively.

Problem 32
Sketch the parabola $x^2 - 2y - 6x = 0$, and show that the focus is located at $(3, -4)$ and that the directrix is located at $y = -5$.

Problem 33
The parabola $y^2 - 2ay + 2x - a^2 = 0$ has its focus on the y-axis above the origin. Find the number a and sketch the graph. Answer: $a = 1/\sqrt{2}$

Problem 34
An up- or down-opening parabola can be generally expressed as $y = ax^2 + bx + c$. In terms of a, b, c, find the vertex and the focus. Answer: $\left(-b/2a, c - b^2/4a\right)$, $1/4a$ above the vertex

3.4 Polar Representation

In polar coordinates, recall that a point (x, y) in the Cartesian plane is represented by

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta)\ , \end{aligned}$$

where r is the distance to the origin and θ is the angular parameter. These can be inverted to solve for r, θ with respect to x, y:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

The definition (1.57) combined with Equations (1.58), (1.59) lends naturally to polar coordinates:

$$1 = \frac{R}{Q} = \frac{r}{p + x} = \frac{r}{p + r\cos(\theta)}$$

Solving for $r(\theta)$, one finds

$$r = \frac{p}{1 - \cos(\theta)}\ . \tag{1.64}$$

Equation (1.64) traces a parabola in the plane.

3.5 Internal Relations

Right Focal Chord

Problem 35
Prove that in a parabola the length of the chord passing through the focus making an angle θ with the axis is equal to $L/\sin^2\theta$, where L is the length of the *right focal chord*, the line that passes through the focus and is perpendicular to the axis. Hint: Use $x^2 = p^2 + 2py$ and then focal chords are given by $y = \cot(\theta)\,x$.

Problem 36
A parabolic segment (i.e. the area bounded by a parabola and a chord perpendicular to the axis) is 32 cm high and its base is 16 cm. How far is the focus from the directrix? Answer: 1 cm

3.6 Tangent Line to the Parabola

At a point $s = (x, y)$ on a parabola, there exists a tangent line AB that represents the instantaneous slope m_s of the parabola as shown in Figure 1.15. The value of m_s is straightforwardly attained by implicit diffentiation of Equation (1.60), which comes out to

$$m_s = \frac{p}{y}\ . \tag{1.65}$$

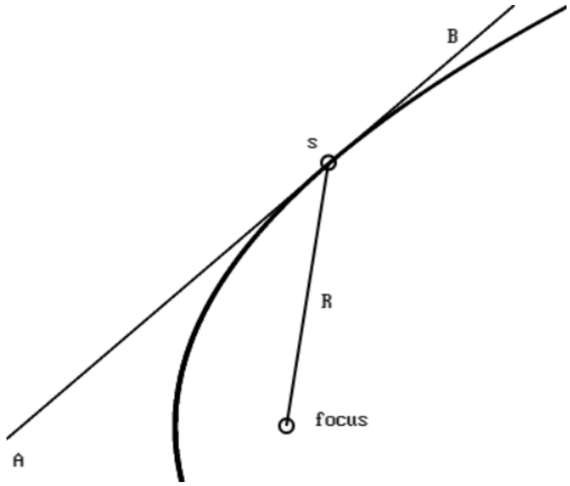

Figure 1.15: A point s on the parabola implies a tangent line AB that represents the instantaneous slope of the parabola. The origin is at the focus.

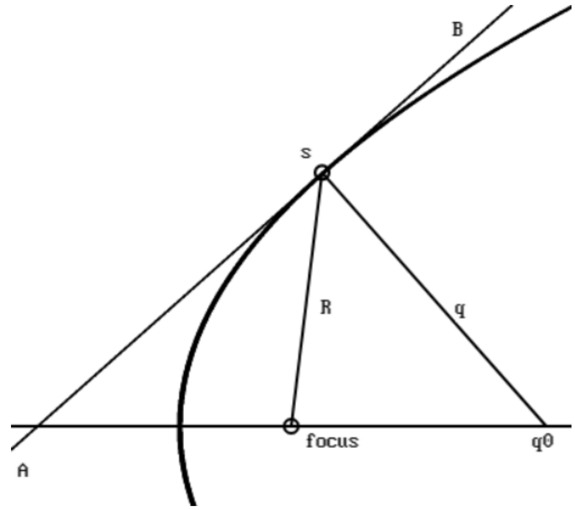

Figure 1.16: A point s on the parabola implies a normal line q that intersects the x-axis at $x = q_0$. The origin is at the focus.

Problem 37

At a point $(\tilde{x}, \tilde{y})$ on the parabola $y^2 = p^2 + 2px$, show that the tangent line is

$$y\tilde{y} = p^2 + p\,(x + \tilde{x}) \ .$$

Problem 38

Determine where the normal line intersects the y-axis.

3.7 Normal Line to the Parabola

Consider a normal line q that is perpendicular to the tangent line at point $s = (x, y)$ on the parabola as shown in Figure 1.16. The slope of the normal line is defined as the negative reciprocal of the tangent's slope, namely $-1/m_s$ given by (1.65). The normal line q can thus be written

$$y_q = -x_q/m_s + b_q \ ,$$

with $b_q = y + x/m_s$. Such a line is more conveniently expressed as

$$y_q = y + (x - x_q)\,/m_s \ . \tag{1.66}$$

The normal line intersects the x-axis at the point $x_q = q_0$, which we determine by setting $y_q = 0$:

$$q_0 = x + m_s y = x + p \tag{1.67}$$

Problem 39

At a point $(\tilde{x}, \tilde{y})$ on the parabola $y^2 = p^2 + 2px$, show that the normal line is

$$\frac{x - \tilde{x}}{p} + \frac{y}{\tilde{y}} = 1 \ .$$

3.8 Reflection Property

Consider the parabola described by $y^2 = p^2 + 2px$ with the origin at the focus. As shown in Figure 1.17, a point $s = (x, y)$ on the parabola implies a tangent line AB, along with a normal line q. The reflection property of the parabola states that *a ray from the focus to the parabola is reflected parallel to the axis*. Reading this backwards, we can also say that *incoming rays parallel to the axis are reflected by the parabola to the focus.*

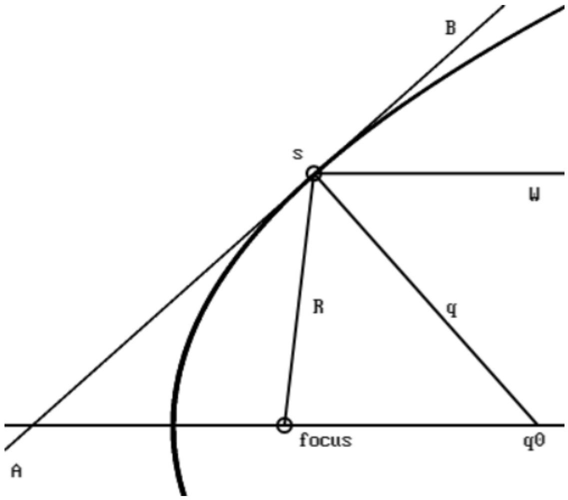

Figure 1.17: A ray emitted from the focus will intersect the parabola at $s = (x, y)$ and reflect parallel to the axis along line sW.

To proceed, refer to Figure 1.17 to define:

$$\begin{aligned}
\text{Angle } Asf &= \alpha_1 \\
\text{Angle } BsW &= \alpha_2 \\
\text{Angle } fsq_0 &= \beta_1 \\
\text{Angle } Wsq_0 &= \beta_2 \\
\text{Line } sW &\propto \hat{x} \\
\text{Slope of } AB &= m_s
\end{aligned}$$

Note that the line sW is parallel to the axis. To establish the reflection property, we must show that either $\alpha_1 = \alpha_2$ or that $\beta_1 = \beta_2$.

Vector Analysis

With the construction on hand, let us write a vector $\vec{q}$ that points from q_0 to s:

$$\vec{q} = \vec{s} - q_0\,\hat{x} \tag{1.68}$$

Note that $\vec{s}$ is equivalent to the position vector $\vec{r}$, which is itself composed of a magnitude r and unit vector $\hat{r}$. To prove that $\beta_1 = \beta_2$, we observe that the unit vector representing $\vec{s}$ projected onto $\vec{q}$ has to equal that of the unit vector $-\hat{x}$ projected onto $\vec{q}$:

$$q\cos(\beta_1) = \hat{r}\cdot\vec{q} = -\hat{x}\cdot\vec{q} = q\cos(\beta_2) \tag{1.69}$$

Substituting $\vec{q}$ from Equation (1.68), we have

$$\hat{r}\cdot(\vec{s} - q_0\,\hat{x}) = -\hat{x}\cdot(\vec{s} - q_0\,\hat{x}) \ ,$$

boiling down to

$$r - \cancel{\left(\frac{x+p}{r}\right)}x = -x + q_0 \ .$$

On the right, the quantity $-x + q_0$ is simply the constant p according to Equation (1.67). The parenthesized quantity on the left resolves to *one* according to definition (1.57), bringing the result to

$$r - x = p \ . \tag{1.70}$$

To check that Equation (1.70) is true we employ polar coordinates, $x = r\cos(\theta)$, and the above quickly resolves to the polar representation (1.64) of the parabola, finishing the proof.

<u>Problem 40</u>
Derive Equation (1.70) from Equation (1.69).

Slope Analysis of the Parabola

The proof that that $\alpha_1 = \alpha_2$ using pure slope analysis is slightly tricky. We first note that the angle formed between R and the x-axis, i.e. θ as used in polar coordinates, is equal to two times α_1, leading us to write

$$\tan(2\alpha_1) = \frac{y}{x} \ .$$

Meanwhile, the slope m_s of line AB is the tangent of α_2:

$$m_s = \tan(\alpha_2)$$

From this point, let us cautiously assume that α_1, α_2 are equal to a common value α and make sure no contradictions arise.

Next, use the angle-sum identity (1.21) for tangent to write

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)} \ ,$$

and replace all trigonometric terms with factors of x, y, and m_s:

$$0 = m_s^2 + 2\frac{x}{y}m_s - 1 \tag{1.71}$$

Complete the square in m_s and then solve for m_s. The result boils down to

$$m_s = \frac{p}{y} \ ,$$

the formula (1.65) for the slope at the point (x, y) on the parabola, validating the assumption $\alpha_1 = \alpha_2$ and completes the proof.

<u>Problem 41</u>
Derive Equation (1.65) from Equation (1.71).

Differential Analysis

Starting from Equation (1.71), multuiply through by y^2 and make the subsitution

$$y^2 = r^2 - x^2 ,$$

where implicit differentiation tells us

$$y m_s = r\frac{dr}{dx} - x .$$

With this, Equation (1.71) reduces to

$$0 = r^2 - r^2 \left(\frac{dr}{dx}\right)^2 ,$$

telling us

$$\frac{dr}{dx} = 1 ,$$

which is integrated to give us r as a function of x up to a constant:

$$r = x + \text{const}$$

Comparing the above to Equation (1.70), the integration constant is essentially p. Arriving at a familiar result without contradiction, we assure again that $\alpha_1 = \alpha_2$.

4 Slicing the Cone

4.1 Conic Sections

Now we address why the ellipse, hyperbola, and parabola are called *conic sections*. It turns out that each of these curves can be produced from the intersection of a cone and a plane. The cone can slice the plane in any way, at any angle, and *only* conic sections are produced. Consider a cone in three dimensions represented by

$$x^2 + y^2 = \alpha z^2 , \tag{1.72}$$

where α is a dimensionless parameter controlling the 'sharpness' of the cone. Next we'll need a plane to slice the cone, which we represent by

$$1 = \frac{x}{x_0} + \frac{z}{z_0} . \tag{1.73}$$

Conspiculously absent from Equation (1.73) is any representation of the y-variable. Due to the axial symmety of the cone, one can always choose a coordinate system where the horizontal coodinate on the the plane aligns perfectly with a cartesian axis. In this case, the plane's intersection with the cone is defined by the angle θ formed between the plane and the horizontal. The special case $\theta = 0$ means the plane slices through the cone's vertex (an infinitely small ellipse).

To proceed, suppose the coordinate system embedded on the plane is labeled u, v (in analog to x, y). From geometry, we can write several observations about this system:

$$u\cos(\theta) = x_0 - x \tag{1.74}$$
$$u\sin(\theta) = z \tag{1.75}$$
$$v = y \tag{1.76}$$
$$\tan(\theta) = \frac{z_0}{x_0} \tag{1.77}$$

In other words, the u-coordinate on the plane corresponds to locations mixing x and z. The v-coordinate is equivalent to the y-coordinate.

Using everything we have on hand, we write an equation

$$z_0 = \tan(\theta)\sqrt{\alpha u^2 \sin^2(\theta) - v^2} + u\sin(\theta) ,$$

implying

$$\begin{aligned} 1 = {} & \frac{u^2}{z_0^2}\sin^2(\theta)\left(\alpha\tan^2(\theta) - 1\right) \\ & + 2\frac{u}{z_0}\sin(\theta) - \tan^2(\theta)\frac{v^2}{z_0^2} \end{aligned} \tag{1.78}$$

and furthermore, after a page of algebra:

$$\begin{aligned} 1 = {} & \left(\frac{u}{z_0}\cos(\theta)\frac{\left(1-\alpha\tan^2(\theta)\right)}{\sqrt{\alpha}} - \frac{\cot(\theta)}{\sqrt{\alpha}}\right)^2 \\ & + \frac{v^2}{z_0^2}\left(\frac{1-\alpha\tan^2(\theta)}{\alpha}\right) \end{aligned} \tag{1.79}$$

<u>Problem 42</u>

Derive Equation (1.78) and Equation (1.79).

Gamma Factor

To help tame the symbolic explosion that has occurred, let us introduce yet another symbol γ such that

$$\gamma = 1 - \alpha\tan^2(\theta) . \tag{1.80}$$

4.2 Parabolic Case

If the quantity $\gamma = 1 - \alpha\tan^2(\theta)$ resolves to zero for some special choice of α, θ, then Equation (1.78) reduces to that of a parabola:

$$1 = 2\frac{u}{z_0}\sin(\theta) - \tan^2(\theta)\frac{v^2}{z_0^2}$$

To assure we're looking at a parabola, note the v-coordinate occurs as v^2, whereas the u-coordinate occurs to the first power.

4.3 Ellipse vs. Hyperbola

Looking again at Equation (1.79), it turns out that the γ factor also dictates whether we're looking at an ellipse versus a hyperbola. For small angles θ, and/or for small stretch factors α, the quantity γ will always remain positive. This corresponds to an elliptical conic section. On the other hand, for large angles θ and/or large stretch factors α, the quantity γ becomes negative, giving rise to the hyperbolic conic section. In either case, we have:

$$1 = \left(\frac{u}{z_0}\cos(\theta)\frac{\gamma}{\sqrt{\alpha}} - \frac{\cot(\theta)}{\sqrt{\alpha}}\right)^2 + \frac{v^2}{z_0^2}\left(\frac{\gamma}{\alpha}\right) \quad (1.81)$$

With u, v playing analogous roles to x, y, we can immediately pick out the major and minor axes a, b which come out to:

$$a = \frac{z_0\sqrt{\alpha}}{\gamma\cos(\theta)}$$
$$b = z_0\sqrt{\frac{\alpha}{\gamma}}$$

Eccentricity

Going by definition, the eccentricity of the ellipse is

$$\begin{aligned} e &= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \gamma\cos^2(\theta)} \\ &= \sin(\theta)\sqrt{1+\alpha}\,. \end{aligned} \quad (1.82)$$

Similarly, the eccentricity of the hyperobla is

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \gamma\cos^2(\theta)} \\ &= \sqrt{2 - \sin^2(\theta)(1+\alpha)}\,. \end{aligned} \quad (1.83)$$

Problem 43

Derive Equation (1.82) and Equation (1.83).

5 General Conic Sections

5.1 Review

By playing certain games with a directrix (line) and a focus (point) in the plane, three distinct species of curve emerge, namely the ellipse, the hyperbola, and the parabola. Each curve has a distict shape and at least one focus as detailed:

	equation	focus
ellipse	$x^2/a^2 + y^2/b^2 = 1$	$(c, 0)$
hyperbola	$y^2/a^2 - x^2/b^2 = 1$	$(0, c)$
parabola	$y = ax^2 + bx + c$	$1/4a$

Eccentricity

A single number called eccentricity, denoted e, classifies whether the curve is an ellipse ($e < 1$), a hyperbola ($e > 1$), or a parabola ($e = 1$), as summarized:

	eccentricity
ellipse	$e = \sqrt{1 - b^2/a^2} < 1$
hyperbola	$e = \sqrt{1 + b^2/a^2} > 1$
parabola	$e = 1$

Polar Representation of Conics

The ellipse, hyperbola, and parabola are siblings in polar coordinates when the origin is at a focus. Remarkably, all three curves are represented by one single equation tuned by the eccentricity:

$$r = \frac{pe}{1 - e\cos(\theta)}$$

5.2 Generalized Conics

In the most general case, a conic section in the Cartesian plane can come to you in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F\,. \quad (1.84)$$

The coefficients A through F not only determine the curve species, but also the placement and *rotation* of the curve via the Bxy term.

5.3 Rotated Coordinates

To analyze Equation (1.84), it helps to use a second coordinate system uv that is rotated with respect to the original xy coordinate system so there is no mixed 'rotation' term. The uv system shares the shame origin as the xy system, but is tuned by θ to align with the curve's principal axes. Such a rotated coordinate system can be written

$$\begin{aligned} u &= x\cos(\theta) + y\sin(\theta) \\ v &= -x\sin(\theta) + y\cos(\theta)\ , \end{aligned}$$

which can be inverted to read

$$\begin{aligned} x &= u\cos(\theta) - v\sin(\theta) \\ y &= u\sin(\theta) + v\cos(\theta)\ . \end{aligned}$$

Note that a positive rotation in θ corresponds to counterclockwise progression of the uv system with respect to the xy system, which makes the curve itself appear to progress clockwise in the uv frame.

To proceed, take the quantity $Ax^2 + Bxy + Cy^2$ and substitute the x- and y-equations above to find

$$\begin{aligned} &Ax^2 + Bxy + Cy^2 \\ &\quad = u^2 \left(A\cos^2(\theta) + C\sin^2(\theta) + B\sin(\theta)\cos(\theta)\right) \\ &\quad + v^2 \left(A\sin^2(\theta) + C\cos^2(\theta) - B\sin(\theta)\cos(\theta)\right) \\ &\quad + uv\left(-A\sin(2\theta) + C\sin(2\theta) + B\cos(2\theta)\right) \end{aligned}$$

In order to eliminate the 'mixed' uv-term in the rotated coordinate system, we find the restriction on θ to be given by

$$B' = -A\sin(2\theta) + C\sin(2\theta) + B\cos(2\theta) = 0\ ,$$

or

$$\tan(2\theta) = \frac{B}{A-C}\ . \tag{1.85}$$

So far then, we can write

$$A'u^2 + C'v^2 + D'u + E'v = F\ , \tag{1.86}$$

where the primed coeffcients A' through E' can be traced back to the original coefficients.

Problem 44

Write explicit formulas for A', B', C', D', E' in terms of θ and the unprimed coefficients.

Problem 45

The equation

$$21x^2 + 31y^2 - \sqrt{300}xy = 144$$

describes a tilted ellipse centered at the origin. Determine the angle θ required to un-tilt the ellipse and write the new equation. (Answer: $2\theta = \tan^{-1}\left(\sqrt{3}\right)$, $u^2/9 + v^2/4 = 1$)

Problem 46

The axis of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is tilted by 45 degrees using the origin as a pivot so that the new axis lies along the line $y = x$. (The axis cuts through both focus points and the rotated hyperbola lives strictly in the first and third quadrants.) Prove that the equation of the tilted hyperbola is

$$v = \left(\frac{a^2+b^2}{a^2-b^2}\right)u \pm \frac{ab}{a^2-b^2}\sqrt{4u^2 - 2\left(a^2-b^2\right)}\ .$$

Problem 47

The axis of the hyperbola $x^2 - y^2 = 2$ is tilted by 45 degrees using the origin as a pivot so that the new axis lies along the line $y = x$. Show that the new equation is $v = 1/u$.

Problem 48

The axis of the parabola $y = x^2 - 1/4$ is tilted by 45 degrees using the focus as a pivot so that the new axis lies along the line $y = x$. Prove that the equation of the tilted parabola is

$$v = u + \frac{1}{\sqrt{2}} \pm \sqrt{2\sqrt{2}u + 1}\ .$$

Classifying Rotated Conics

With the mixed uv-term squelched out, the type of curve described by Equation (1.86) is indicated by the signs on the A' and C' terms. If either A' or C' is zero, the curve is parabolic. The curve is elliptical if A and C agree each positive, and so on. If both A and C are zero, the curve at best linear.

5.4 Discrimanant of a Conic

It is possible to determine the type of curve described by the general Equation (1.84) without manually rotating coordinates. To do this, we must calculate the *discrimanant* of the conic, defined by

$$\mathcal{D} = B^2 - 4AC\ . \tag{1.87}$$

It turns out that $\mathcal{D}$ resolves to the same value regardless of the rotation angle of the coordiante system, making $\mathcal{D}$ an *invariant* quantity. To prove this, let us calculate

$$\mathcal{D}' = \left(B'\right)^2 - 4A'C'$$

and check if $\mathcal{D}' = \mathcal{D}$ in the general case. To get started, we'll calculate the square of B' and the product $-4A'C'$ separately:

$$\begin{aligned} \left(B'\right)^2 &= \left(A^2 + C^2 - 2AC\right)\sin^2(2\theta) \\ &\quad + B^2\cos^2(2\theta) + B\left(C - A\right)\sin(4\theta) \\ -4A'C' &= \left(-A^2 - C^2 + B^2\right)\sin^2(2\theta) \\ &\quad - B\left(C - A\right)\sin(4\theta) \\ &\quad - 4AC + 2AC\sin^2(2\theta) \end{aligned}$$

Taking the sum of the two results, we see that all of the ugly terms cancel out and the form $B^2 - 4AC$ emerges:

$$\left(B'\right)^2 - 4A'C' = B^2 - 4AC \tag{1.88}$$

More succintly, we see $\mathcal{D}' = \mathcal{D}$ for any angle θ. Not surprisingly, the coefficients D, D', E, E', F, are not involved in the discrimanant or the classification of the curve.

Problem 49

Derive Equation (1.88).

5.5 Classifying General Conics

Having shown that the discrimanant $B^2 - 4AC$ of a general conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

is invariant with respect to coordinate system rotations, we are free to choose the system that tunes for $B' = 0$ in accordance with Equation (1.85) to write

$$\mathcal{D} = B^2 - 4AC = -4A'C' \ . \tag{1.89}$$

Recall that the parabolic case corresponds to either of A' or C' being zero, causing $\mathcal{D} = 0$. If A' and C' agree in sign, the curve is elliptical and $\mathcal{D}$ remains negative. If A' and C' disagree in sign, the curve is hyperbolic and $\mathcal{D}$ is positive.

	discrimanant
ellipse	$B^2 - 4AC < 0$
hyperbola	$B^2 - 4AC > 0$
parabola	$B^2 - 4AC = 0$

Problem 50

Consider the hyperbola given by $xy = 1$. Use Equation (1.89) to express the same hyberbola with no mixing term.

Using the Discriminant

It's possible to show using calculus that the area of the ellipse is given by

$$\text{Area} = \pi ab \ ,$$

where a, b are the major and minor axes. With this information, we can determine the area of the ellipse

$$Ax^2 + Bxy + Cy^2 = 1 \ .$$

Choose a second uv-coordinate system whose orientation satisfies Equation (1.85), and the same ellipse takes the form

$$A'u^2 + C'v^2 = 1 \ .$$

Comparing this to the usual equation of an ellipse, it seems that A' is the inverse square of the major axis, and similarly for C' and the minor axis. The area of this ellipse is thus

$$\text{Area} = \frac{\pi}{\sqrt{A'C'}} \ .$$

Now involve the discriminant via Equation (1.89)

$$B^2 - 4AC = -4A'C'$$

to replace the primed terms with the original variables and the problem is finished:

$$\text{Area} = \frac{\pi}{\sqrt{4AC - B^2}}$$

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