Complex-Algebra
MANUSCRIPT

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Chapter 1

Complex Algebra

1 History of Complex Numbers

The story of *complex numbers* begins more than a century before calculus, in a time when mathematicians were still puzzling through what we would now consider high school algebra.

The issue of solving depressed cubic equations

$$x^3 + bx = c$$

was especially prescient in mid-1500s Italy, and eventually a pair of mathematicians would derive the *del Ferro-Tartaglia* formula

$$x_0 = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}\,,$$

allowing a single solution to the depressed cubic to be attained.

While equation the above works for a certain class of depressed cubic problems, it is still peculiar in the sense that negative numbers may end up embedded under the radical symbols. Staying within the rules of algebra, the un-treatable quantity always boils down to $\sqrt{-1}$, making x_0 impossible to simplify as such.

Aware of this, mathematician Rafael Bombelli had the 'wild thought' to work with factors of $\sqrt{-1}$ *anyway*. This means to suppose the 'ugliness' of the cube-root terms in x_0 could be split away from the well-behaved part such that

$$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} = U + \sqrt{-1}\,V$$
$$\sqrt[3]{\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} = U - \sqrt{-1}\,V$$

for two unknown coefficients U and V. Then, when we assemble x_0 again, the V-terms cancel,

$$x_0 = U + \cancel{\sqrt{-1}\,V} + U - \cancel{\sqrt{-1}\,V} = 2U\,,$$

which is guaranteed to come out to a 'clean' number.

Real, Imaginary, Complex

To avoid using terms like 'clean' numbers, it is generally meant that numbers containing no factors of

$\sqrt{-1}$ are called *real*. Any real number multiplied by $\sqrt{-1}$ is *imaginary* number. The sum of a real number and an imaginary number is a *complex* number.

Imaginary Component

Synonyms for the 'real part' and 'imaginary part' of a complex number, respectively are the *components* of the number.

The present task is to solve for the previously-defined components U, V in terms of the coefficients b, c. Do this by raising each of side of the above to the third power to find

$$\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}} = U\left(U^2-3V^2\right)+\sqrt{-1}\,V\left(3U^2-V^2\right) ,$$

availing the connection:

$$\text{Real part of}\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right) = U\left(U^2-3V^2\right)$$
$$\text{Imag. part of}\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+\frac{b^3}{27}}\right) = V\left(3U^2-V^2\right)$$

Pursing Bombelli's method, we end up with a pair of two equations with two unknowns (neither contain $\sqrt{-1}$). Faced with this, Bombelli had a second wild thought: perhaps U and V need to be *positive integers* (we won't dwell much on this detail). Solving for U, V completes the recipe for cooking the first solution x_0.

With x_0 in hand, the term $x-x_0$ can be factored out of the depressed cubic equation, yielding the form

$$x^3+bx-c=(x-x_0)\left(x^2+x_0x+\frac{c}{x_0}\right) .$$

Since the remaining term is quadratic in x, there are at most two more solutions $x_{1,2}$ to the equation that can be found with the quadratic formula on:

$$x^2+x_0x+\frac{c}{x_0}=0$$

Worked Example

Putting these ideas to work, suppose we need to find all solutions to

$$x^3-30x-36=0 .$$

Identifying $b=-30$ and $c=36$, the del Ferro-Tartaglia formula tells us

$$x_0=\sqrt[3]{18+\sqrt{-1}\,26}+\sqrt[3]{18-\sqrt{-1}\,26} .$$

By Bombelli's reasoning, we seek integer solutions to U, V such that

$$18=U\left(U^2-3V^2\right)$$
$$26=V\left(3U^2-V^2\right) .$$

The prime factorization of 18 is $2\times3\times3$, suggesting that $U=3$, and thus $V=1$. Suddenly we've got a solution:

$$x_0=2U=6$$

This is quite a remarkable achievement, since by writing

$$\sqrt[3]{18+\sqrt{-1}\,26}+\sqrt[3]{18-\sqrt{-1}\,26}=6 ,$$

we see undeniably that, regardless of how one may feel about $\sqrt{-1}$, it is useful for problem solving.

<u>Problem 1</u>

With $x_0=6$ as a known solution to

$$x^3-30x-36=0 ,$$

show that the other two solutions are

$$x_1=3+\sqrt{3}$$
$$x_2=3-\sqrt{3} .$$

2 Complex Numbers

2.1 Definition

Complex Numbers

Formally, let the complex number z exist as as an ordered pair of two (real) numbers called *components* a, b such that

$$z=(a,b) . \tag{1.1}$$

Complex Conjugate

For every complex number z, there exists the *complex conjugate*, also a complex number, denoted $\overline{z}$ or z^*, by flipping the sign on b:

$$\overline{z}=z^*=(a,-b) . \tag{1.2}$$

Relationship to Real Numbers

A subtlety worth highlighting is that the complex number $z=(a,0)$ is the same as the scalar $z=a$:

$$(a,0)\leftrightarrow a \tag{1.3}$$

<u>Problem 2</u>

Check that the complex conjugate of the complex conjugate recovers the complex number:

$$\overline{\overline{z}}=z \tag{1.4}$$

2.2 Complex Arithmetic

Scalar Multiplication

Complex numbers can be 'scaled' by real numbers called *scalars*:

$$\lambda z = (\lambda a, \lambda b) \tag{1.5}$$

Complex Addition

For two complex numbers $z_1 = (a_1, b_1)$, $z_2 = (a_1, b_1)$, their sum is

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2) \ . \tag{1.6}$$

Complex Multiplication

For two complex numbers $z_1 = (a_1, b_1)$, $z_2 = (a_1, b_1)$, their product is simultaneously defined by a commutation relation and a conjugate relation:

$$z_1 \cdot z_2 = z_2 \cdot z_1 \tag{1.7}$$
$$\overline{z_1} \cdot \overline{z_2} = \overline{z_1 \cdot z_2} \tag{1.8}$$

Conspicuously absent from the list of axioms is an explicit formula for complex multiplication. For completeness, the formula for complex multiplication reads

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \ , \tag{1.9}$$

but this is does not need to be axiomatic unless you're in a hurry. Instead, we'll soon derive Equation (1.9) from the equations preceding it.

2.3 Properties of Addition

Isolating Complex Components

Given a complex number $z = (a, b)$ and its complex conjugate $\overline{z} = (a, -b)$, take their sum and difference, respectively, to write a pair of relations that 'solve for' a, b:

$$(a, 0) = \frac{z + \overline{z}}{2} \tag{1.10}$$
$$(0, b) = \frac{z - \overline{z}}{2} \tag{1.11}$$

Note that a and b in isolation are each real numbers. As they appear in $z = (a, b)$, a is the so-called 'real part', and b is the 'imaginary part'.

Commutation and Conjugation

Two relationships readily verifiable from the axioms are the respective *commutation* and *conjugation* relations for addition:

$$z_1 + z_2 = z_2 + z_1 \tag{1.12}$$
$$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2} \tag{1.13}$$

Problem 3

Verify Equations (1.12) and (1.13) using any of the axioms.

2.4 Properties of Multiplication

Finally we encounter the first piece of hard work, which is to derive the formula for complex multiplication. Most texts simply take (1.9) as an axiom to avoid a slightly dry derivation, and you are welcome to do so now as well.

Derivation

As a starting point, propose the product $z_1 \cdot z_2$ to result in a new complex number comprised of every order-two combination of $a_{1,2}$, $b_{1,2}$

$$z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (q, r) \ ,$$

where

$$q = \alpha \, a_1 a_2 + \beta \, a_1 b_2 + \gamma \, a_2 b_1 + \delta \, b_1 b_2$$
$$r = \tilde{\alpha} \, a_1 a_2 + \tilde{\beta} \, a_1 b_2 + \tilde{\gamma} \, a_2 b_1 + \tilde{\delta} \, b_1 b_2 \ ,$$

and each Greek index α, β, etc. (eight in total) resolves to 1, 0, or -1. We shall nail these down in several steps:

Impose the conjugation relation (1.8), causing all b-terms to flip sign. The q-term must remain the same under this change, but the r-term must flip sign. This can only hold if

$$\tilde{\alpha} = \beta = \gamma = \tilde{\delta} = 0 \ ,$$

so we now have:

$$q = \alpha \, a_1 a_2 + \delta \, b_1 b_2$$
$$r = \tilde{\beta} \, a_1 b_2 + \tilde{\gamma} \, a_2 b_1 \ ,$$

Impose the commutation relation (1.7), causing all 1- and 2-subscripts to swap. The q-equation remains unchanged, but the r equation demands

$$\tilde{\beta} = \tilde{\gamma} \neq 0 \ .$$

Swap all a- and b-symbols. Doing so should completely change the results, however the r-equation is invariant with respect to the swap. It follows that α and δ in the q-equation must disagree in sign:

$$\alpha = -\delta \neq 0$$

Boiling everything down:

$$\begin{aligned} q &= \alpha\left(a_1a_2 - b_1b_2\right) \\ r &= \tilde{\beta}\left(a_1b_2 + a_2b_1\right) \end{aligned}$$

Let $b_2 = 0$. The corresponding product becomes

$$z_1 \cdot z_2 = \left(\alpha\, a_1a_2, \tilde{\beta}\, a_2b_1\right) = a_2\left(\alpha\, a_1, \tilde{\beta}\, b_1\right) \,,$$

which looks much like the scalar multiplication λz_1 with $\lambda = a_2$. For the sake of keeping complex multiplication consistent with scalar multiplication, let us finally set

$$\alpha = \tilde{\beta} = 1 \,,$$

finishing the derivation.

Associative Property

Complex numbers $z_j = (a_j, b_j)$ with $j = 1, 2, 3$ obey the *associative property*

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) \,, \tag{1.14}$$

shown by brute force:

$$\begin{aligned} (z_1 \cdot z_2) \cdot z_3 &= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) \cdot (a_3, b_3) \\ &= (a_1a_2a_3 - b_1b_2a_3 - a_1b_2b_3 - b_1a_2b_3, \\ &\qquad a_1a_2b_3 - b_1b_2b_3 + a_1b_2a_3 + b_1a_2a_3) \\ &= (a_1\left(a_2a_3 - b_2b_3\right) - b_1\left(b_2a_3 + a_2b_3\right), \\ &\qquad a_1\left(a_2b_3 + b_2a_3\right) + b_1\left(b_2b_3 - a_2a_3\right)) \\ &= (a_1, b_1) \cdot (a_2a_3 - b_2b_3, a_2b_2 + b_2a_3) \\ &= z_1 \cdot (z_2 \cdot z_3) \end{aligned}$$

Distributive Property

Complex numbers $z_j = (a_j, b_j)$ with $j = 1, 2, 3$ also obey the *distributive property*:

$$z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_2 \cdot z_3) \,, \tag{1.15}$$

also shown by brute force:

$$\begin{aligned} z_1 \cdot (z_2 + z_3) &= (a_1, b_1) \cdot (a_2 + a_3, b_2 + b_3) \\ &= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3, \\ &\qquad a_1b_2 + a_1b_3 + a_2b_1 + a_3b_1) \\ &= (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) + \\ &\qquad (a_1a_3 - b_1b_3, a_1b_3 + a_3b_1) \\ &= (z_1 \cdot z_2) + (z_1 \cdot z_3) \end{aligned}$$

2.5 Complex Magnitude

The quantity

$$|z| = \sqrt{z \cdot \overline{z}} \tag{1.16}$$

is called the *magnitude* of z, and is always a non-negative real number:

$$\begin{aligned} |z| &= \sqrt{z \cdot \overline{z}} = \sqrt{(a, b) \cdot (a, -b)} \\ &= \sqrt{(a^2 + b^2, 0)} = \sqrt{a^2 + b^2} \end{aligned}$$

This result foreshadows *some* kind of geometric interpretation of complex numbers in the sense that $|z|$ is the hypotenuse of a right triangle with sides a, b.

2.6 Complex Division

The last standard arithmetic operation is complex division, which seems inoperable at face value:

$$d = \frac{z_1}{z_2} = \frac{(a_1, b_1)}{(a_2, b_2)}$$

To make a useful complex number from this, multiply the top and bottom by $\overline{z}_2$:

$$d = \frac{(a_1, b_1)}{(a_2, b_2)}\frac{(a_2, -b_2)}{(a_2, -b_2)} = \frac{z_1 \cdot \overline{z}_2}{|z_2|^2} \tag{1.17}$$

Explicitly, the above means:

$$d = \frac{1}{a_2^2 + b_2^2}\left(a_1a_2 + b_1b_2, -a_1b_2 + a_2b_1\right) \tag{1.18}$$

Problem 4

Prove that the complex division operation obeys its own conjugate relation:

$$\overline{z}_1 / \overline{z}_2 = \overline{z_1 / z_2} \tag{1.19}$$

2.7 Generalized Complex Arithmetic

If we seek to write a tighter set of axioms, one could start with the general equations

$$\begin{aligned} z_1 \star z_2 &= z_2 \star z_1 \\ \overline{z_1} \star \overline{z_2} &= \overline{z_1 \star z_2} \end{aligned}$$

for a generalized operator $\star$. As we've seen, the multiplication operator results from seeking all order-two combinations of $a_{1,2}$, $b_{1,2}$. By the same token, the addition operator results from seeking all order-one combinations of $a_{1,2}$, $b_{1,2}$.

Problem 5

Derive the addition operation (1.6) using only Equations (1.12) and (1.13).

2.8 Imaginary Unit

Having established the notion of complex multiplication, now consider the product:

$$(0,1)\cdot(0,1) = (0-1\cdot 1, 0) = -1$$

The left side has two instances of $(0,1)$, which ought to mean:

$$\sqrt{(0,1)\cdot(0,1)} = (0,1)$$

Suddenly though, without ever asking, we have an answer for the meaning of $\sqrt{-1}$. Apply the square root operation across the whole equation to find

$$(0,1) = \sqrt{-1}\,,$$

known as the *imaginary unit*. Often, the fundamental unit is denoted as i, meaning

$$i^2 = -1\,.$$

Problem 6

Let z_1 be any complex number. Find all other complex numbers z_2 that satisfy $z_1 \cdot z_2 = 0$.

Problem 7

Let z_1 be any complex number. Find all other complex numbers z_2 that satisfy $z_1 \cdot z_2 = z_1$.

3 Complex Plane

3.1 Complex Numbers as Operators

For a complex number $z = (a,b)$, consider the 'identity' statement

$$(1,0)\cdot(a,b) = (a,b)\ . \tag{1.20}$$

Equation (1.20) is the 'least invasive' operation in which z_1 can particpate, which is to merely multiply by one.

Seeking a corresponding identity for $(0,1)$, we find, by complex multiplication,

$$(0,1)\cdot(a,b) = (-b,a)\ . \tag{1.21}$$

The 'operator' $(0,1)$ swaps a with b and introduces a negative sign as shown. This amounts to another hint that the real and imaginary components of z are somehow 'orthogonal', meaning there could be some geometric interpretation of complex numbers.

Going on this hunch, let us think of $(1,0)$ and $(0,1)$ as 'basis vectors', and write a linear combination with two undetermined coefficients α, β:

$$\alpha\,(1,0) + \beta\,(0,1) = (\alpha,0) + (0,\beta) = (\alpha,\beta)$$

Almost obviously, such an operation is nothing more than a complex number (α,β). Trying this 'operator' on a different complex number $z = (a,b)$, we simply have

$$(\alpha,\beta)\cdot(a,b) = (\alpha a - \beta b, \alpha b + \beta a)\ , \tag{1.22}$$

or more concisely,

$$(\alpha,\beta)\cdot z = z' = (a',b')\ .$$

3.2 Rotation Operator

Let us find an operator (just a complex number) $z_\phi = (\alpha_\phi, \beta_\phi)$ that acts on $z = (a,b)$ such that the components change but the magnitude does not. From Equation (1.22), we write

$$\begin{aligned}|z'| &= \sqrt{(a')^2 + (b')^2}\\ &= \sqrt{(\alpha_\phi\, a - \beta_\phi\, b)^2 + (\beta_\phi\, a + \alpha_\phi\, b)^2}\,,\end{aligned}$$

simplifying nicely to

$$|z'| = |z|\sqrt{\alpha_\phi^2 + \beta_\phi^2}\,. \tag{1.23}$$

In the special case that z_ϕ has $\alpha_\phi^2 + \beta_\phi^2 = 1$, then z_ϕ qualifies as a rotation operator. The locus of α_ϕ, β_ϕ describes a 'complex unit circle', begging the parameterization

$$\alpha_\phi = \cos(\phi) \tag{1.24}$$
$$\beta_\phi = \sin(\phi)\ , \tag{1.25}$$

where ϕ is a real continuous parameter. As a sanity check, one can see that $\phi = 0$, $\phi = \pi/2$ correspond to the respective operators $(1,0)$, $(0,1)$. Let us therefore take the *complex rotation operator* to be the complex number

$$z_\phi = (\cos(\phi), \sin(\phi))\ . \tag{1.26}$$

3.3 Radius and Phase

The rotation operator allows us to interpret complex numbers in a curious way. Given the real number r, any complex number z with magnitude

$$|z| = r \tag{1.27}$$

is the product

$$z = r z_\phi = (r\cos(\phi), r\sin(\phi)) = (a,b)\ . \tag{1.28}$$

Borrowing terminology from polar coordinates, the magnitude of a complex number is equivalent to the *radius*, and the angle parameter is called the *phase*.

In terms of these, the components of a complex number read

$$a = r\cos(\phi) \tag{1.29}$$
$$b = r\sin(\phi)\ , \tag{1.30}$$

easily inverted:

$$r = |z| = \sqrt{a^2 + b^2} \tag{1.31}$$
$$\phi = \arctan\left(\frac{b}{a}\right) \tag{1.32}$$

As a matter of terminology, recall that the a- and b- components of z are the real and imaginary parts, i.e.

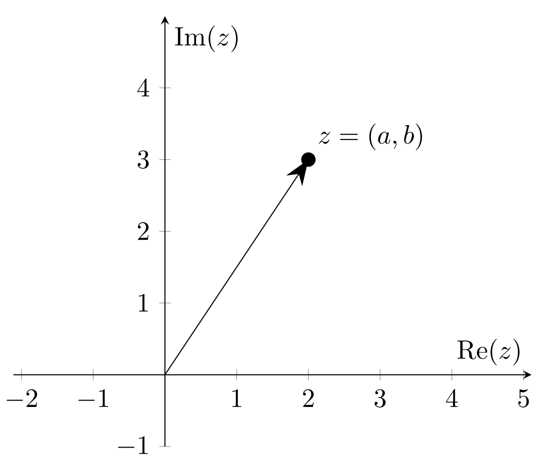
$$a = \mathrm{Re}(z)$$
$$b = \mathrm{Im}(z)\ .$$

The phase angle ϕ is often denoted $\mathrm{Arg}(z)$, or 'argument of z'. All together, an equivalent statement of (1.32) reads

$$\mathrm{Arg}(z) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)\ . \tag{1.33}$$

3.4 Real and Imaginary Axes

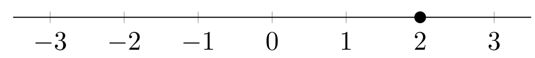
The r, ϕ interpretation of complex numbers leads to an almost-identical mathematical apparatus needed for plane polar coordinates. Complex numbers occupy a space that is analogous to the Cartesian xy-plane, except the x-axis is replaced by the real axis, and the y-axis is replaced by the imaginary axis. In this connection we speak freely of the *complex plane*:



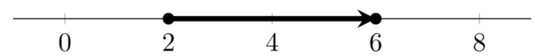
Any 'location' on the $\mathrm{Im}(z) = 0$ line is a purely real number $z = (a, 0) = a$, whereas anywhere on the $\mathrm{Re}(z) = 0$ is a purely imaginary number $z = (0, b)$. Any off-axis location is a complex number $z = (a, b)$.

3.5 Number as Location

Thinking for a moment about classical arithmetic, it's convenient to represent any real number, such as $x = 2$, on a number line:



Any operation performed on x, such as multiplying by three, can be represented as a spatial displacement on the line:

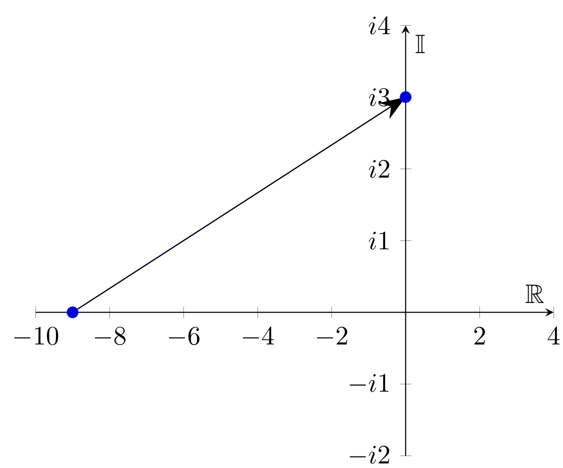


In fact, all of the 'well-behaved' operations that one could possibly perform on x will land *somewhere* on the number line.

The first hint that the number line is somehow incomplete arises when we ask, much like the renaissance-era mathematicians asked, 'what happens when we take the square root of a negative number?' While this issue was a sign for most to turn around, recall that Bombelli, in a moment of 'wild thought' had the idea to separate the real component from the imaginary component. He understood that numbers like

$$x^2 = -9$$
$$x = \sqrt{-9} = \sqrt{-1} \times \sqrt{9} = 3\sqrt{-1}$$

must be represented off of the number line, even without foreknowledge of the complex plane. Since we *do* know about the the complex plane though, the core of Bombelli's insight can now be visualized without ambiguity:



3.6 Complex Numbers and Vectors

Since there is much talk of two-component objects and their relationship to a plane, it's worthwhile to ask how closely complex numbers resemble vectors. Begin this inquiry by considering two complex numbers z_1, z_2, and solve (1.29)-(1.30) for the respective trigonometry terms:

$$\cos(\phi_1) = \frac{a_1}{|z_1|}$$
$$\sin(\phi_1) = \frac{b_1}{|z_1|}$$

$$\cos(\phi_2) = \frac{a_2}{|z_2|}$$
$$\sin(\phi_2) = \frac{b_2}{|z_2|}$$

Multiply the cos-terms and the sin-terms respectively

$$\cos(\phi_1)\cos(\phi_2) = \frac{a_1 a_2}{|z_1||z_2|}$$
$$\sin(\phi_1)\sin(\phi_2) = \frac{b_1 b_2}{|z_1||z_2|} \; ,$$

and combine the results:

$$\cos(\phi_1 - \phi_2) = \frac{a_1 a_2 + b_1 b_2}{|z_1||z_2|} \tag{1.34}$$

Had z_1 and z_2 been vectors $\mathbf{z}_1$ and $\mathbf{z}_2$, the quantity $a_1a_2 + b_1b_2$ stands out as the dot product $\mathbf{z}_1 \cdot \mathbf{z}_2$. Evidently, we have

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = |z_1||z_2|\cos(\phi_1 - \phi_2) \; . \tag{1.35}$$

Note that the quantity $a_1a_2 + b_1b_2$ can be written yet another way, namely

$$a_1 a_2 + b_1 b_2 = \frac{1}{2}\left(\bar{z}_1 \cdot z_2 + z_1 \cdot \bar{z}_2\right) \; , \tag{1.36}$$

allowing a tight relationship to be written:

$$\mathbf{z}_1 \cdot \mathbf{z}_2 = \frac{1}{2}\left(\bar{z}_1 \cdot z_2 + z_1 \cdot \bar{z}_2\right) \tag{1.37}$$

Problem 8

Write equations analogous to (1.34), (1.35), and (1.36) to derive the cross product analog to (1.37):

$$\mathbf{z}_1 \times \mathbf{z}_2 = \frac{1}{2}\left(\bar{z}_1 \cdot z_2 - z_1 \cdot \bar{z}_2\right) \tag{1.38}$$

Problem 9

Derive:

$$\bar{z}_1 \cdot z_2 = \left(\mathbf{z}_1 \cdot \mathbf{z}_2, \mathbf{z}_1 \times \mathbf{z}_2\right) \tag{1.39}$$

Problem 10

Using vectors as an analogy, derive the triangle inequality for complex numbers:

$$||z_1| - ||z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2| \tag{1.40}$$

3.7 Embedded Complex Numbers

As a matter of curiosity, one may wonder if it makes sense to work with 'embedded complex numbers' such as

$$((A, B), C) \qquad (A, (B, C)) \; .$$

In the same way that equation (1.3) permits the association $(A, 0) \leftrightarrow A$, let us extend this by writing

$$((A, B), 0) \leftrightarrow (A, B) \; .$$

Using the above, we find for the first case:

$$\begin{aligned} ((A, B), C) &= ((A, B), 0) + (0, C) \\ &= (A, B) + (0, C) \\ &= (A, B + C) \end{aligned}$$

Evidently, the 'embedded' complex number flattens down to the 'ordinary' complex number:

$$((A, B)\, C) \to (A, B + C)$$

Working out the second case in similar fashion:

$$\begin{aligned} (A, (B, C)) &= (A, 0) + (0, (B, C)) \\ &= (A, 0) + (0, 1) \cdot ((B, C), 0) \\ &= (A, 0) + (0, 1) \cdot (B, C) \\ &= (A, 0) + (-C, B) \\ &= (A - C, B) \end{aligned}$$

Once again, the embedded complex number flattens down to a (different) ordinary complex number. As a corollary to the above, we can stack the two results to write the more general statement

$$((A, B), (C, D)) = (A - D, B + C) \; .$$

Now consider two complex numbers

$$z_{1,2} = (a_{1,2}, b_{1,2}) \; ,$$

and also let $A = a_1a_2$, $D = b_1b_2$, $B = a_1b_2$, $C = b_1a_2$. The right side of the above becomes identical to the formula for the complex nultiplication $z_1 \cdot z_2$, giving an equivalent representation as an embedded number:

$$z_1 \cdot z_2 = ((a_1a_2, a_1b_2), (b_1a_2, b_1b_2))$$

Since each component $a_{1,2}$, $b_{1,2}$ is scalar, the above becomes

$$\begin{aligned} z_1 \cdot z_2 &= (a_1(a_2, b_2), b_1(a_2, b_2)) \\ &= a_1a_2 + (0, a_1b_2) + (0, b_1a_2) + (0, b_1(0, b_2)) \\ &= (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2) \; , \end{aligned}$$

and the standard notation is recovered.

4 Euler's Formula

Now we derive the most important equation in all of complex analysis, called *Euler's formula.*

Repeated Rotations

To kick things off, recall how the complex rotation operator (1.26) acts on any complex number $z = (a, b)$ to produce a new complex number z' such that

$$z' = (\cos(\phi), \sin(\phi)) \cdot z \ ,$$

where the parameter ϕ is an arbitrary real number.

Next, suppose that ϕ is the sum of two arbitrary angles θ_1, θ_2, and the above becomes

$$z' = (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)) \cdot z \ ,$$

simplifying nicely to

$$z' = (\cos(\theta_1), \sin(\theta_1)) \cdot (\cos(\theta_2), \sin(\theta_2)) \cdot z \ .$$

Evidently, the *effective* rotation operator representing ϕ becomes the product of two rotation operators representing θ_1, θ_2.

Generalizing this pattern, suppose instead that ϕ is the sum of n copies of the same angle θ such that

$$\phi = \theta_1 + \theta_2 + \theta_3 + \cdots = n\theta \ ,$$

so a rotation can be written

$$z' = \left(\cos\left(\frac{\phi}{n}\right), \sin\left(\frac{\phi}{n}\right)\right)^n \cdot z \ .$$

(Don't let the presence of an exponent throw you off - this is just shorthand for $n - 1$ complex multiplications of the same number.)

Summarizing our progress so far, the rotation operator can be interpreted as a repetition of small rotations:

$$(\cos(\phi), \sin(\phi)) = \left(\cos\left(\frac{\phi}{n}\right), \sin\left(\frac{\phi}{n}\right)\right)^n \tag{1.41}$$

Infinite Rotations

Given the re-interpred rotation operator (1.41), the inevitable question is, what happens when n is extremely large, or perhaps infinitely large? Looking at the cos- and sin-terms, we're left to evaluate

$$\cos(\phi/n)$$
$$\sin(\phi/n)$$

as the argument ϕ/n goes to zero. Borrowing from 'elementary' trigonometry and precalculus though, the relations

$$\lim_{n\to\infty} \cos\left(\frac{\phi}{n}\right) = 1 \tag{1.42}$$

$$\lim_{n\to\infty} \sin\left(\frac{\phi}{n}\right) = \frac{\phi}{n} \tag{1.43}$$

apply, letting us write

$$\lim_{n\to\infty} \left(\cos\left(\frac{\phi}{n}\right), \sin\left(\frac{\phi}{n}\right)\right)^n = \lim_{n\to\infty} \left(1, \frac{\phi}{n}\right)^n \ .$$

Summarizing again, we now see that the rotation operator is an infinite repetition of small rotations:

$$(\cos(\phi), \sin(\phi)) = \lim_{n\to\infty} \left(1, \frac{\phi}{n}\right)^n \tag{1.44}$$

Invoking Euler's Constant

Looking again at the complex number $(1, \phi/n)$, split the components via

$$\left(1, \frac{\phi}{n}\right) = (1, 0) + \left(0, \frac{\phi}{n}\right) \ .$$

First, note that $(1, 0)$ is equivalent to 1, so the complex notation can be dropped from the first term. Next, factor n^{-1} from the second term, and we have

$$\left(1, \frac{\phi}{n}\right) = 1 + \frac{1}{n}(0, \phi) \ .$$

Plugging this back into (1.44) gives

$$(\cos(\phi), \sin(\phi)) = \lim_{n\to\infty} \left(1 + \frac{1}{n}(0, \phi)\right)^n \ . \tag{1.45}$$

The right-side of the above should remind us of another notion from precalculus, namely Euler's constant, defined as:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \ ,$$

or in more useful form,

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

for any real number x.

Euler's Formula

Euler reasoned that the formula for e^x could be modified (it's *his* formula, after all) to receive complex arguments:

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

If this is the case, the right side of (1.45) can be written as en exponential

$$\lim_{n\to\infty}\left(1+\frac{1}{n}(0,\phi)\right)^n = e^{(0,\phi)} ,$$

which is amazing, because it means the *left* side of (1.45) is an exponential! Putting these results together, we write

$$(\cos(\phi), \sin(\phi)) = e^{(0,\phi)} , \tag{1.46}$$

hailed as Euler's formula. At face value, Euler's formula is an economical way to write the rotation operator.

It's worth pausing to write Euler's formula in terms of the so-called imaginary unit, which is to associate

$$(0,1) = \sqrt{-1} = i ,$$

thus the above becomes:

$$\cos(\phi) + i\sin(\phi) = e^{i\phi}$$

Setting $\phi = \pi$ reveals a remarkable connection between key players of mathematics:

$$0 = 1 + e^{i\pi}$$

4.1 Polar Form

Let us revisit the radius-and-phase construction of a complex number. For a complex number $z = (a, b)$, recall that equations (1.29)-(1.32)

$$\begin{aligned} a &= r\cos(\phi) \\ b &= r\sin(\phi) \\ r &= |z| = \sqrt{a^2+b^2} \\ \phi &= \arctan(b/a) \end{aligned}$$

allow us to understand z as a rotation of the real number r 'up into' the complex plane by an angle ϕ:

$$z = (a, b) = r(\cos(\phi), \sin(\phi))$$

With Euler's formula in hand, we may replace the trigonometry terms via (1.46), and arrive at the most elegant expression of a complex number in terms of radius and phase, the so-called polar form:

$$z = r\, e^{(0,\phi)} \tag{1.47}$$

Problem 11

Show that:

$$\cos(n\arccos(x)) = \sum_{k=0}^{n/2}\binom{n}{2k}x^{n-2k}\left(x^2-1\right)^k$$

Hint: Let $x = \cos(\theta)$ and arrive at

$$\cos(n\theta) = \mathrm{Re}\left((\cos(\theta) + i\sin(\theta))^n\right) .$$

4.2 Multiplication and Division

Given the polar form (1.47) of complex numbers, the multiplication formula (1.9) and the division formula (1.18) can be revised. For two complex numbers $z_1(r_1, \phi_1)$, $z_2(r_2, \phi_2)$, we have

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2\, e^{(0,\phi_1+\phi_2)} \\ &= r_1 r_2 (\cos(\phi_1+\phi_2), \sin(\phi_1+\phi_2)) \end{aligned} \tag{1.48}$$

for multiplication, and for division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}\, e^{(0,\phi_1-\phi_2)} \\ &= \frac{r_1}{r_2}(\cos(\phi_1-\phi_2), \sin(\phi_1-\phi_2)) \end{aligned} \tag{1.49}$$

Problem 12

Derive (1.48) and (1.49).

4.3 Trigonometric Functions

Euler's formula lends to a curious representation of the standard trigonometric functions. Start with (1.46) and let $\phi \to -\phi$ to write

$$(\cos(\phi), -\sin(\phi)) = e^{(0,-\phi)} ,$$

and then add both equations (also divide by two) to isolate the cosine:

$$(\cos(\phi), 0) = \frac{1}{2}\left(e^{(0,\phi)} + e^{(0,-\phi)}\right) \tag{1.50}$$

An analogous procedure isolates the sine:

$$(0, \sin(\phi)) = \frac{1}{2}\left(e^{(0,\phi)} - e^{(0,-\phi)}\right) \tag{1.51}$$

Recalling that the complex rotation operator can be written

$$z_\phi = (\cos(\phi), \sin(\phi)) ,$$

note that the above can expressed more tightly in accordance with (1.10)-(1.11):

$$\begin{aligned} (\cos(\phi), 0) &= \frac{z_\phi + \overline{z}_\phi}{2} \\ (0, \sin(\phi)) &= \frac{z_\phi - \overline{z}_\phi}{2} \end{aligned}$$

4.4 Hyperbolic Functions

Continuing the discussion of trigonometric functions, isolate the real number ϕ and apply the operator $(0,1)$, resulting in $(0,\phi)$ without question. Next take the complex number $(0,\phi)$ and apply the same operator to find

$$(0,1)\cdot(0,\phi) = -\phi\,.$$

The reason we do this is to start with Euler's formula (1.46), and multiply each instance of ϕ by the complex number $(0,1)$, giving

$$\left(\cos\left((0,\phi)\right),\sin\left((0,\phi)\right)\right) = e^{-\phi}\,.$$

A similar set of steps leads to a version with $-\phi \to \phi$, or

$$\left(\cos\left((0,\phi)\right),-\sin\left((0,\phi)\right)\right) = e^{\phi}\,.$$

Notice the quantity on the right is a real number, which must mean

$$\left(0,\sin\left((0,\phi)\right)\right) = (0,1)\cdot\sin\left((0,\phi)\right)$$

is also real, thus $\sin\left((0,\phi)\right)$ by itself is imaginary.

The above can be written:

$$\begin{aligned}\cos\left((0,\phi)\right) + (0,1)\cdot\sin\left((0,\phi)\right) &= e^{-\phi}\\ \cos\left((0,\phi)\right) - (0,1)\cdot\sin\left((0,\phi)\right) &= e^{\phi}\end{aligned}$$

Take the sum, and the sin-term cancels, allowing $\cos\left((0,\phi)\right)$ to be isolated:

$$\cos\left((0,\phi)\right) = \frac{1}{2}\left(e^{\phi}+e^{-\phi}\right)$$

Similarly, take the difference and the cos-term vanishes:

$$-\left(0,1\right)\cdot\sin\left((0,\phi)\right) = \frac{1}{2}\left(e^{\phi}-e^{-\phi}\right)$$

It turns out that these 'complex trigonometry' terms above have special names, the *hyperbolic cosine* and *hyperbolic sine*, respectively:

$$\cosh\left(\phi\right) = \cos\left((0,\phi)\right) = \frac{1}{2}\left(e^{\phi}+e^{-\phi}\right) \tag{1.52}$$

$$\sinh\left(\phi\right) = -\left(0,1\right)\cdot\sin\left((0,\phi)\right) = \frac{1}{2}\left(e^{\phi}-e^{-\phi}\right) \tag{1.53}$$

5 Roots and Branches

5.1 Complex Natural Logarithm

Start again with Euler's polar form (1.47)

$$z = r\,e^{(0,\phi)}\,,$$

and let us dissect this equation apart along a new seam. Recall that one way to adjust the 'position' of a point in the complex plane is to change the phase ϕ, but nothing special happens if the phase wanders outside of $[0:2\pi)$. Euler's formula has this fact built-in, as

$$\begin{aligned}e^{(0,\phi\pm 2\pi n)} &= \left(\cos\left(\phi\pm 2\pi n\right),\sin\left(\phi\pm 2\pi n\right)\right)\\ &= \left(\cos\left(\phi\right),\sin\left(\phi\right)\right)\\ &= e^{(0,\phi)}\end{aligned}$$

holds for any integer n.

Now things get interesting. The presence of e in (1.47) beckons trying the natural logarithm (ln) on both sides, resulting in:

$$\ln\left(z\right) = \ln\left(r\right) + (0,\phi) \tag{1.54}$$

Unlike $e^{(0,\phi)}$, the complex natural logarithm does not 'reset' at 2π. The phase term $(0,\phi)$ in (1.54) is not modified by a trigonometry function, thus ϕ keeps accumulating value across $[0:2\pi)$. The complex natural logarithm is unique for *every* phase.

5.2 Complex Square Root

Consider the square root of $z\left(r,\phi\right)$, which by Euler's formula reads

$$z^{1/2} = \left(r\,e^{(0,\phi)}\right)^{1/2} = r^{1/2}\,e^{(0,\phi/2)}\,. \tag{1.55}$$

The square root always seems to cause trouble in mathematics, and the complex square root is no exception. Examine two representations of the same point in the complex plane, for instance (r,ϕ) and $(r,\phi+2\pi)$.

Calculating $z^{1/2}$ for each case gives

$$\begin{aligned}\sqrt{z\left(r,\phi\right)} &= \sqrt{r}\,e^{(0,\phi/2)}\\ \sqrt{z\left(r,\phi+2\pi\right)} &= -\sqrt{r}\,e^{(0,\phi/2)}\,,\end{aligned}$$

and we see the phase causes an abrupt sign flip. By convention, the positive solution is called the *principal root*.

5.3 Complex nth Root

Consider the so-called nth root problem

$$z^n = a \tag{1.56}$$

for integer n, and complex numbers z and a. The question is, which value(s) of z make this statement true? The answer is made easy with Euler's formula (1.47), where we recast each variable as

$$\begin{aligned}z &= \left|z\right|e^{(0,\phi)}\\ a &= \left|a\right|e^{(0,\theta+2\pi m)}\,.\end{aligned}$$

To proceed most generally, a phase factor of $2\pi m$ is slipped into the phase of a, knowing full well this doesn't actually change its value, where m is any positive or negative integer. Rewriting (1.56) gives

$$|z|^n e^{(0,n\phi)} = |a| e^{(0,\theta+2\pi m)} .$$

Then, the radial and angular components on each side are equal:

$$\begin{aligned} |z|^n &= |a| \\ n\phi &= \theta + 2\pi m \end{aligned}$$

Solving the above for $|z|$, ϕ, we have

$$|z| = |a|^{1/n} \tag{1.57}$$

$$\phi = \frac{\theta}{n} + \frac{2\pi m}{n} , \tag{1.58}$$

where

$$m = 0, 1, 2, \ldots, (n-1) .$$

Note that in order to stay on one branch, the integer m is restricted to produce unique solutions for ϕ. The principal root is in general defined as the solution with the greatest real component.

5.4 Complex Exponent

The same multi-value problem that arises with the complex logarithm and complex roots applies to complex exponents. Consider two complex numbers $z = (a, b)$, $w = (c, d)$. Starting with the polar expression $z = |z| e^{(0,\phi)}$, the complex exponent z^w calculation is slightly nontrivial:

$$\begin{aligned} z^w &= \left(|z| e^{(0,\phi)}\right)^{(c,d)} \\ &= e^{\ln|z|(c,d)} e^{(-d\phi,0)} e^{(0,c\phi)} \\ z^w &= \exp\left(c \ln|z| - d\phi, d \ln|z| + c\phi\right) \end{aligned} \tag{1.59}$$

6 Complex Functions

The complex exponential (1.47), complex natural logarithm (1.54), along with complex roots and exponents all qualify as *complex functions*, usually denoted $w(z)$ for complex numbers $z(x, y)$. In the general case, a complex function produces a complex number

$$w(z) = (u(x, y), v(x, y)) , \tag{1.60}$$

having respective components $u(x, y)$, $v(x, y)$, each a real function.

6.1 Notion of Inverse

If a given function $w(z)$ can be inverted into an equation for $z(w)$, then we have

$$z(w) = (f(u, v), g(u, v)) . \tag{1.61}$$

The functions $f(u, v)$, $g(u, v)$ contain all of the gritty details of actually inverting w.

6.2 Branch Cuts

The peculiar behavior of the complex natural logarithm (1.54) and the complex square root (1.55) suggest that care must be taken when 'stepping across' the boundary $0 \leftrightarrow 2\pi n$.

In general, discontinuity in $w(z)$ arises anywhere in the complex plane that involves a sudden an abrupt jump in phase. Such 'fault lines' are curves called *branch cuts*.

Complex Natural Logarithm

For the natural logarithm, we choose the branch on the line $\phi = \pm\pi$ (not 2π, by convention).

The very small 'wedge' centering on $\phi = \pm\pi$ is where the phase of $\ln(z)$ jumps abruptly, i.e. the branch cut.

Complex Square Root

The square root function (1.55) has two branches, characterized by $\pm\mathrm{Im}(\sqrt{z})$, separated by the branch cut $(-\infty, 0)$. Choosing the positive branch allows us to unambiguously associate $\sqrt{a^2}$ with $+a$, the principal root.

6.3 Riemann Surface

The multi-valued nature of certain complex functions cannot be completely represented in a two-dimensional plot on the complex plane. To deal with functions exhibiting branching behavior, a third dimension representing the phase of $w(z)$ is required.

With the notion of $z(w) = (f, g)$ on hand, generating three-dimensional plots called *Riemann surfaces* is a standard exercise in plotting. The parameters f, g become analogous to x and y in a standard plot, and the off-plane direction is often associated with $\mathrm{Im}(w)$, but only by convention. It can be just as informative to plot $\mathrm{Re}(w)$ in the third dimension.

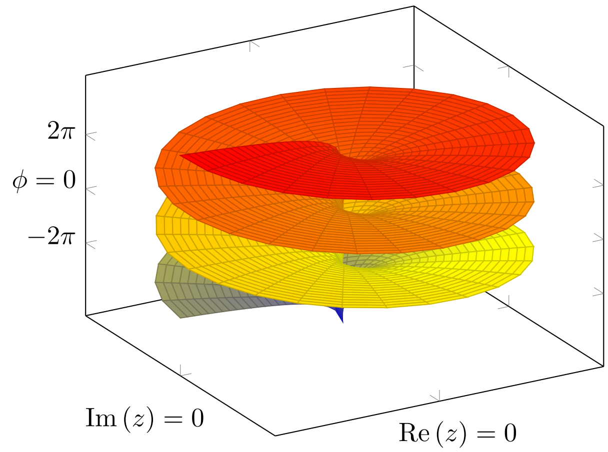


Figure 1.1: Complex natural logarithm $\ln(z) = \ln(r) + (0, \phi)$.

Complex Natural Logarithm

To visualize the complex natural logarithm, start with $w(z) = \ln(z)$, esaily inverted:

$$z(w) = e^w = e^{(u,v)} = (e^u \cos(v), e^u \sin(v))$$

In this case, it makes sense to choose a polar parameterization with $r = e^u$:

$$X(r, v) = r\cos(v)$$
$$Y(r, v) = r\sin(v)$$
$$Z(r, v) = \ln(r) + v$$

The 'plot variables' are uppercase symbols X, Y, Z.

Plotting this system with $(r > 0, v \in [-3\pi : 3\pi])$ leads to Fig. 1.1. For constant r, the complex logarithm traces out a *helix* for varying ϕ. The family of all helices made this way comprise a 'ramp' spiraling around the origin as shown in Fig. 1.1. The resulting Riemann surface is most generally called a *manifold*.

It's also possible to choose just one branch for two-dimensional plotting. For the complex natural logarithm, this amounts to cutting the 'middle' sheet from the stack in Fig. 1.1, and flatting it down onto the complex plane as shown in Figure 1.2. (Color may vary.)

The bold-shaded lines indicate integer r, ϕ, respectively. Specifically, the circle corresponds to $r = 1$ (with $r = 2$ outside of the plot), and the straight spokes correspond to

$$\phi = 0, 1, 2, 3, -3, -2, -1\,,$$

listing counterclockwise from $\phi = 0$.

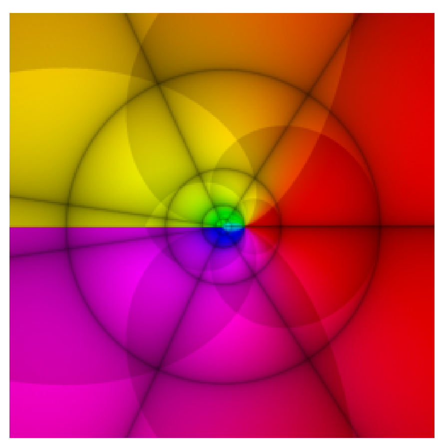


Figure 1.2: Complex logarithm on the branch $(-\pi : \pi]$.

Complex Square Root

The Riemann surface recipe also applies to plotting the square root function $w(z) = z^{1/2}$. Seeking a form

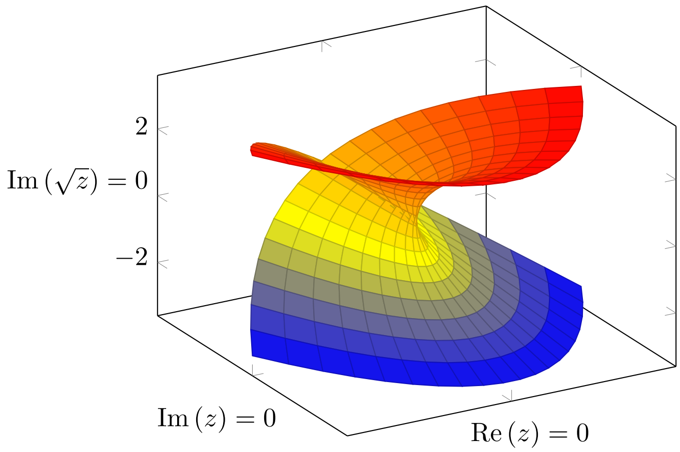


Figure 1.3: Complex square root $z^{1/2}$.

like (1.61), we easily write

$$z\,(w) = w^2 = (u, v) \cdot (u, v) = \left(u^2 - v^2, 2uv\right) \ .$$

Turning this into a three-dimensional system, we write

$$\begin{aligned} X\,(u, v) &= u^2 - v^2 \\ Y\,(u, v) &= 2uv \\ Z\,(u, v) &= u \ , \end{aligned}$$

while treating u, v as parameters. Plotting this system near $(u = 0, v = 0)$ leads to Fig. 1.3. The phase of $z^{1/2}$ abruptly jumps at the branch cut $\phi = \pm\pi$.

Choosing the principal root and plotting the square root in the complex plane leads to Figure 1.4. The bold-shaded lines correspond to positive integer outputs of $z^{1/2}$. Below the line $\mathrm{Re}\,(z) = 0$, the sign on $\mathrm{Im}\,(z)$ flips, but $\mathrm{Re}\,(z)$ remains positive.

This is quickly tested by letting let $n = 2$ to find $|z| = \sqrt{|a|}$, and $\phi_0 = \theta/2$, $\phi_1 = \theta/2 + \pi$, corresponding to two complex solutions

$$\begin{aligned} z_0 &= \sqrt{|a|}\, e^{(0,\theta/2)} \\ z_1 &= -\sqrt{|a|}\, e^{(0,\theta/2)} \ . \end{aligned}$$

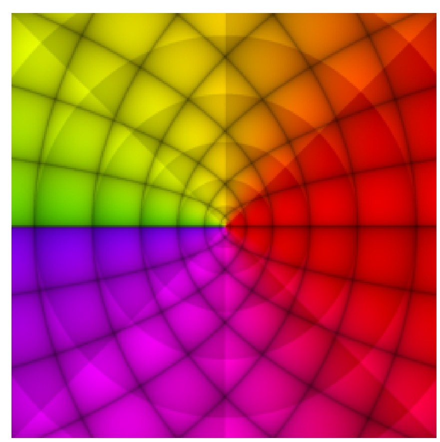


Figure 1.4: Complex square root on the branch $(-\pi : \pi]$.

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