Collatz Fractal
MANUSCRIPT

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Contents

1 Introduction

In 1937, Lothar Collatz pointed out a pattern that has been bugging mathematicians and hobbyists ever since. Start with any integer $n > 1$. If n is even, change n to $n/2$. If n is odd, change n to $3n+1$. Repeat this until n changes to 1. The so-called *Collatz conjecture* states that *any* integer $n > 1$ will eventually reduce down to 1. So far so good, but where's the proof?

Any acceptable proof the Collatz conjecture has remained elusive to the most accomplished mathematicians. This problem, while gaining a slew of nicknames along the way, namely (but not limited to) the $3n+1$ problem, the hailstone sequence, the hailstone numbers, and the wondrous numbers, has nonetheless been been 'proved' by a number of optimists. We will make no attempt to wrap our heads around the any proofs here, or to debunk any crackpot analysis. Rather, the plan is to further deepen the mystery by involving complex numbers.

1.1 Collatz Function

The Collatz conjecture can be contained in a rather ugly equation:

$$f\left(x\right) = \begin{cases} x/2 & x \text{ even} \\ 3x+1 & x \text{ odd} \end{cases} \tag{1}$$

Note that there is no 'escape condition' for equation (1), thus the infinite cycle $1-4-2-1-4-2-\cdots$ awaits any integer that satisfies the conjecture. A more utilitarian version of the above would contain a way to escape from such a cycle when $x = 1$ first occurs.

1.2 Continuous Collatz Function

In order to study the Collatz conjecture on the complex plane, it makes sense to generalize $f\left(x\right)$ into a new function $c\left(z\right)$ that is (i) smooth and differentiable, and (ii) reducible to $f\left(x\right)$ for real integers. In the most general case then, we can write

$$c\left(z\right) = \left(\frac{z}{2}\right) g_{\text{even}}\left(z\right) + \left(3z+1\right) g_{\text{odd}}\left(z\right) \;, \tag{2}$$

where the functions $g_{\text{even}}\left(z\right)$, $g_{\text{odd}}\left(z\right)$ obey:

$$g_{\text{even}}\left(z\right) = \begin{cases} 1 & z \text{ even} \\ 0 & z \text{ odd} \end{cases}$$

$$g_{\text{odd}}\left(z\right) = \begin{cases} 0 & z \text{ even} \\ 1 & z \text{ odd} \end{cases}$$

Common literature[1] on the subject suggests trying

$$g_{\text{even}}(z) = \cos^2\left(\frac{\pi z}{2}\right)$$
$$g_{\text{odd}}(z) = \sin^2\left(\frac{\pi z}{2}\right) ,$$

and inserting these into equation (2) gives, after simplifying,

$$c(z) = \frac{7z+2}{4} - \frac{5z+2}{4}\cos(\pi z) \ . \tag{3}$$

One can easily check that equation (3) satisfies equation (2) by checking a few real integers. Moreover, $c_1(z)$ is a smooth and differentiable function, which translates to 'much more appealing to mathematicians than $f(x)$'.

It is instructive to go through a second derivation of equation (3) by choosing[2] $g_{\text{even}}(z)$, $g_{\text{odd}}(z)$ such that

$$g_{\text{even}}(z) = \frac{(-1)^z + 1}{2}$$
$$g_{\text{odd}}(z) = -\frac{(-1)^z - 1}{2} \ .$$

As required, one function 'turns off' as the other 'turns on' for any given real integer z. For non integer z, the above become complex numbers, and may be reconciled via Euler's formula

$$(-1)^z = e^{i\pi z} = \cos(\pi z) + i\sin(\pi z) \ .$$

We see the term $\cos(\pi z)$ does the heavy lifting, as the imaginary term $\sin(\pi z)$ is precisely zero for real integer z and can be omitted.

The provisional formula for $c(z)$ so far reads

$$c(z) = \frac{z}{2}\left(\frac{\cos(\pi z)+1}{2}\right) + (3z+1)\left(\frac{1-\cos(\pi z)}{2}\right) ,$$

and simplifying this leads to equation (3). Interestingly, note that either g-function can be raised to an integer power without overstaying its welcome in $c(z)$. For integers q_e, q_o, this means

$$c(z) = \frac{z}{2}\left(\frac{\cos(\pi z)+1}{2}\right)^{q_e} + (3z+1)\left(\frac{1-\cos(\pi z)}{2}\right)^{q_0} \tag{4}$$

is another valid Collatz function.

[1] 'Collatz conjecture'. *Wikipedia*. December 22 2022. `https://en.wikipedia.org/wiki/Collatz/conjecture`

[2] Xander. 'The Collatz Fractal'. Rhapsody in Numbers. January 12 2012. `https://yozh.org/2012/01/12/the_collatz_fractal/`

1.3 Time Analysis

Convergence Time

If we take a real integer z and recursively apply $z \to c(z)$, the number of iterations required to reach $z = 1$ is the *convergence time* of z. For instance, starting with $z = 3$ we would have the sequence $3 - 10 - 5 - 16 - 8 - 4 - 2 - 1$, having convergence time $t = 7$. The special case $z = 1$ has convergence time $t = 0$. Certain numbers, especially odd integers, can have unexpectedly long convergence times. One example is $z = 27$, requiring $t = 111$ iterations to converge.

Following are the convergence times of the first forty positive integers (increasing left to right):

0	1	7	2	5
8	16	3	19	6
14	9	9	17	17
4	12	20	20	7
7	15	15	10	23
10	111	18	18	18
106	5	26	13	13
21	21	21	34	8

Escape Time

While it seems that all integers *probably* satisfy the Collatz conjecture, we still need to account for what happens when z is a decimal, a complex number, or something else that fails to satisfy the conjecture. It will turn out that most of the complex plane must be treated this way, and the converging cases are more rare than common.

When either (i) the magnitude of z exceeds some hard-coded maximum, or (ii) the number of iterations exceeds a cutoff number, then z has reached an *escape time*. In the former case, z is striding toward infinity, and the escape time indicates 'how quickly' this occurs. In the latter case, z is wandering around the complex plane with no apparent destiny. These are considered to converge infinitely slowly.

2 Collatz Set

The *Collatz set* is comprised of numbers z_0 in the complex plane that reach a convergence time in a finite number of $z \to c(z)$ applications, where $c(z)$ is the continuous Collatz function (2) or a generalization thereof. Points characterized by an escape time are outside of the set.

2.1 Discrete Plot

Plotting Set Members

For visualizing the Collatz set, we can start by assigning a distinct value - namely color - to each convergence time. Any complex numbers z_0 that converge in the same number t steps get the same color, and different numbers z' with different convergence times t' get a different color, and so on. While the particular choice of colors does not quite matter, it helps to have a diversity of shades with noticeable contrast between boundaries.

Going right to some results, Figure 1 shows a heat map of convergence times over the complex plane centered at $x = 15$. On the positive real axis are the (randomly chosen) shades assigned to the convergence time for the integer at that position.

Plotting Set Non-Members

To represent points outside the Collatz set, we choose a two-color gradient to indicate escape time. Inputs that escape slower than others are shaded black, whereas points that jump right of the picture are white-shifted toward beige. Figure 1 shows such a gradient hugging the real axis. Note that the vertical stripe (white) in the Figure corresponds to inputs $z = (1, y)$, and these inputs automatically satisfy the Collatz conjecture without iteration. The origin is centered one notch leftward of the vertical stripe.

The last category corresponds to complex inputs that fail to converge and diverge, eventually exceeding the cutoff number. These are assigned a flat color (green). As you may imagine, the negative integers fit this description.

2.2 Continuous Plot

With an idea of how the Collatz set appears for integers, now we construct the set over the continuous complex plane. Shown in Figure 2 is the continuous analog to Figure 1, and some stark differences are visible. Most notably (i) the integers with a convergence time seem to be *gone*, (ii) the gradient has a 'spiky' appearance, undoubtedly due to the cosine term in $c(z)$, and (iii) inputs near the origin seemingly don't converge.

3 Collatz Fractal

The barely-visible boundary separating members of the Collatz set from non-members is called the *Collatz fractal*. Shown in Figure 3 shows a detailed view of the Collatz fractal generated using equation (3) near the origin. Note that the subtle vertical line at $x = 1$ is not magnified by zooming in - this line has zero thickness and renders as one pixel at best.

3.1 Self Similarity

Part of the definition of 'fractal' is the property of *self-similarity*, which means a piece of the set reflects the whole set. This means, for instance, that we can center near any 'green blob' and magnify, and the resulting image should contain elements from Figure 3. The Collatz fractal surely demonstrates this, as shown in Figures 4, 5. The former Figure, zoomed in at the point $z = (2.01, 0.368)$, shows a near-ninety degree rotation base fractal. Choosing another point on which to center and zoom, particularly $z = (2.04, 0.415)$, we produce the latter Figure, containing a varied view of the base fractal.

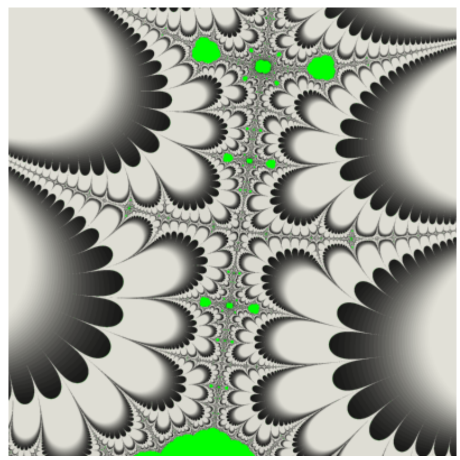

Figure 4: Self-similarity of the Collatz set. Centered at $(2.01, 0.368)$. (Zoom: 32)

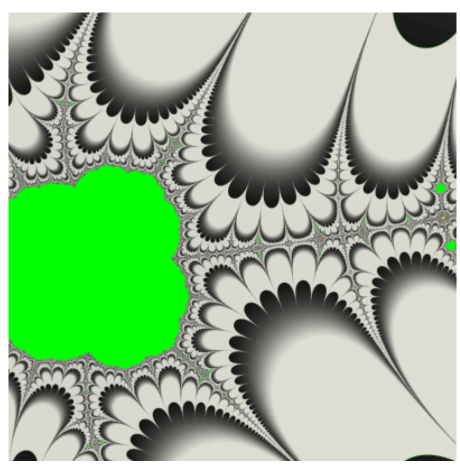

Figure 5: Self-similarity of the Collatz set. Centered at $(2.04, 0.415)$. (Zoom: 256)

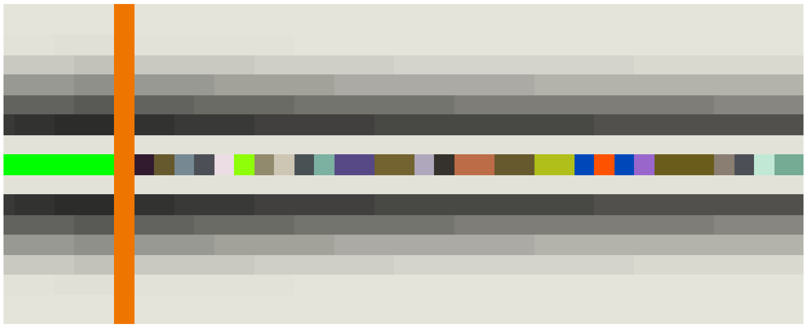

Figure 1: Collatz set over complex integer inputs. Vertical stripe corresponds to $x = 1$. (Center: $z = 15$, Zoom: 0.125)

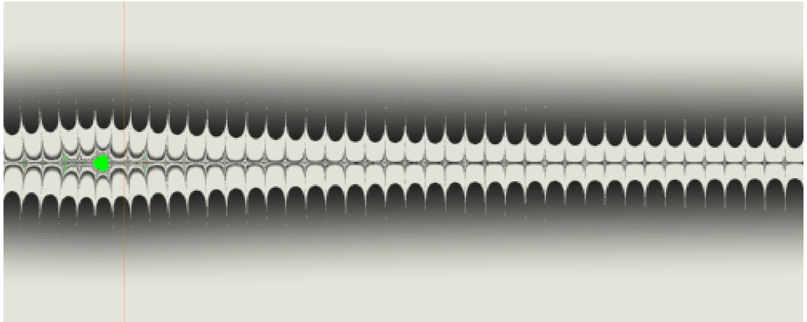

Figure 2: Collatz set over complex continuous inputs. Vertical stripe corresponds to $x = 1$. (Center: $z = 15$, Zoom: 0.125)

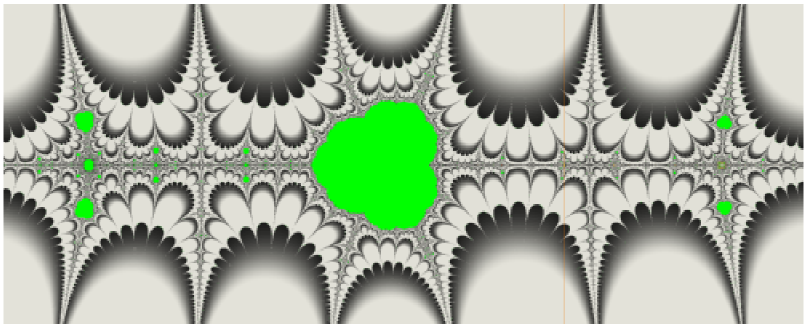

Figure 3: Collatz set over complex continuous inputs centered on $z = 0$ in a window of width ≈ 5. The boundary of the green regions is the Collatz fractal. (Zoom: 1)

4 Collatz Gems

The positive integers, which are the testing ammunition for the Collatz conjecture on the real line, bear minuscule representation in Collatz fractal as depicted in Figure 3. The integers are surely in the set, but are the integers represented by infinitesimal points now? The answer is a resounding 'no', and we must zoom in a bit to appreciate what happens on and near the integers.

4.1 Anatomy of a Gem

Shown in Figures 6, 7 are two images of the Collatz 'gem' (a word we'll keep) located at $z = 1$. The boundary of the gem is part of the Collatz fractal, and the interior is colored according to the convergence time of the point tested. Note the subtle presence of the infinitesimal line $z = (1, y)$ streaking down the center of each image. Outside of the gem is an infinitude of self-similarity of the fractal, as evident by the green speckles scattered about.

From what we know of the discrete Collatz set, the integer $z = 1$ should be associated with the same shade of orange that occupies the vertical stripe in Figure 1. In the Collatz gem, we have to magnify profoundly to verify this as shown in Figures 8, 9. Many narrow bands of color are passed on the way there.

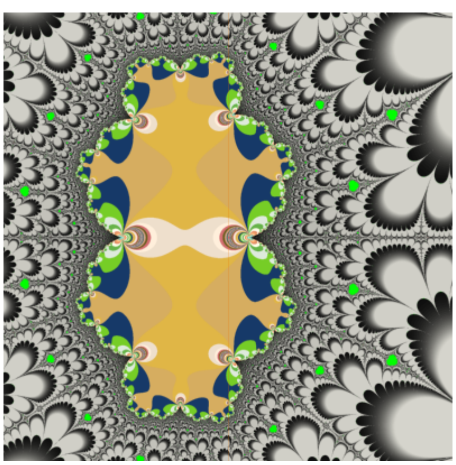

Figure 6: Collatz gem at $z = 1$. (Zoom: 128)

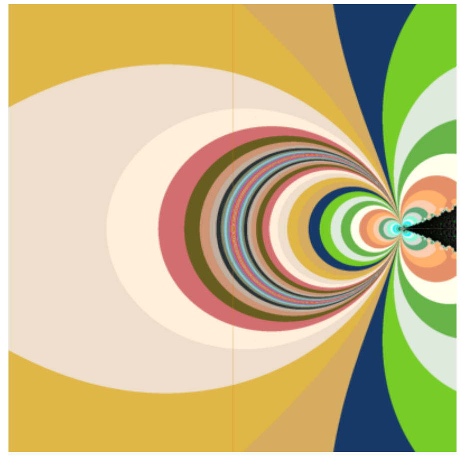

Figure 7: Collatz gem at $z = 1$. (Zoom: 1024)

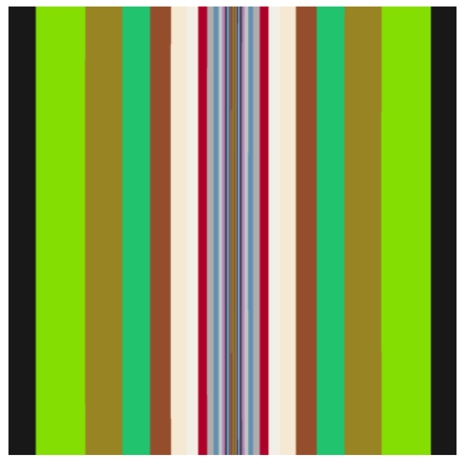

Figure 8: Collatz gem at $z = 1$. (Zoom: 4194304)

Figure 9: Collatz gem at $z = 1$. (Zoom: 536870912)

4.2 Collatz Gems $z > 1$

Collatz gems occur at (seemingly) all integer positions $z > 1$, and can be studied in similar fashion to the $z = 1$ case. The color of the gem varies throughout its shape, but the shade pinpointed on the exact integer always corresponds to the discrete case. A gem's interior shading near the boundaries accentuates the Collatz fractal occurring there. Interestingly too, the gem is never (or rarely) centered on the integer value itself. Figures 10-15 survey the Collatz gems from $z = 2$ to $z = 7$.

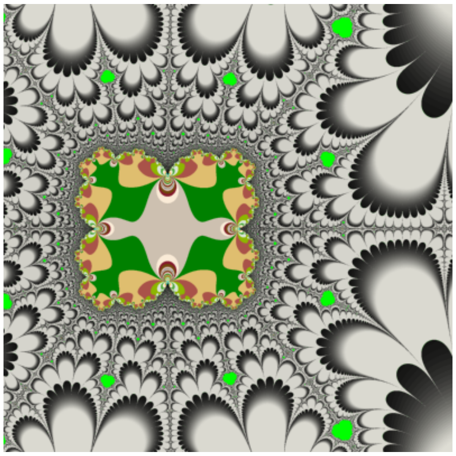

Figure 10: Collatz gem at $z = 2$. (Zoom: 32)

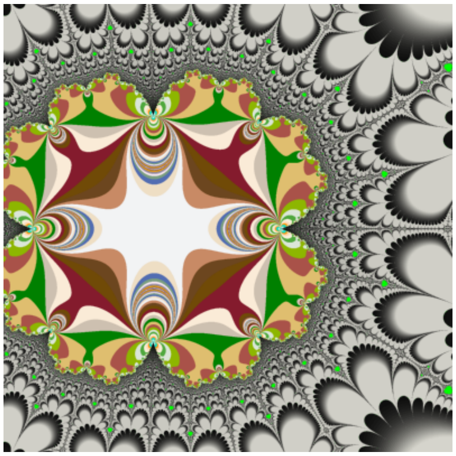

Figure 11: Collatz gem at $z = 3$. (Zoom: 512)

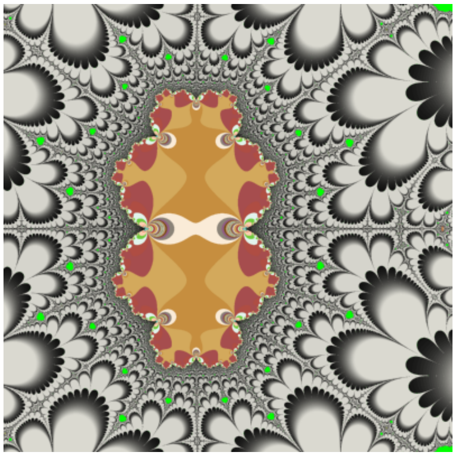

Figure 12: Collatz gem at $z = 4$. (Zoom: 32)

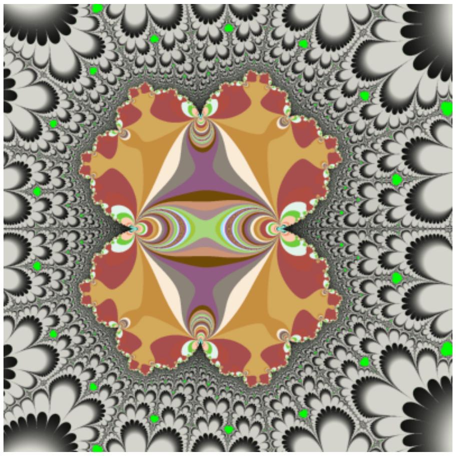

Figure 13: Collatz gem at $z = 5$. (Zoom: 256)

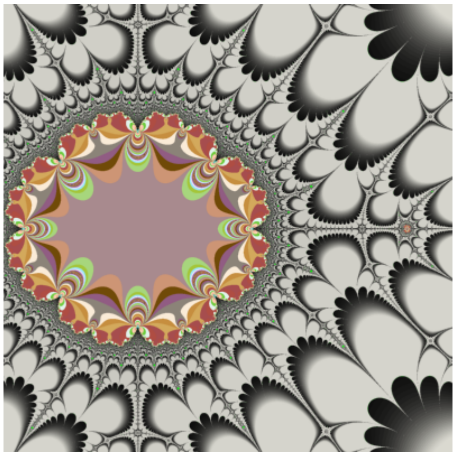

Figure 14: Collatz gem at $z = 6$. (Zoom: 128)

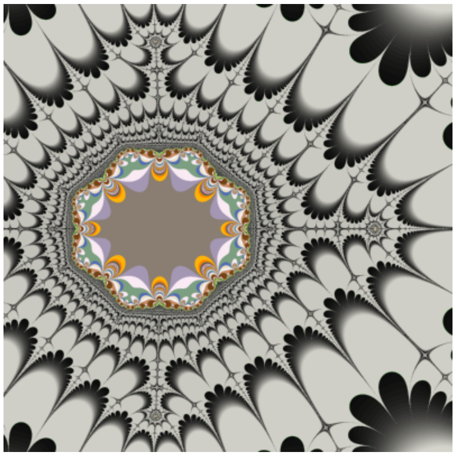

Figure 15: Collatz gem at $z = 7$. (Zoom: 1024)

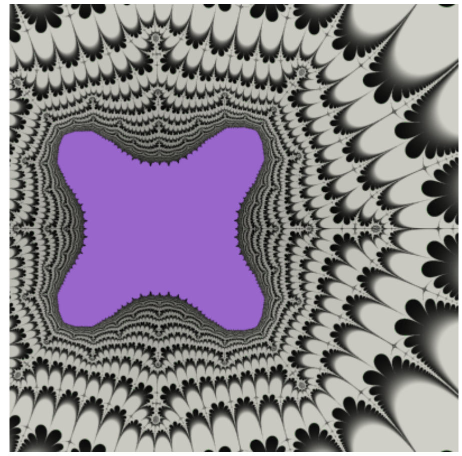

Figure 16: Collatz gem at $z = 27$. (Zoom: 4096)

4.3 Pathological Gem $z = 27$

The number 27 is arguably the most quirky input one can send to the Collatz function. As mentioned, it takes 111 iterations to finally converge, begging the question of whether the Collatz gem at $z = 27$ tells a related story. Recursing on values near $c(27)$ and looking up the corresponding gem, we get the rather uninteresting shape shown in Figure 16. It is of uniform shade with index 111 corresponding to the convergence time. The boundary is still a Collatz fractal, but the overall appearance is unlike other gems.

Let us chase the progression of 27 through the Collatz pipeline, i.e. $c(27) = 82$, and then $c(82) = 41$, etc., and view the corresponding gems. For the 82-case we produce Figure 17. The shading is again flat and few features are discernible. This pattern continues to the next integer in the sequence, i.e. 41, as shown in Figure 18.

Going the other direction, we can check power-of-two multiples of 27 and see the same 'flat color' effect. For instance, trying $z = 27 \times 2^5 = 864$ leads to Figure 16, and the interior gem shading is still flat. Finally when we try $z = 27 \times 2^{10} = 27648$ and magnify like crazy, we see the alternating color bands in a typical Collatz gem shown in Figure 21. The view had to be centered slightly off-integer to capture the image.

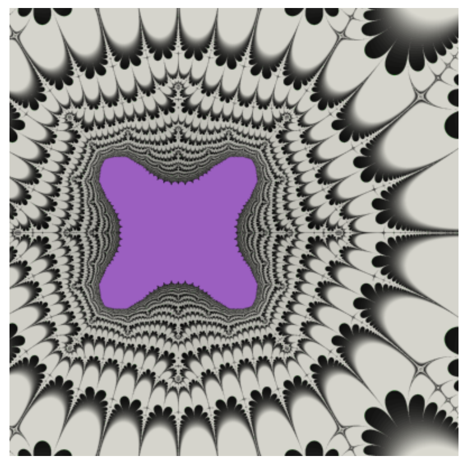

Figure 17: Collatz gem at $z = 82$. (Zoom: 1024)

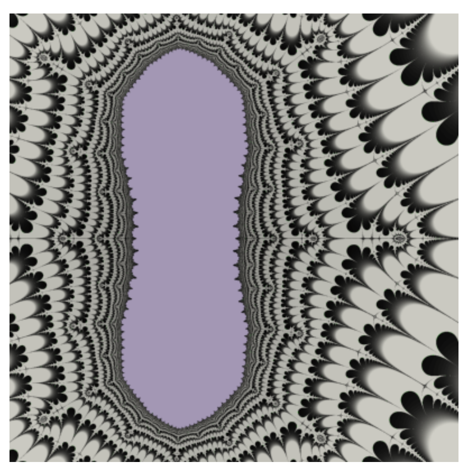

Figure 18: Collatz gem at $z = 41$. (Zoom: 4096)

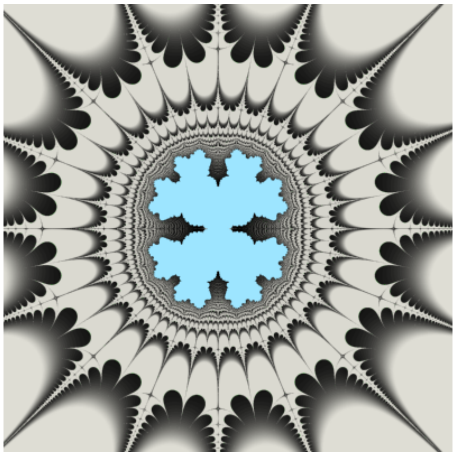

Figure 19: Collatz gem at $z = 864$. (Zoom: 2048)

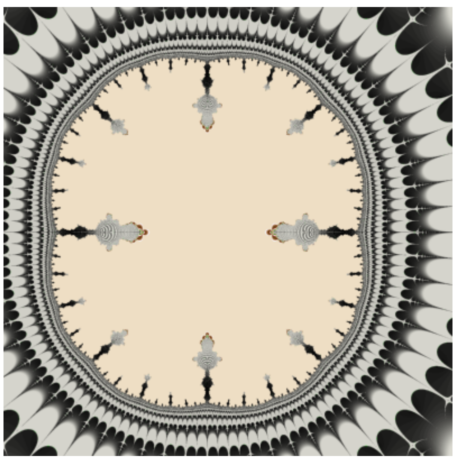

Figure 20: Collatz gem at $z = 27648$. (Zoom: 131072)

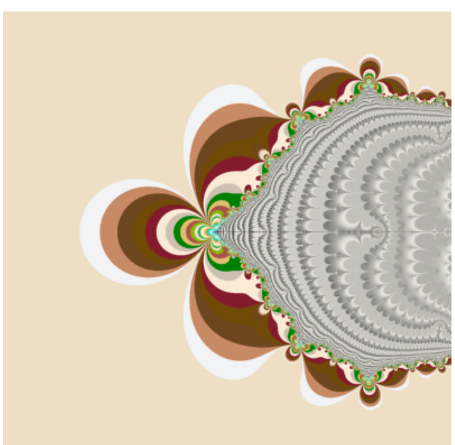

Figure 21: Collatz gem at $z = 27648$. (Zoom: 2097152, Off-centered)

4.4 Non-Integer Members

The Collatz set, and any gem that helps fill it, is not limited to the postitive real axis. We find interesting structure at decimal z and at negative points on the real axis as shown in Figures 22, 23. We see another Collatz gem in the former case, as evident by the alternating bands of color representing convergence time. In the ladder case we find with a clover-like glyph, yet another face of the Collatz fractal.

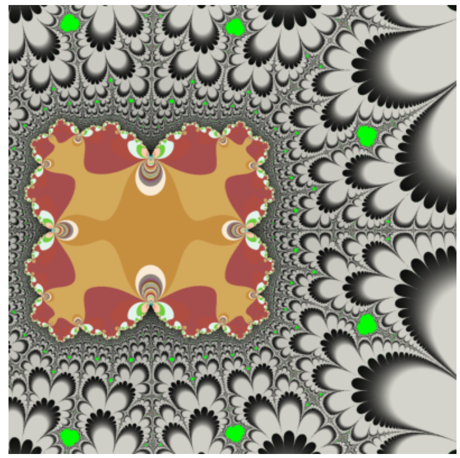

Figure 22: Collatz gem. Center: $z = (0.61082, 0)$. (Zoom: 256)

Figure 23: Collatz set. Center: $z \approx (-1.2656, 0)$. (Zoom: 8192)

4.5 Further Study

All views of the Collatz fractal shown above, along with all embeded gems, have been generated using equation (3), an order-one generalization of equation (4). The study could be repeated using equation (4) using any set of integers q_e, q_o, which will undoubtedly have some bearing on the appearnce of the fractal and it's gems.

Shown in Figures 24, 25 are simplified views of the familiar 'order-one' Collatz fractal, followed by a pure order-two representation. The qualitative views are different in each, and the gem at $z = 2$ is visible in the latter case.

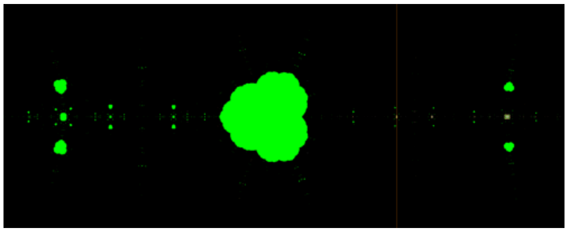

Figure 24: Simplified view of order-one Collatz fractal.

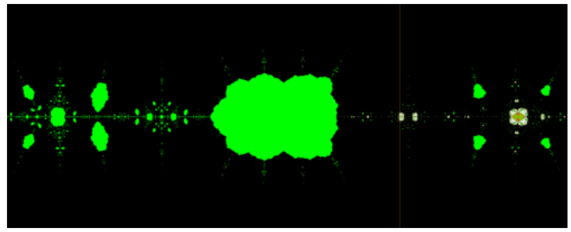

Figure 25: Simplified view of order-two Collatz fractal.