Complex Analysis
MANUSCRIPT

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Chapter 1

Complex Analysis

1 Complex Algebra Review

1.1 Complex Number

Let any complex number z be represented as

$$z = x + iy \,,$$

where x and y are real numbers, and i is the imaginary unit satisfying

$$i^2 = -1 \,.$$

The x-component is called the 'real part' of z, written

$$x = \mathrm{Re}\,(z) \,,$$

and the y-component is the 'imaginary part' of z:

$$y = \mathrm{Im}\,(z)$$

Complex Conjugate

Any complex number z has a complex conjugate $\overline{z} = z^*$, also a complex number, defined such that

$$\bar{z} = z^* = x - iy \,.$$

That is, the complex conjugate simply reverses the sign on the imaginary component.

Scalar Multiplication

A complex number z can be scaled by a dimensionless real number λ by multiplying λ into each component of z:

$$\lambda z = \lambda x + i\lambda y$$

1.2 Complex Arithmetic

For two complex numbers z_1, z_2, the equations of complex arithmetic can be generated from two statements

$$\begin{aligned} z_1 * z_2 &= z_2 * z_1 \\ \overline{z_1} * \overline{z_2} &= \overline{z_1 * z_2} \,, \end{aligned}$$

where the generalized operator represents either complex addition (+) or complex multiplication ($\cdot$).

From the above, one finds

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

and

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1) \,.$$

It's straightforward to show that complex addition and multiplication follow the standard associative and distributive properties:

$$(z_1 z_2)\, z_3 = z_1\, (z_2 z_3)$$
$$z_1\, (z_2 + z_3) = (z_1 z_2) + (z_1 z_3)$$

Complex Magnitude

The complex magnitude

$$|z| = \sqrt{z\bar{z}} = \sqrt{(x+iy)\,(x-iy)} = \sqrt{x^2+y^2}$$

is a real number that measures the distance from z to the origin in the complex plane.

Complex Division

For two complex number z_1, z_2, the ratio is defined as:

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z_2}}{|z_2|^2}$$

One can readily check that this definition readily satisfies $\overline{z_1} * \overline{z_2} = \overline{z_1 * z_2}$, where the generalized arithmetic operator ($*$) is replaced by the division symbol (/).

1.3 Complex Plane

Polar Representation

Complex numbers lend naturally to polar representation

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \\ r &= |z| = \sqrt{x^2+y^2} \\ \phi &= \arctan(y/x)\ , \end{aligned}$$

where ϕ is the complex phase of z.

This setup is identical to that of plane polar coordinates with the real numbers along the x-axis and the imaginary numbers along the y-axis. The complex number z can be written

$$z = r\,(\cos(\phi) + i\sin(\phi))\ .$$

Rotations

The special complex number z_θ with $|z_\theta| = 1$ and any phase θ is a rotation operator for complex numbers:

$$z_\theta = \cos(\theta) + i\sin(\theta)$$

This is because the product $z_\theta z$ for any $z\,(r,\phi)$ results in:

$$z_\theta z = r\,(\cos(\phi+\theta) + i\sin(\phi+\theta))$$

That is multiplying by z_θ has the effect of a change of phase $\phi \to \phi + \theta$.

1.4 Euler's Formula

Repeated Rotations

Making repeated use of the rotation operator z_θ, suppose we start with a complex number $z\,(r,\phi)$ and make n identical rotations by the angle θ:

$$z_\theta^n z = r\,(\cos(\phi+n\theta) + i\sin(\phi+n\theta))$$

Without loss of generality, we can suppose the original complex number z is the real number $z = 1$, meaning $r = 1$ and $\phi = 0$. This yields a new way to write the rotation operator as

$$z_\theta = \left(\cos\left(\frac{\theta}{h}\right) + i\sin\left(\frac{\theta}{h}\right)\right)^h\ ,$$

where $h = 1/n$.

Pressing the limit $h \to \infty$, the quantity θ/h tends to zero, warranting the small-angle approximation to replace both trigonometry terms:

$$z_\theta = \lim_{h\to\infty}\left(1 + \frac{i\theta}{h}\right)^h$$

The right side is precisely the definition of Euler's constant e raised to the power $i\theta$. In summary, we have found

$$z_\theta = e^{i\theta} = \cos(\theta) + i\sin(\theta)\ ,$$

one of the msot useful relationships in mathematics.

Calculus-based Derivation

Begin with the polar representation of a complex number

$$z = r\,(\cos(\phi) + i\sin(\phi))\ ,$$

and compute the differential dz. From calculus, we know

$$dz = \frac{\partial z}{\partial r}\,dr + \frac{\partial z}{\partial \phi}\,d\phi\,,$$

where the ∂ symbol denotes a partial derivative. Evaluating this and simplifying, find

$$\frac{dz}{z} = \frac{dr}{r} + i\,d\phi\,.$$

With all variables separated, integrate the above to find

$$\ln(z) = \ln(r) + i\phi + \cancel{C}\,,$$

dropping the integration constant. This result is simply the natural log of Euler's formula:

$$z = r\,e^{i\phi}$$

Multiplication and Division

Euler's formula makes quick work of multiplication and division of complex numbers. For any two complex numbers $z_1(r_1, \phi_1)$, $z_2(r_2, \phi_2)$, we always have

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \, e^{i(\phi_1+\phi_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} \, e^{i(\phi_1-\phi_2)} \end{aligned}$$

Complex Logarithm

Consider a general complex number

$$z(r, \phi) = r \, e^{i\phi} \, ,$$

which admits a logarithmic form

$$\ln(z) = \ln(r) + i\phi \, .$$

Branches

Unlike $z(r, \phi)$, the complex logarithm of z has a complex phase term $i\phi$ that does not 'reset' outside the interval $[0 : 2\pi)$. This raises an important subtlety called branches, or branch cuts, which are apparent phase discontinuities in complex functions projected onto the complex plane.

Complex Exponent

For two complex numbers $z(r, \phi)$, $w(\alpha, \beta)$, the exponent calculation z^w proceeds as:

$$\begin{aligned} z^w &= \left(r \, e^{i\phi}\right)^{\alpha+i\beta} \\ &= r^\alpha r^{i\beta} \, e^{i\phi\alpha} \, e^{-\phi\beta} \\ &= e^{\ln(r)\alpha} \, e^{\ln(r)i\beta} \, e^{i\phi\alpha} \, e^{-\phi\beta} \\ &= e^{\alpha\ln(r)-\beta\phi} \, e^{i(\beta\ln(r)+\alpha\phi)} \\ &= \exp\left((\ln(r) + i\phi)(\alpha + i\beta)\right) \\ &= e^{w\ln(z)} \end{aligned}$$

2 Solving Classic Systems

Complex numbers are a pathway to many abilities some would consider to be unusual.

2.1 Velocity and Acceleration

If a complex number z depends on time via

$$z(t) = r(t) \, e^{i\theta(t)} \, ,$$

we may take derivatives to write equations for 'velocity' and 'acceleration' in the complex plane. Letting

$$\begin{aligned} \frac{dr}{dt} &= \dot{r} \\ \frac{d\theta}{dt} &= \dot{\theta} = \omega \, , \end{aligned}$$

one finds:

$$\begin{aligned} \frac{d}{dt} z(t) &= \dot{r} e^{i\theta} + i e^{i\theta} r\omega \\ \frac{d^2}{dt^2} z(t) &= e^{i\theta}\left(\ddot{r} - r\omega^2\right) + i e^{i\theta}\left(2\dot{r}\omega + r\dot{\omega}\right) \end{aligned}$$

The results for $\dot{z}$ and $\ddot{z}$ are each complex numbers, carrying real and imaginary components. By associating

$$\begin{aligned} e^{i\theta} &\to \hat{r} \\ i e^{i\theta} &\to \hat{\theta} \, , \end{aligned}$$

we discover a shortcut for the velocity and acceleration vectors in polar coordinates.

2.2 Simple Harmonic Oscillator

Near a stable point, many classical systems are characterized by a position $x(t)$ that obeys

$$\ddot{x} + \omega_0^2 \, x = 0 \, ,$$

where the 'double-dot' operator implies two time derivatives, i.e. $\ddot{x} = d^2x/dt^2$, and the angular frequency ω_0 is a real-valued constant. This is the so-called simple harmonic oscillator having known solutions based on trigonometric functions.

Setup

The SHO problem is elegantly solved using complex numbers. Let us pose the same problem for a complex-valued, time-dependent variable $w(t)$:

$$\ddot{w} + \omega_0^2 \, w = 0$$

As a complex number w, decomposes to

$$w(t) = x(t) + iy(t) \, ,$$

or in terms of real and imaginary components

$$\begin{aligned} x(t) &= \mathrm{Re}\left(w(t)\right) \\ y(t) &= \mathrm{Im}\left(w(t)\right) \, , \end{aligned}$$

specifically

$$\begin{aligned} \ddot{x} + \omega_0^2 \, x &= 0 \\ \ddot{y} + \omega_0^2 \, y &= 0 \, . \end{aligned}$$

Solution

Next, we propose solutions to $w(t)$ as

$$w(t) = A\, e^{\lambda t} ,$$

where A is the amplitude constant and λ is a frequency constant. Substitution of $w(t)$ into the SHO differential equation quickly reveals

$$\lambda^2 + \omega_0^2 = 0 ,$$

telling us

$$\lambda = \pm i\omega_0 .$$

Since the SHO differential equation is linear, it follows that the general solution is the sum of partial solutions

$$w(t) = A_1\, e^{i\omega_0 t} + A_2\, e^{-i\omega_0 t}$$

Proceed by writing the complex constants A_1, A_2 in polar form

$$A_1 = a_1\, e^{i\phi_1}$$
$$A_2 = a_2\, e^{i\phi_2} ,$$

where a_1, a_2 are real-valued, and ϕ_1, ϕ_2 are phase constants.

Using Euler's formula to expand the exponential terms, we find

$$\begin{aligned} w(t) &= a_1 \cos(\phi_1 + \omega_0 t) + a_2 \cos(\phi_2 - \omega_0 t) \\ &\quad + i a_1 \sin(\phi_1 + \omega_0 t) + i a_2 \sin(\phi_2 - \omega_0 t) , \end{aligned}$$

or, splitting the real from the imaginary parts:

$$\begin{aligned} x(t) &= a_1 \cos(\phi_1 + \omega_0 t) + a_2 \cos(\phi_2 - \omega_0 t) \\ y(t) &= a_1 \sin(\phi_1 + \omega_0 t) + a_2 \sin(\phi_2 - \omega_0 t) \end{aligned}$$

These solutions are identical up to the phase constants ϕ_1, ϕ_2. For instance, let $\phi_2 \to \phi_2 + \pi/2$ to transform the sines into cosines. Proceeding with the $x(t)$-equation, let

$$\begin{aligned} a &= a_1 \cos(\phi_1) + a_2 \cos(\phi_2) \\ b &= -a_1 \sin(\phi_1) + a_2 \sin(\phi_2) , \end{aligned}$$

and we get

$$x(t) = a \cos(\omega_0 t) + b \sin(\omega_0 t) .$$

Or, to make the solution even tighter, let

$$\begin{aligned} a &= R \cos(\phi_0) \\ b &= R \sin(\phi_0) , \end{aligned}$$

and $x(t)$ becomes

$$x(t) = R \cos(\omega_0 t - \phi_0) .$$

Note that the number of free constants is down to two, which should be the case for a second-order differential equation.

Damped Harmonic Oscillator

Now we consider the differential equation for the damped harmonic oscillator given by

$$\ddot{x} + b\dot{x} + \omega_0^2\, x = 0 ,$$

where b is the camping coefficient.

Following the same procedure that applies to the SHO case, we replace all $x(t)$ with $w(t)$ and consider complex solutions

$$w(t) = A\, e^{\lambda t} ,$$

immediately leading to

$$\lambda^2 + b\lambda + \omega_0^2 = 0 .$$

The two solutions for λ come out as

$$\lambda_\pm = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \omega_0^2} .$$

The way b relates to ω_0 dictates the overall character of the solution.

Overdamped Harmonic Oscillator

In the special case $b/2 > \omega_0$, the damping term is strong enough to overwhelm the system's tendency to oscillate, and the solution decays exponentially without oscillating. (All relevant variables in the problem are real-valued.) Up to arbitrary constants determined by initial conditions, the overdamped oscillator obeys

$$x(t) = A_1\, e^{\lambda_+ t} + A_2\, e^{\lambda_- t} .$$

Underdamped Harmonic Oscillator

If we instead have $b/2 < \omega_0$, the λ-terms become

$$\lambda_\pm = -\frac{b}{2} \pm i\sqrt{\omega_0^2 - \frac{b^2}{4}} ,$$

now including an imaginary component. Utilizing the SHO analysis above, the general solution to this case reads

$$w(t) = e^{-bt/2} \left(a_1\, e^{\tilde{\omega} t + \phi_1} + a_2\, e^{-\tilde{\omega} t + \phi_2} \right) ,$$

where

$$\tilde{\omega} = \sqrt{\omega_0^2 - b^2/4} .$$

The damping term causes the amplitude to decay exponentially in time. Note the oscillatory portion of the solution is identical to that of the simple harmonic oscillator, and can be reduced to sine and/or cosine terms with two arbitrary constants. Note that the effective angular frequency $\tilde{\omega}$ depends on the damping term.

Critically-Damped Harmonic Oscillator

The behavior of the damped oscillator depends chiefly on the quantity $\omega_0^2 - b^2/4$, giving rise to either damped or oscillatory motion. The special case $\omega_0^2 = b^2/4$ is the criteria for *critical damping*. Physically, a critically-damped system returns to its equilibrium position in the shortest time.

Returning to the proposed solution $w(t) = A\exp(\lambda t)$ under the condition

$$\omega_0^2 - \frac{b^2}{4} = 0\,,$$

we see there is only one choice for λ, namely $\lambda = b/2$. As such, this would mean there is only one arbitrary parameter A in the the solution, which is one too few parameters to qualify as a general solution to a second-order differential equation. In other words, something is wrong with our guess for $w(t)$.

Starting the problem over again by introducing the shorthand notation

$$\frac{d}{dt}x(t) = \dot{x} = d_t x(t)\ ,$$

the differential equation of the damped oscillator can be written

$$\left[d_{tt} + b d_t + \omega_0^2\right] x(t) = 0\,,$$

where the quantity in square brackets is an object that operates on $x(t)$. Proceed by 'completing the square' within the operator to land at

$$\left[d_t + \frac{b}{2}\right]^2 x(t) = \left(\frac{b^2}{4} - \omega_0^2\right) x(t)\ .$$

For convenience, use the shorthand

$$D = \left[d_t + \frac{b}{2}\right]$$

$$-\tilde{\omega}^2 = \frac{b^2}{4} - \omega_0^2$$

to write

$$D^2 x(t) = -\tilde{\omega}^2 x(t)\ .$$

Next, let

$$x(t) = e^{-bt/2}\tilde{x}(t)\ ,$$

and the above becomes

$$e^{-bt/2} D^2 \tilde{x}(t) + \tilde{x}(t) D^2 e^{-bt/2} = -\tilde{\omega}^2 e^{-bt/2}\tilde{x}(t)\ .$$

Dealing with the middle term first, notice

$$\begin{aligned} D^2 e^{-bt/2} &= \left[d_{tt} + b d_t + \frac{b^2}{4}\right] e^{-bt/2} \\ &= \left(\frac{b^2}{4} - \frac{b^2}{2} + \frac{b^2}{4}\right) e^{-bt/2} = 0\,, \end{aligned}$$

and the above simplifies to

$$D^2 \tilde{x}(t) = -\tilde{\omega}^2 \tilde{x}(t)\ .$$

At this point, we finally apply the case of critical damping, in where $\tilde{\omega} = 0$. This reduces the above to

$$D^2 \tilde{x}(t) = 0\,,$$

having general solution

$$\tilde{x}(t) = a_1 + a_2 t\,.$$

To see this quickly, you may treat D as a derivative operator and mentally integrate both sides of the equation. If skeptical, formally unpack the operation via $D = d_t + b/2$ and find the same result. Finally, we assemble the solution to the critically-damped oscillator:

$$x(t) = e^{-bt/2}(a_1 + a_2 t)$$

Driven Harmonic Oscillator

It's also possible to analyze the so-called driven oscillator, generally described by

$$\ddot{x} + b\dot{x} + \omega_0^2 x = f(t)\ .$$

Due to the linearity in the left hand side, we know already that the solution to the above takes the form

$$x(t) = x_h(t) + x_p(t)\ ,$$

i.e. the sum of a homogeneous part $x_h(t)$ and a particular part $x_p(t)$. Knowing $x_h(t)$ already to be

$$x_h(t) = R\, e^{-bt/2}\cos(\omega_0 t - \phi_0)\ ,$$

the task is reduced to finding a particular solution $x_p(t)$.

In general, finding the particular solution to the the above with arbitrary $f(t)$ is as difficult as it sounds, so we make the job easier by considering a sinusoidal driving function

$$f(t) = \gamma\cos(\alpha t)\ .$$

To solve the problem on hand, it's convenient to convert all variables to polar form, and then take only the real part of the solution. Proceeding this way, we have, for a complex variable $w(t)$,

$$\ddot{w} + b\dot{w} + \omega_0^2 w = \gamma\, e^{i\alpha t}\,.$$

Next, we postulate complex solutions of the form

$$w(t) = A\, e^{i\alpha t}$$

for some arbitrary constant A, and the above becomes

$$A\left(-\alpha^2+ib\alpha+\omega_0^2\right)\cancel{e^{i\alpha t}}=\gamma\,\cancel{e^{i\alpha t}}\,.$$

Solving for A gives

$$A=\frac{\gamma\left(\omega_0^2-\alpha^2\right)-i\gamma b\alpha}{\left(\omega_0^2-\alpha^2\right)^2+b^2\alpha^2}\,,$$

where if we let

$$\begin{aligned}u&=\omega_0^2-\alpha^2\\ v&=\beta\alpha\,,\end{aligned}$$

a complex number q can be written as

$$q=u-iv=q_0\,e^{-i\phi_0}\,,$$

with

$$q_0=\sqrt{\left(\omega_0^2-\alpha^2\right)^2+b^2\alpha^2}\,.$$

Rewriting A, we have:

$$A=\frac{\gamma q_0\,e^{-i\phi_0}}{q_0^2}=\frac{\gamma\,e^{-i\phi_0}}{\sqrt{\left(\omega_0^2-\alpha^2\right)^2+b^2\alpha^2}}$$

Finally, the particular solution to the driven oscillator reads

$$\begin{aligned}w\left(t\right)&=\frac{\gamma\,e^{i(\alpha t-\phi_0)}}{\sqrt{\left(\omega_0^2-\alpha^2\right)^2+b^2\alpha^2}}\\ \phi_0&=\tan^{-1}\left(\frac{b\alpha}{\omega_0^2-\alpha^2}\right)\end{aligned}$$

Note that this particular solution lacks a term like $\exp\left(bt/2\right)$, which, much unlike the homogeneous solution, doesn't decay over long times.

The amplitude A depends on how ω_0 relates to α. Calculating

$$\frac{d}{d\alpha}\left|A\left(\alpha\right)\right|=0$$

indicates the special α_R for which the amplitude is maximal. Performing this calculation, one finds

$$\alpha_R^2=\omega_0^2-\frac{b^2}{2}\,,$$

indicating

$$A_R=\frac{\gamma/b}{\sqrt{\omega_0^2-b^2/2}}\,.$$

Note that when $\alpha=\alpha_R$, the system is said to be in *resonance.*

2.3 Particle in Magnetic Field

Consider a particle of mass m and charge q in the presence of a uniform magnetic field $\vec{B}$. The force incident on the particle is given by

$$\vec{F}=q\,\vec{v}\left(t\right)\times\vec{B}\,,$$

where $\vec{v}\left(t\right)$ is the instantaneous velocity. Equations of motion are determined by applying Newton's second law

$$\vec{F}=m\frac{d}{dt}\vec{v}\left(t\right)\ .$$

Being a three-dimensional problem, let us choose to align the magnetic field with the positive z-axis. Then, by eliminating $\vec{F}$, we have

$$m\frac{d}{dt}\vec{v}\left(t\right)=qB\,\vec{v}\left(t\right)\times\hat{z}\,,$$

or

$$\begin{aligned}\frac{d}{dt}\vec{v}\left(t\right)&=\omega_0\,\vec{v}\left(t\right)\times\hat{z}\\ \omega_0&=\frac{qB}{m}\,.\end{aligned}$$

The unpacks into three equations:

$$\begin{aligned}\frac{d}{dt}v_x\left(t\right)&=\omega_0v_y\left(t\right)\\ \frac{d}{dt}v_y\left(t\right)&=-\omega_0v_x\left(t\right)\\ \frac{d}{dt}v_z\left(t\right)&=0\end{aligned}$$

Noting that $v_z\left(t\right)$ is constant in time, we immediately know the solution for $z\left(t\right)$, namely

$$z\left(t\right)=z_0+v_zt\,,$$

and we may focus entirely on what occurs in the xy-plane. Defining a complex variable

$$w\left(t\right)=v_x\left(t\right)+iv_y\left(t\right)\ ,$$

we capture the information written above using the derivative

$$\frac{d}{dt}w\left(t\right)=\omega_0v_y\left(t\right)-i\omega_0v_x\left(t\right)\ ,$$

simplifying to

$$i\frac{d}{dt}w\left(t\right)=\omega_0w\left(t\right)\ .$$

This first-order differential equation is easily solved by

$$w\left(t\right)=Ae^{-i\omega_0t}=\left|A\right|e^{i(\phi_0-\omega_0t)}\,,$$

where A is an arbitrary complex constant. Taking the real and imaginary parts of the above, the equations for $v_x(t)$, $v_y(t)$ emerge:

$$v_x(t) = \mathrm{Re}(w(t)) = |A|\cos(\omega_0 t - \phi_0 t)$$
$$v_y(t) = \mathrm{Im}(w(t)) = -|A|\sin(\omega_0 t - \phi_0)$$

While the above can be integrated separately to attain equations of motion $x(t)$, $y(t)$, we can integrate $w(t)$ directly by introducing a variable

$$R = x(t) + iy(t) \ ,$$

whose derivative is $w(t)$:

$$\frac{d}{dt}R(t) = w(t)$$

Then, the integral of the above can be written

$$R(t) = R_0 + |A|\int_0^t e^{i\left(\phi_0 - \omega_0 t'\right)} dt' \ ,$$

which can be solved with relative ease:

$$\begin{aligned} R(t) &= R_0 + |A| e^{i\phi_0}\int_0^t e^{-i\omega_0 t'} dt' \\ &= R_0 + |A| e^{i\phi_0}\frac{1}{-i\omega_0}e^{-i\omega_0 t'}\Big|_0^t \\ &= R_0 + |A| e^{i\phi_0}\frac{i}{\omega_0}\left(e^{-i\omega_0 t} - 1\right) \\ &= R_0 + |A| e^{i\phi_0 - i\omega_0 t}\frac{i}{\omega_0} - |A| e^{i\phi_0}\frac{i}{\omega_0} \end{aligned}$$

Repacking the integration constants together, we write

$$\tilde{R}_0 = R_0 - |A| e^{i\phi_0}\frac{i}{\omega_0} \ ,$$

the solution reads

$$\begin{aligned} R(t) &= \tilde{R}_0 + |A| e^{i(\phi_0 - \omega_0 t)}\frac{i}{\omega_0} \\ &= \tilde{R}_0 + \frac{|A|}{\omega_0}\left(i\cos(\omega_0 t - \phi_0) + \sin(\omega_0 t - \phi_0)\right) \ . \end{aligned}$$

Finally, we find

$$x(t) = \mathrm{Re}(R(t)) = \tilde{x}_0 + \frac{|A|}{\omega_0}\sin(\omega_0 t - \phi_0)$$
$$y(t) = \mathrm{Im}(R(t)) = \tilde{y}_0 + \frac{|A|}{\omega_0}\cos(\omega_0 t - \phi_0) \ ,$$

where

$$\tilde{x}_0 = x_0 + \frac{|A|}{\omega_0}\sin(\phi_0)$$
$$\tilde{y}_0 = y_0 - \frac{|A|}{\omega_0}\cos(\phi_0) \ .$$

This is the exact circular motion for a charged particle in a uniform magnetic field.

3 Complex Differentiation

A complex function $w(z)$ of a single variable $z = x + iy$ has the structure

$$w(z) = u(x, y) + iv(x, y) \ ,$$

where u and v are the respective real and imaginary components of the function.

If the components of w depend on time, we've seen that taking derivatives of $w(z(t))$ have utility in problem solving. However, a treatment of complex derivatives require care in the general case.

3.1 Partial Derivatives

Begin by calculating the differential of a function $w(u, v)$ while substituting dx and dy for their representations in terms of dz and $d\bar{z}$:

$$\begin{aligned} dw(x, y) &= dx\frac{\partial w}{\partial x} + dy\frac{\partial w}{\partial y} \\ &= \frac{1}{2}(dz + d\bar{z})\frac{\partial w}{\partial x} + \frac{1}{2i}(dz - d\bar{z})\frac{\partial w}{\partial y} \\ &= \frac{dz}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) + \frac{d\bar{z}}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \end{aligned}$$

Meanwhile, the same function w can be written $w(z, \bar{z})$, having differential version

$$\delta w(z, \bar{z}) = \delta z\frac{\partial w}{\partial z} + \delta\bar{z}\frac{\partial w}{\partial \bar{z}} \ .$$

Comparing the two equations provides a definition for $\partial w/\partial z$ and $\partial w/\partial \bar{z}$:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)$$
$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)$$

Polar Frame

In place of x and y, we may cast complex functions in terms of r and θ, leading to the following derivative operators:

$$\begin{aligned} r\frac{\partial}{\partial r} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial \theta} &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} \\ z\frac{\partial}{\partial z} &= \frac{1}{2}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial \theta}\right) \\ \bar{z}\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right) \end{aligned}$$

To derive the these, let $x = r\cos\theta$ and $y = r\sin\theta$, and then use the chain rule on $w(r, \theta)$ and $w(z, \bar{z})$.

3.2 Total Derivative

While partial derivatives of complex functions are straightforward, the total derivative is trouble. Proceeding in a calculus-101 analogy, we write

$$\frac{dw(z,\bar{z})}{dz} = \lim_{\Delta z \to 0} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} \frac{\Delta \bar{z}}{\Delta z},$$

where if $\Delta z = |\Delta z| e^{i\theta}$, then the ratio $\Delta \bar{z}/\Delta z$ becomes $e^{-2i\theta}$, which can have *any* phase θ as $\Delta z \to 0$. The best thing we can do about the total derivative is to restrict w to have no explicit $\bar{z}$-dependence, eliminating the second term altogether. We therefore take the following two equations as criteria of the total derivative:

$$\frac{dw}{dz} = \frac{\partial w}{dz}$$
$$\frac{\partial w}{\partial \bar{z}} = 0$$

3.3 Analytic Functions

We have seen that a complex function's simultaneous dependence on the complex point z and its complex conjugate $\bar{z}$ can have an ambiguous total derivative. Such functions where z and $\bar{z}$ appear are denoted as $w(z,\bar{z})$, with the letter w reserved.

Many interesting complex functions only depend on z (with $\bar{z}$ absent), denoted $f(z)$. If the derivative df/dz exists then the function is called *analytic*. Points where df/dz does not exist are called *singular*, where isolated singular points are called *poles*. An *entire* function has no singular points in its domain.

To demonstrate, the following three functions are *not* analytic for all z:

$$f_1(z) = x^2 - y^2$$
$$f_2(z) = x^2 + iy^2$$
$$f_3(z) = r^2(\cos(\theta) + i\sin(\theta))$$

Check this by calculating derivatives of each:

$$\frac{\partial}{\partial \bar{z}} f_1(z) = \frac{\partial}{\partial \bar{z}}(z\bar{z}) = z \neq 0$$
$$\frac{\partial}{\partial \bar{z}} f_2(z) = \frac{1}{2}\left(\frac{\partial f_2}{\partial x} + i\frac{\partial f_2}{\partial y}\right) = x - y \neq 0$$
$$\frac{\partial}{\partial \bar{z}} f_3(z) = \frac{1}{2\bar{z}}\left(r\frac{\partial f_3}{\partial r} + i\frac{\partial f_3}{\partial \theta}\right) = \frac{2rz - irz}{2\bar{z}} \neq 0$$

On the other hand, the following functions are analytic for all z:

$$f_4(z) = x^2 + 2ixy - y^2 = z^2$$
$$f_5(z) = \ln(r) + i\theta = \ln(z)$$
$$f_6(z) = r^\alpha e^{i\alpha\theta} = z^\alpha$$

You can puzzle these out from algebra alone. If $\bar{z}$ vanishes from the function, the function is likely analytic.

3.4 Cauchy-Riemann Conditions

Complex functions with no explicit $\bar{z}$-dependence follow an analogy from vector calculus. Start with $\partial w/\partial \bar{z} = 0$, and take the derivative of

$$f(x,y) = u(x,y) + iv(x,y)$$

using

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)$$

to write the *Cauchy-Riemann conditions*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Level Curves

For a complex function

$$f(x,y) = u(x,y) + iv(x,y)$$

obeying the Cauchy-Riemann conditions, note that for two constants c_1 and c_2, the level curves

$$u(x,y) = c_1$$
$$v(x,y) = c_2$$

are orthogonal, as

$$\begin{aligned}\nabla u \cdot \nabla v &= \left(\frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y}\right) \cdot \left(\frac{\partial v}{\partial x}\hat{x} + \frac{\partial v}{\partial y}\hat{y}\right) \\ &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0\,.\end{aligned}$$

Harmonic Functions

For a complex function

$$f(x,y) = u(x,y) + iv(x,y)$$

obeying the Cauchy-Riemann conditions, the Laplacian always vanishes, as

$$\begin{aligned}\nabla^2 f &= \nabla \cdot \nabla f(x,y) \\ &= \frac{\partial^2 u}{\partial_{xx}} + \frac{\partial^2 v}{\partial_{xx}} + \frac{\partial^2 u}{\partial_{yy}} + \frac{\partial^2 v}{\partial_{yy}} \\ &= \frac{\partial^2 v}{\partial_{xy}} - \frac{\partial^2 u}{\partial_{xy}} - \frac{\partial^2 v}{\partial_{xy}} + \frac{\partial^2 u}{\partial_{xy}} = 0\,,\end{aligned}$$

or more strongly,

$$\nabla^2 u(x,y) = 0$$
$$\nabla^2 v(x,y) = 0 \,.$$

Conventionally, $u(x,y)$ and $v(x,y)$ that satisfy the above are called *harmonic* functions.

Connection to Electromagnetism

In two dimensions, consider two real vector fields $\vec{A}$ and $\vec{B}$ defined in terms of harmonic functions $u(x,y)$, $v(x,y)$:

$$\vec{A} = u\,\hat{x} - v\,\hat{y}$$
$$\vec{B} = v\,\hat{x} + u\,\hat{y}$$

Imposing the Cauchy-Riemann equations onto $\vec{A}$ and $\vec{B}$, we see that the divergence and curl of each field resemble Maxwell's equations in charge-free two-dimensional space:

$$\vec{\nabla}\cdot\vec{A} = 0$$
$$\vec{\nabla}\times\vec{A} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = 0$$

Approaching this differently, it turns out that two problems from electromagnetism are solved by the complex function:

$$f(z) = \frac{1}{z - z_0}$$

Letting

$$x - x_0 = \rho\cos\theta$$
$$y - y_0 = \rho\sin\theta\,,$$

$f(z)$ my be written

$$\begin{aligned} f(z) &= \frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2} \\ &\quad - i\frac{y-y_0}{(x-x_0)^2 + (y-y_0)^2} \\ &= \frac{1}{\rho}(\cos\theta - i\sin\theta) \\ &= u + iv\,, \end{aligned}$$

where

$$u = \rho^{-1}\cos\theta$$
$$v = -\rho^{-1}\sin\theta\,.$$

Then, the fields $\vec{A} = u\,\hat{x} - v\,\hat{y}$ and $\vec{B} = v\,\hat{x} + u\,\hat{y}$ respectively tell us

$$\vec{A} = \frac{\cos\theta\,\hat{x} + \sin\theta\,\hat{y}}{\rho} = \frac{\hat{r}}{\rho}$$
$$\vec{B} = \frac{-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}}{\rho} = \frac{\hat{\theta}}{\rho}\,.$$

Explicitly, $\vec{A}$ is proportional to the electric field vector due to a line of charge, whereas $\vec{B}$ is proportional to the magnetic field vector due to a line of current.

4 Contour Integrals

Now we develop the notion of integration in the complex plane. Consider a contour C that begins and ends at the respective points z_a and z_b in the complex plane. The integral of a function $f(z)$ over C can be recast using a real parameter t via the chain rule:

$$\int_C f(z)\,dz = \int_{t_a}^{t_b} f(z(t))\,\frac{dz(t)}{dt}\,dt$$

Substituting

$$f(z) = u(x,y) + iv(x,y)$$

and using prime notation for derivatives via

$$z' = x'(t) + iy'(t)\ ,$$

the contour integral splits into real and imaginary parts:

$$\int_C f(z)\,dz = \int_C (ux' - vy')\,dt + i\int_C (vx' + uy')\,dt$$

Re-using the notation $\vec{A} = u\hat{x} - v\hat{y}$, $\vec{B} = v\hat{x} + u\hat{y}$, write the integral in vector notation

$$\int_C f(z)\ dz = \int_C \vec{A}\cdot d\vec{l} + i\int_C \vec{B}\cdot d\vec{l}\,,$$

where $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$. Evidently, the the integral of a function $f(z)$ in the complex plane decomposes into a pair of integrals involving the fields $\vec{A}$, $\vec{B}$.

4.1 Cauchy's Integral Theorem

Starting with the result above, we may consider closed contours C and apply Stokes's theorem to transform each line integral into an area integral in the complex plane. Denoting the the off-plane direction $\hat{k}$, we have

$$\begin{aligned} \oint_C f(z)\ dz &= \int_\Omega \hat{k}\cdot\left(\vec{\nabla}\times\vec{A}\right)\ dx\,dy \\ &\quad + i\int_\Omega \hat{k}\cdot\left(\vec{\nabla}\times\vec{B}\right)\ dx\,dy\,, \end{aligned}$$

which, by the rules of vector calculus, resolves to *zero* when region the Ω is completely enclosed by C.

This is the essence (and the proof) of the Cauchy Integral Theorem, formally stating that if a function $f(z)$ is analytic in a simply-connected region R, then the integral along a closed path C in R equals zero.

4.2 Defects

Singular points in the integration region that cause $f(z)$ to become non-analytic must be 'stepped around' to be excluded from the contour C.

Consider the integral

$$I_n^{(0)} = \oint_C \frac{dz}{(z - z_0)^n} ,$$

where n is an integer and z_0 is a singular point interior to C. Since the integration contour may be arbitrarily shaped, we may choose a unit circular path around the point z_0, running counter-clockwise by convention, with

$$\begin{aligned} z(\theta) &= z_0 + e^{i\theta} \\ z'(\theta) &= ie^{i\theta} . \end{aligned}$$

Substituting $z(\theta)$ into the above and using the delta function

$$\delta(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha t} \, d\alpha ,$$

the above integral becomes:

$$I_n^{(0)} = 2\pi i \, \delta(n-1) = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

4.3 Cauchy Integral Formula

Generalizing the analysis of defects, we consider the integral

$$I_n = \oint_C \frac{f(z) \, dz}{(z - z_0)^n}$$

about a circular path of arbitrarily-small radius

$$z(\theta) = \lim_{r \to 0} z_0 + re^{i\theta} .$$

The act of taking $r \to 0$ is equivalent to expanding $f(z_0)$ by Taylor series to discard high-order terms, provided that derivatives of $f(z)$ exist. Using the expansion

$$f(z) = \sum_{q=0}^{\infty} \frac{f^{(q)}(z_0)}{q!} (z - z_0)^q ,$$

the above simplifies to

$$I_n = \sum_{q=0}^{\infty} \frac{f^{(q)}(z_0)}{q!} I_{n-q}^{(0)} .$$

The term $I_{n-q}^{(0)}$ contains a delta function in the quantity $n - q - 1$, which nukes all terms in the sum except the one satisfying $q = n - 1$, and thus

$$I_n = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0) .$$

Rewriting the integral gives us the (very important) Cauchy integral formula:

$$\oint_C \frac{f(z) \, dz}{(z - z_0)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$$

Speaking to the pedantic reader, it's possible to prove that the existence of all derivatives $f^{(n)}(z)$ is guaranteed by the Cauchy integral formula.

4.4 Analytic Continuation

Taylor series is convergent until the contour C touches a singular point z_0, where the *radius of convergence* corresponds to the largest contour C_0. In a process called *analytic continuation*, we may choose a point z_1 inside C_0 where Taylor expansion is valid, implying a new contour C_1 centered on z_1 with its own radius of convergence, in which another Taylor expansion for $f(z)$ applies. The non-overlapping part of C_1 is new 'territory' that the z_0-centered approximation doesn't cover. Iterating this process, we may cover the whole complex plane, as long as singular points (and regions) are stepped around.

It readily follows that a closed contour integral that encloses singularities is equal to the sum of elementary integrals around the singularities. If the region of analyticity is a multiply-connected surface due to singularities, $f(z)$ may be multi-valued.

4.5 Laurent Series

A generalization of the Taylor series that includes both positive and negative exponents is the *Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Bringing the Cauchy integral formula

$$I_n = \oint_C \frac{f(z) \; dz}{(z - z_0)^n}$$

into the mix, we find, by direct substitution:

$$\begin{aligned} I_n &= \oint_C \sum_{m=-\infty}^{\infty} \frac{a_m \left(z - z_0\right)^m dz}{\left(z - z_0\right)^n} \\ &= \sum_{m=-\infty}^{\infty} a_m \oint_C \frac{dz}{\left(z - z_0\right)^{n-m}} \end{aligned}$$

The remaining integral is simply $I_{n-m}^{(0)}$, so we have

$$\begin{aligned} I_n &= \sum_{m=-\infty}^{\infty} a_m I_{n-m}^{(0)} \\ &= \sum_{m=-\infty}^{\infty} a_m 2\pi i \, \delta\left(n - m - 1\right) , \end{aligned}$$

so the surviving term satisfies $m = n - 1$. Simplifying quickly gives $I_n = a_{n-1} 2\pi i$, or $I_{n+1} = a_n 2\pi i$.

Solving for a_n, we find a formula for the Laurent series coefficients:

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_\Gamma \frac{f\left(z\right) dz}{\left(z - z_0\right)^{n+1}} \\ n &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Note that Γ is a contour topologically equivalent to C_0.

Example: Annulus

Consider a function $f\left(z\right)$ that is analytic in the annulus

$$R_1 \leq \left|z - z_0\right| \leq R_0$$

centered on z_0. Begin by writing the $n = 0$ case of the Cauchy integral formula

$$2\pi i \, f\left(z\right) = \oint_C \frac{f\left(\tilde{z}\right) d\tilde{z}}{\tilde{z} - z} ,$$

where the function $f\left(z\right)$ is approximated by Laurent series, and contour C encloses z.

Next, stretch C so as to wrap the inside of the annulus, with a tight 'bridge' of canceling paths that connect the enclosing radii. The resulting contours are C_1 and C_0 with opposing directions of integration. The contour integral along C_0 corresponding to R_0 was solved previously and generates the $n \geq 0$ terms:

$$a_{n\geq 0} = \frac{1}{2\pi i} \oint_{C_0} \frac{f\left(z\right) dz}{\left(z - z_0\right)^{n+1}}$$

Along contour C_1, the fraction $1/\left(\tilde{z} - z\right)$ may be expanded via geometric series

$$\begin{aligned} \frac{1}{\tilde{z} - z} &= \frac{1}{\tilde{z} - z_0 + z_0 - z} \\ &= \frac{1}{z_0 - z} \frac{1}{1 + \frac{\tilde{z} - z_0}{z_0 - z}} = \sum_{m=0}^{\infty} \frac{\left(z_0 - \tilde{z}\right)^m}{\left(z_0 - z\right)^{m+1}} , \end{aligned}$$

which guarantees convergence as

$$\left|z - z_0\right| > \left|\tilde{z} - z_0\right| = R_1 .$$

To discover the restriction on a_n along C_1, replace $\tilde{z} - z$ and $f\left(\tilde{z}\right)$ in the integral formula as follows:

$$\begin{aligned} 2\pi i \, f\left(z\right) &= \oint_{C_1} \frac{f\left(\tilde{z}\right) \, d\tilde{z}}{\tilde{z} - z} \\ &= \oint_{C_1} \sum_{n=-\infty}^{\infty} a_n \left(\tilde{z} - z_0\right)^n \\ &\quad \times \sum_{m=0}^{\infty} \frac{\left(z_0 - \tilde{z}\right)^m}{\left(z_0 - z\right)^{m+1}} \, d\tilde{z} \end{aligned}$$

Keep condensing terms to write

$$\begin{aligned} 2\pi i \, f\left(z\right) = &\sum_{n=-\infty}^{\infty} a_n \sum_{m=0}^{\infty} \left(z - z_0\right)^{-(m+1)} \\ &\times \oint_{C_1} \left(\tilde{z} - z_0\right)^{m+n} \, d\tilde{z} , \end{aligned}$$

and further

$$\begin{aligned} \cancel{2\pi i} \, f\left(z\right) = \cancel{2\pi i} &\sum_{n=-\infty}^{\infty} a_n \\ &\times \sum_{m=0}^{\infty} \left(z - z_0\right)^{-(m+1)} \delta\left(-\left(m + n\right) - 1\right) . \end{aligned}$$

The delta functions tells us $m + n = -1$, and we find

$$f\left(z\right) = \sum_{n=-1}^{-\infty} a_n \left(z - z_0\right)^n .$$

Evidently, only the negative n-terms have survived on contour C_1. We conclude that

$$a_{n<0} = \frac{1}{2\pi i} \oint_{C_1} \frac{f\left(z\right) dz}{\left(z - z_0\right)^{n+1}} .$$

5 Residue Calculus

Our study of contour integrals in the complex plane has yielded several useful results. First, the Cauchy integral theorem tells us that an analytic function $f\left(z\right)$ integrated along a closed contour C free of singularities always resolves to zero.

Isolated singularities (poles) $z_0^{(p)}$ are handled by expanding $f\left(z\right)$ as a Laurent series

$$f\left(z\right) = \sum_{n=-\infty}^{\infty} a_n^{(p)} \left(z - z_0^{(p)}\right)^n ,$$

where the coefficients a_n were found to be

$$a_n^{(p)} = \frac{1}{2\pi i} \oint_{C_0^{(p)}} \frac{f(z)\ dz}{\left(z - z_0^{(p)}\right)^{n+1}}$$
$$n = 0, \pm 1, \pm 2, \dots\ .$$

Take the special case $n = -1$ to write

$$2\pi i\, a_{-1}^{(p)} = \oint_{C_0^{(p)}} f(z)\, dz\ ,$$

telling us that the integral of $f(z)$ around a pole is equal to the constant $2\pi i\, a_{-1}^{(p)}$.

This process may repeat for each pole inside the contour C, resulting in the sum

$$2\pi i \sum_p a_{-1}^{(p)} = \oint_C f(z)\, dz\ .$$

This amazing result says that solving contour integrals is reduced to finding the Laurent coefficients $a_{-1}^{(p)}$ at each pole $z_0^{(p)}$. The coefficient $a_{-1}^{(p)}$ is called the *residue* at $z_0^{(p)}$:

$$2\pi i \sum_p \text{Res}\left[f\left(z_0^{(p)}\right)\right] = \oint_C f(z)\, dz$$

Calculating Residue(s)

The *order* of a pole $z_0^{(p)}$ is the lowest (most negative) index of the Laurent series expansion of $f(z)$ around the pole. A *simple* pole has a lowest index of -1. In general, a pole z_0 of order m has corresponding Laurent series

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n\ .$$

Now we introduce the function $g(z)$ such that

$$g(z) = (z - z_0)^m f(z) = \sum_{q=0}^{\infty} a_{q-m} (z - z_0)^q\ ,$$

which bumps z_0 to the numerator. Since the sum index begins at zero, $g(z)$ is simply a Taylor series, meaning

$$a_{q-m} = \frac{g^{(q)}(z_0)}{(q)!}\ ,$$

where the case $q - m = -1$ gives the residue of $f(z)$ at z_0:

$$\text{Res}\left[f(z_0)\right] = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

For functions containing only simple poles, the above reduces to

$$\text{Res}\left[f(z_0)\right] = g(z_0)\ .$$

What we see is an integral on the left, and a simple function evaluation on the right. In practice, this means that whole families of integrals can be cheated by choosing an integration contour that makes the residue easy to calculate.

5.1 Ratios

For functions of the form

$$f(z) = \frac{p(z)}{q(z)}\ ,$$

containing a simple pole in the denominator, i.e. $q(z_0) = 0$, it's simple to show that the residue calculation always resolves to

$$\text{Res}\left[f(z_0)\right] = \frac{p(z_0)}{q'(z_0)}\ .$$

To prove this, write the definition of $q'(z)$ and simplify to produce

$$\lim_{z \to z_0} \frac{z - z_0}{q(z)} = \frac{1}{q'(z)}\ ,$$

and then eliminate $z - z_0$ using

$$g(z) = (z - z_0) f(z) = (z - z_0) \frac{p(z)}{q(z)}$$

to get

$$\lim_{z \to z_0} g(z) = \frac{p(z)}{q'(z)} = \text{Res}\left[f(z_0)\right]\ .$$

5.2 Polynomial Functions

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

whose domain is the real number line. If we connect $x = \infty$ to $x = -\infty$ with a (counterclockwise) semicircular arc, the resulting contour encloses the upper half of the complex plane.

Factoring the denominator to clearly see the singular points, the integral becomes

$$I = \oint_C \frac{dz}{(z - i)(z + i)}\ ,$$

which encloses one simple pole $z_0 = i$. The pole at $z = -i$ is outside the integration contour and is ignored.

Proceed by writing

$$g(z) = (z-i) f(z)$$

and quickly find

$$\text{Res}[f(i)] = \frac{1}{2i},$$

and the integral resolves to

$$I = 2\pi i \,\text{Res}[f(i)] = \pi$$

Example 1
Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

Contour contains two poles.

$$I = \oint_C \frac{dz}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$g_1(z) = \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z+i)(z^2+4)}$$

$$g_2(z) = \frac{\cancel{(z-2i)}}{(z^2+1)\cancel{(z-2i)}(z+2i)}$$

$$I = 2\pi i (g_1(i) + g_2(2i)) = \frac{\pi}{6}$$

(Or use partial fractions.)

Example 2
Evaluate:

$$I = \int_0^{\infty} \frac{x^2\, dx}{(x^2+4)(x^2+9)}$$

Contour contains two poles.

$$2I = \oint_C \frac{z^2\, dz}{(z+2i)(z-2i)(z+3i)(z-3i)}$$

$$g_1(z) = \frac{z^2\cancel{(z-2i)}}{(z+2i)\cancel{(z-2i)}(z^2+9)}$$

$$g_2(z) = \frac{z^2\cancel{(z-3i)}}{(z^2+4)(z+3i)\cancel{(z-3i)}}$$

$$I = \frac{2\pi i}{2}(g_1(2i) + g_2(3i)) = \frac{\pi}{10}$$

Example 3
Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$$

Contour contains one order-two pole.

$$I = \oint_C \frac{dz}{((z+i)(z-i))^2}$$

$$g(z) = \frac{\cancel{(z-i)^2}}{((z+i)\cancel{(z-i)})^2}$$

$$I = 2\pi i \left(\frac{d}{dz} g(z) \bigg|_{z=i} \right) = \frac{\pi}{2}$$

Example 4
Evaluate:

$$I = \int_0^{\infty} \frac{dx}{(4x^2+1)^3}$$

Contour contains one order-3 pole.

$$2I = \oint_C \frac{dz}{4^3 \left(z+\frac{i}{2}\right)^3 \left(z-\frac{i}{2}\right)^3}$$

$$g(z) = \frac{\cancel{\left(z-\frac{i}{2}\right)^3}}{4^3 \left(z+\frac{i}{2}\right)^3 \cancel{\left(z-\frac{i}{2}\right)^3}}$$

$$I = \frac{2\pi i}{2} \left(\frac{1}{2} \frac{d^2}{dz^2}\, g(z) \bigg|_{z=i/2} \right) = \frac{3\pi}{32}$$

5.3 Jordan's Lemma

Starting with the Fourier transform integral

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx,$$

we carry the problem to the complex plane under three assumptions: (i) $k > 0$ and is a real number, (ii) $f(z)$ is analytic in the upper-half plane with the exception of simple poles, (iii) $\lim_{|z|\to\infty} f(z) = 0$.

Jordan's lemma for Fourier transform states that the integration path can be closed by an infinite semicircle in the upper-half plane. For th $k < 0$ case, the path would enclose the lower half-plane.

5.4 Sine and Cosine in Polynomial

For a real variable $a > 0$, the integral

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx \\ &= \text{Re} \int_{-\infty}^{\infty} \frac{e^{ikx}\, dx}{x^2+a^2} = \oint_C \frac{e^{ikz}\, dz}{z^2+a^2} \end{aligned}$$

is easily recast as a contour integral using Jordan's lemma.

Using $g(z) = e^{ikz}/(z+ia)$, we quickly find $I = (\pi/a)\, e^{-ka}$. Generalizing this, one finds:

$$\int f(x)\cos(kx)\ dx = \mathrm{Re}\int f(x)\, e^{ikx}\ dx$$
$$\int f(x)\sin(kx)\ dx = \mathrm{Im}\int f(x)\, e^{ikx}\ dx$$

Example 5
Evaluate:

$$I = \int_0^\infty \frac{\cos 2x}{9x^2+4}\ dx$$

Contour contains one pole.

$$I = \frac{1}{18}\ \mathrm{Re}\oint_C \frac{e^{2iz}\ dz}{\left(z+\frac{2i}{3}\right)\left(z-\frac{2i}{3}\right)}$$

$$g(z) = \frac{e^{2iz}}{z+\frac{2i}{3}}$$

$$g(2i/3) = \frac{3}{4i}\ e^{-4/3}$$

$$I = \frac{2\pi i}{18}\frac{3}{4i}\ e^{-4/3} = \frac{\pi}{12}\ e^{-4/3}$$

Example 6
Evaluate:

$$I = \int_0^\infty \frac{\cos 2x}{\left(9x^2+4\right)^2}\ dx$$

Contour contains one order-two pole.

$$I = \frac{1}{2\cdot 9^2}\ \mathrm{Re}\oint_C \frac{e^{2iz}\ dz}{\left(z+\frac{2i}{3}\right)^2\left(z-\frac{2i}{3}\right)^2}$$

$$g(z) = \frac{e^{2iz}\cancel{\left(z-\frac{2i}{3}\right)^2}}{\left(z+\frac{2i}{3}\right)^2\cancel{\left(z-\frac{2i}{3}\right)^2}}$$

$$g^{(1)}(2i/3) = -i\left(\frac{14}{3}\cdot\frac{3^3}{4^3}\right)e^{-4/3}$$

$$I = 2\pi i\left(g^{(1)}(2i/3)\right) = \frac{7\pi}{288}\ e^{-4/3}$$

Example 7
Evaluate:

$$I = \int_{-\infty}^\infty \frac{\cos kx}{\left(x^2+a^2\right)^2}\ dx$$

Contour contains one order-two pole.

$$I = \mathrm{Re}\oint_C \frac{e^{ikz}\ dz}{\left(z^2+a^2\right)^2}$$

$$g(z) = \frac{e^{ikz}}{\left(z+ia\right)^2}$$

$$g^{(1)}(ia) = \frac{i}{4\,e^{ka}}\frac{ka+1}{a^3}$$

$$I = \frac{\pi}{2\,e^{ka}}\frac{ka+1}{a^3}$$

Example 8
Evaluate:

$$I = \int_{-\infty}^\infty \frac{x\sin kx}{x^2+a^2}\ dx$$

Contour contains one pole.

$$I = \mathrm{Im}\oint_C \frac{z\,e^{ikz}\ dz}{z^2+a^2}$$

$$g(z) = \frac{z\,e^{ikz}}{z+ia}$$

$$g(ia) = \frac{1}{2}e^{-ka}$$

$$I = \mathrm{Im}\ \frac{2\pi i}{2\,e^{ka}} = \frac{\pi}{e^{ka}}$$

Example 9
Evaluate:

$$I = \int_{-\infty}^\infty \frac{x\sin kx}{\left(x^2+a^2\right)^2}\ dx$$

Exploit a previous example to ease calculations.

$$I = -\frac{\partial}{\partial k}\int_{-\infty}^\infty \frac{\cos kx}{\left(x^2+a^2\right)^2}\ dx$$
$$= -\frac{\partial}{\partial k}\left(\frac{\pi}{2\,e^{ka}}\frac{ka+1}{a^3}\right) = \frac{\pi k}{2a\,e^{ka}}$$

or, by standard means:

$$I = \mathrm{Im}\oint_C \frac{z\,e^{ikz}\ dz}{\left(z^2+a^2\right)^2} = \mathrm{Im}\ \frac{2\pi i k}{4a\,e^{ka}} = \frac{\pi k}{2a\,e^{ka}}$$

Example 10
Evaluate:

$$I = \int_{-\infty}^\infty \frac{x\sin x}{x^2+4x+5}\ dx$$

Contour contains one pole.

$$I = \mathrm{Im}\oint_C \frac{z\,e^{ikz}\ dz}{(z+2+i)(z+2-i)}$$

$$g(z) = \frac{z\,e^{ikz}\cancel{(z+2-i)}}{(z+2+i)\cancel{(z+2-i)}}$$

$$g(-2+i) = \frac{(-2+i)\ e^{i(-2+i)}}{2i}$$

$$I = \frac{\pi}{e}\left(2\sin 2+\cos 2\right)$$

5.5 Trigonometric Functions

Integrals of the form

$$I = \int_0^{2\pi} f(\cos(\theta), \sin(\theta))\, d\theta$$

can be recast by expressing all θ-terms in terms of z and choosing an integration contour C_0 as the unit circle centered on $z = 0$. Using the equations of complex trigonometry, and also noting that $dz = ire^{i\theta}\, d\theta$, the above becomes

$$I = -i \oint_{C_0} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{z} .$$

Example 11

Evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$$

Contour contains one pole.

$$\begin{aligned} I &= -i \oint_{C_0} \frac{dz}{z} \frac{1}{13 + (5/2i)(z - \bar{z})} \\ &= \oint_{C_0} \frac{dz}{\frac{5}{2}\left(z + \frac{i}{5}\right)(z + 5i)} \end{aligned}$$

$$g(z) = \frac{\cancel{\left(z + \frac{i}{5}\right)}}{\frac{5}{2}\cancel{\left(z + \frac{i}{5}\right)}(z + 5i)}$$

$$g\left(\frac{-i}{5}\right) = \frac{-i}{12}$$

$$I = 2\pi i \left(\frac{-i}{12}\right) = \frac{\pi}{6}$$

Example 12

Evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$$

Contour contains one pole.

$$\begin{aligned} I &= -i \oint_{C_0} \frac{dz}{z} \frac{1}{5 + 2i(z - \bar{z})} \\ &= -\oint_{C_0} \frac{dz}{2\left(z - \frac{i}{2}\right)(z - 2i)} \end{aligned}$$

$$g(z) = \frac{-1 \cdot \cancel{\left(z - \frac{i}{2}\right)}}{\cancel{\left(z - \frac{i}{2}\right)} 2(z - 2i)}$$

$$g\left(\frac{i}{2}\right) = \frac{-i}{3}$$

$$I = 2\pi i \left(\frac{-i}{3}\right) = \frac{2\pi}{3}$$

Example 13

For $a > |b|$, evaluate:

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

Contour contains one pole.

$$\begin{aligned} I &= -i \oint_{C_0} \frac{2\, dz}{2az + b(1 + z^2)} \\ &= -i \oint_{C_0} \frac{2\, dz}{b\left(z + \frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}\right)\left(z + \frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}\right)} \end{aligned}$$

$$z_0 = -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}$$

$$g(z) = \frac{2\cancel{(z - z_0)}}{\cancel{b\left(z + \frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}\right)}\left(z + \frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b}\right)}$$

$$g(z_0) = \frac{1}{\sqrt{a^2 - b^2}}$$

$$I = 2\pi i \left(\frac{-i}{\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Example 14

For $a > 1$, evaluate:

$$I = \int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2}$$

Contour contains one order-two pole.

$$\begin{aligned} I &= -i \oint_{C_0} \frac{2z\, dz}{(2az + 1 + z^2)^2} \\ &= -i \oint_{C_0} \frac{2z\, dz}{\left(z + a - \sqrt{a^2 - 1}\right)^2 \left(z + a + \sqrt{a^2 - 1}\right)^2} \end{aligned}$$

$$z_0^+ = -a + \sqrt{a^2 - 1}$$

$$z_0^- = -a - \sqrt{a^2 - 1}$$

$$g(z) = \frac{2z\cancel{\left(z - z_0^+\right)^2}}{\cancel{\left(z - z_0^+\right)^2}\left(z - z_0^-\right)^2}$$

$$g^{(1)}(z) = \frac{-2\left(z + z_0^-\right)}{\left(z - z_0^-\right)^3}$$

$$g^{(1)}\left(z_0^+\right) = \frac{a}{2\left(a^2 - 1\right)^{3/2}}$$

$$I = 2\pi i \left(\frac{-ia}{2\left(a^2 - 1\right)^{3/2}}\right) = \frac{\pi a}{\left(a^2 - 1\right)^{3/2}}$$

5.6 Two-Contour Trick

The infinite complex plane (or a fraction of it) need not be enclosed by a semicircular contour. Rectangles are just as valid, which are an ideal application of the *two-contour trick*. This entails noticing when the integral of $f(z)$ on two enclosing contours C_1 and C_2 is the same up to a complex factor.

To illustrate, the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}\, dx}{1+e^x}$$

with $0 > a > 1$ may be rewritten

$$I = \int_{C_1} \frac{e^{az}\, dz}{1+e^z} ,$$

where contour C_1 is the real number line. Next, introduce a second contour C_2 that is shifted upward into the imaginary numbers but still parallel to the real line such that

$$z = x + 2\pi i .$$

Integrating 'backwards' along C_2, we have

$$-I\, e^{2i\pi a} = \int_{C_2} \frac{e^{az}\, dz}{1+e^z} .$$

Of course, any contributions to the integral at $x = \pm\infty$ are zero, so we combine C_1 and C_2 to close the integration contour:

$$I\left(1-e^{2i\pi a}\right) = \oint_C \frac{e^{az}\, dz}{1+e^z}$$

To finish the calculation above, we note the pole at $z_0 = i\pi$, and then

$$g(z) = \frac{(z-i\pi)\, e^{az}}{1+e^z} .$$

However, $g(z_0)$ leads to $0/0$, thus L'hopital's rule is needed, leading to

$$g(z_0) = \frac{e^{i\pi a}}{e^{i\pi}} .$$

Finally,

$$I\left(1-e^{2i\pi a}\right) = 2\pi i \frac{e^{i\pi a}}{e^{i\pi}} ,$$

and

$$I = \frac{\pi}{\sin(\pi a)} .$$

Example 15

Use a pizza slice contour bounded by the positive real line and $z = r\, e^{2\pi i/n}$ (with vanishing crust at infinity) to evaluate

$$I = \int_0^{\infty} \frac{dx}{1+x^n} .$$

Contour contains one pole.

$$I\left(1-e^{2\pi i/n}\right) = \oint_C \frac{dz}{1+z^n}$$

$$z_0^n = -1 \to z_0 = (-1)^{-n} = e^{i\pi/n}$$

$$g(z) = \frac{\left(z-e^{i\pi/n}\right)}{1+z^n}$$

$$g(z_0) \propto \frac{0}{0} \to \text{Need L'hopital.}$$

$$g(z_0) = -\frac{e^{i\pi/n}}{n}$$

$$I\left(1-e^{2\pi i/n}\right) = -2\pi i \frac{e^{i\pi/n}}{n}$$

$$I = \frac{\pi/n}{\sin(\pi/n)}$$

5.7 Regularization

Principal Value

Consider the *principal value* integral

$$I = \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\; dx}{x-x_0} ,$$

where by shifting $x \to z$, we assume that $f(z)$ is analytic except for a finite number of poles, and that $|f| \to 0$ on the upper (or lower) infinite semicircle in the complex plane.

Since the pole x_0 lies on the real axis, the integration contour cuts directly through x_0. This is handled by *regularization* of the denominator, which entails introducing a small factor $\delta > 0$ as

$$\begin{aligned} I &= \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\; dx}{x-x_0} \\ &= \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{(x-x_0)\, f(x)\; dx}{(x-x_0)^2+\delta^2} \\ &= \lim_{\delta\to 0} \oint_C \frac{(z-z_0)\, f(z)\; dz}{(z-x_0)^2+\delta^2} . \end{aligned}$$

After a little complex algebra, find

$$\begin{aligned} I = &\lim_{\delta\to 0} \oint_C \frac{f(z)\; dz}{z-x_0+i\delta} \\ &+ \lim_{\delta\to 0} \oint_C i\delta \frac{f(z)\; dz}{(z-x_0-i\delta)(z-x_0+i\delta)} , \end{aligned}$$

which indicates one simple pole $z_0 = x_0 + i\delta$ inside the upper-half plane. The first integral in fact *excludes* the pole, so x_0 is skipped in subsequent residue calculations. (Use the δ-term as a reminder to skip x_0.) The second integral is solved by standard residue calculus, i.e., let $g(z) = f(z)/(z - x_0 + i\delta)$, resulting in $\pi i f(x_0)$.

Pulling the results together, we write

$$I^+ = \pi i f(x_0) + \lim_{\delta\to 0} \oint_C \frac{f(z)\ dz}{z - x_0 + i\delta}\ ,$$

where if we started with $\delta < 0$ instead, the integration contour would flip to the lower-half plane, resulting in

$$I^- = -\pi i f(x_0) + \lim_{\delta\to 0} \oint_C \frac{f(z)\ dz}{z - x_0 - i\delta}\ .$$

In tighter notation (regardless of path or the sign of δ), one may write

$$I = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)\ dx}{x - x_0} = \mathrm{P} \oint_C \frac{f(z)\ dz}{z - x_0}\ ,$$

reminding us to *include* x_0 inside integration contour, but take the residue with a factor of $1/2$.

Example 16

Evaluate:

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\ dx$$

This is a straightforward principal value integral:

$$I = \mathrm{Im}\left(\pi i\, e^{i\cdot 0} + \cancel{\lim_{\delta\to 0} \oint_C \frac{e^{iz}\ dz}{z + i\delta}}\right) = \pi$$

Example 17

Evaluate each of:

$$I = \mathrm{P} \int_{-\infty}^{\infty} \frac{\cos kx}{(x - x_0)(x^2 + 2)}\ dx$$

$$J = \mathrm{P} \int_{-\infty}^{\infty} \frac{\sin kx}{(x - x_0)(x^2 + 2)}\ dx$$

$$\begin{aligned}
K &= \mathrm{P} \oint_C \frac{e^{ikz}}{(z - x_0)(z^2 + 2)}\ dz \\
&= i\pi \frac{e^{ikx_0}}{x_0^2 + 2} + 2\pi i \left(\frac{e^{-k\sqrt{2}}}{(\sqrt{2}i - x_0)(2\sqrt{2}i)}\right) \\
&= \left(\frac{-\pi}{x_0^2 + 2}\left(\sin kx_0 + \frac{x_0\ e^{-\sqrt{2}k}}{\sqrt{2}}\right)\right) \\
&\quad + i\left(\frac{\pi}{x_0^2 + 2}\left(\cos kx_0 - e^{-\sqrt{2}k}\right)\right) \\
&= I + iJ
\end{aligned}$$

Dispersion Relations

One special case for $f(z)$ occurs when the upper-half plane contains no singularities, making the contour integral the I^+-equation resolve to zero. By decomposing f into real and imaginary components u and v, respectively, we derive the *Kramers-Kroing* relations:

$$u(x_0) = \frac{1}{\pi}\ \mathrm{P} \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}\ dx$$

$$v(x_0) = \frac{-1}{\pi}\ \mathrm{P} \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}\ dx$$

5.8 Branch Cuts

Non-Integer Powers and Logarithms

Complex numbers involving exponents and logarithms follow plainly from Euler's formula:

$$\begin{aligned} z^a &= r^a\ e^{ai\theta} \\ \ln z &= \ln r + i\theta \end{aligned}$$

Of course, the periodicity of θ leads to certain functions behaving non-smoothly as the line defined by $\theta = 0$ is crossed. For instance, the value of the logarithm

$$\begin{aligned} \ln z(r, 0) &= \ln r \\ \ln z(r, 2\pi) &= \ln r + 2\pi i \end{aligned}$$

at two equal points in the complex plane can disagree with itself by (at least) $2\pi i$.

For another example, the square root $z^{1/2} = r^{1/2}\ e^{i\theta/2}$ is also multi-valued, as

$$\begin{aligned} z^{1/2}(r, 0) &= r^{1/2} \\ z^{1/2}(r, 2\pi) &= -r^{1/2}\ . \end{aligned}$$

All of this is troublesome for contour integrals, so the line on which $f(z)$ is ill-behaved, called a *branch cut*, must be stepped around. To proceed generally we denote an initial phase θ_0 that defines a branch cut $z = r\ e^{i\theta_0}$, and then define

$$\theta_0 + 2\pi N \le \theta < \theta_0 + 2\pi(N + 1)$$

for an integer N, called the *branch*, that indexes multiples of 2π.

Example 18

Use

$$\bar{z}\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)$$

to show that the complex power and logarithm functions are analytic everywhere except for the branch θ_0.

$$\bar{z}\frac{\partial}{\partial\bar{z}}\left(z^a\right) = \frac{a}{2}\,e^{ia\theta}\left(r^a - r^a\right) = 0$$
$$\bar{z}\frac{\partial}{\partial\bar{z}}\left(\ln z\right) = \left(\frac{r}{r} + i^2\right) = 0$$

Products with Powers

We next consider the integral

$$I = \int_0^\infty f\left(x\right) x^a\,dx\;,$$

where $f\left(x\right)$ is well-behaved and non-singular on the real line, and the presence of x^a demands a branch but on $\theta_0 = 0$.

To proceed we move the integral to the complex plane and go along three contours: (i) C_+, corresponding to $z = x + i\delta$ just above the real line, (ii) C_R, a nearly-full trip around the complex plane with $R \to \infty$, and (iii) C_-, coming from infinity back zero just below the real line with $z = x - i\delta$.

Only the $C_\pm$ contours contribute to the integral, so in the limit $\delta \to 0$ we have:

$$I = \int_{C_+} f\left(z\right) z^a\,dz$$
$$-e^{2i\pi a}\,I = \int_{C_-} f\left(z\right) z^a\,dz$$

Solving for I, the final answer pops out:

$$\int_0^\infty f\left(x\right) x^a\,dx = \frac{1}{1-e^{2i\pi a}}\oint_C f\left(z\right) z^a\,dz$$

Note that the integration contour surrounds the whole complex plane minus the branch cut. Don't forget to include all poles in residue calculations.

Example 19

Evaluate:

$$I = \int_0^\infty \frac{x^a\,dx}{\left(1+x\right)^2}$$

Contour contains one order-two pole.

$$\oint_C \frac{z^a\,dz}{\left(1+z\right)^2} = 2\pi i\sum_p \operatorname{Res}\left[f\left(z_0^{(p)}\right)\right]$$

$$g\left(z\right) = \frac{z^a\cancel{\left(1+z\right)^2}}{\cancel{\left(1+z\right)^2}}$$
$$g^{(1)}\left(z\right) = a\,z^{a-1}$$
$$g^{(-1)}\left(z\right) = a\,1^{a-1}\,e^{i\pi a}\,e^{-i\pi} = -a\,e^{i\pi a}$$
$$I = \frac{-2\pi i a\,e^{i\pi a}}{1-e^{2i\pi a}} = \frac{\pi a}{\sin\left(\pi a\right)}$$

Positive Real Domain

The general problem

$$I = \int_0^\infty f\left(x\right)\,dx$$

exploits a branch cut spectacularly. We begin by considering a different integral

$$\tilde{I} = \int_0^\infty f\left(x\right)\ln\left(x\right)\,dx$$

on the same contours C_+ and C_- used above, on the branch $0 \le \theta < 2\pi$. This gives us

$$\int_{C_+} f\left(z\right)\ln\left(z\right)dz = \tilde{I}$$
$$\int_{C_-} f\left(z\right) z^a\,dz = -\tilde{I} - 2\pi i\int_0^\infty f\left(x\right)dx\;,$$

which sum together to perfectly cancel $\tilde{I}$, provided the usual assumptions that allow the integral with $R \to \infty$ to vanish. Solving for the original I, we find:

$$\int_0^\infty f\left(x\right)\,dx = \frac{i}{2\pi}\int_C f\left(z\right)\ln\left(z\right)dz$$

Example 20

Evaluate:

$$I = \int_0^\infty \frac{dx}{\left(x+2\right)\left(x+1\right)^2}$$

Contour contains two poles.

$$I = \frac{i}{2\pi}\cdot 2\pi i\left(\frac{d}{dz}\frac{\ln z}{z+2}\bigg|_{z=-1} + \frac{\ln z}{\left(z+1\right)^2}\bigg|_{z=-2}\right)$$
$$= 1 - \ln 2$$