Central Forces
MANUSCRIPT

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Chapter 1

Central Forces

1 Planetary Motion

Early Progress

The 'modern' understanding of planetary motion arguably began with Johannes Kepler (1571 - 1630), whose career predates the invention of calculus and Newton's laws of motion by decades. Already familiar with the Heliocentric model of the solar system, Kepler studied meticulously-recorded charts of night sky measurements recorded by Tycho Brahe (1546 - 1601).

Paying attention to the positions of observable planets in the night sky, Kepler astonishingly figured out that planetary orbits were elliptical in shape with the sun at a focus. This became known as Kepler's first law, which survives to this day among two other laws written by Kepler.

Aware of Kepler's first law, Newton proposed the existence of a law of mutual Earth-sun attraction that gives rise to elliptical planetary orbits. In the modern vector notation, he began with something like

$$\vec{F} = F\left(r\right)\hat{r} \; ,$$

and the quest was to find whatever $F\left(r\right)$ is.

Using the calculus of his own invention, Newton found the answer to be a unified force depending on the masses involved and the inverse square of the distance separating them. We know this as Newton's law of universal gravitation.

The plan here is to develop the equations of planetary motion using a similar approach, at least in spirit, to Newton.

Shell Theorem

One assumption we'll make early on, which happens to be *true*, and will be proven with triple integration, is *any object can be considered as a point mass located at the object's center of mass.* For instance, if we need to calculate the gravitational attraction between two asteroids, the shape of each does not matter. Only the center-to-center distance and the mass of each body is important.

Newton's Second Law

The one-dimensional version of Newton's second law

$$m\frac{d^2}{dt^2}x\left(t\right) = -\frac{d}{dx}U\left(x\right)$$

generalizes to more dimensions where the force and acceleration become vectors:

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

I avoided saying exactly how $-dU/dx$ becomes $\vec{F}$. Note that in one dimension,

$$F = -\frac{dU}{dx}$$

is true by definition, but the three dimensional version of this requires a vector derivative operator. The exact details aren't needed in order to proceed.

Newton's Third Law

The classic phrase, *for every action, there is an equal an opposite reaction*, is Newton's third law. It means that the force from object 1 onto object 2 is exactly opposite of the force from object 2 onto object 1. This is concisely stated via vectors:

$$\vec{F}_{12} = -\vec{F}_{21}$$

1.1 Two-Body Problem

Consider two bodies in space, one of mass m_1 at position $\vec{r}_1\left(t\right)$, and the other of mass m_2 at position $\vec{r}_2\left(t\right)$. The force imposed onto body 1 by body 2 is given by

$$m_1\frac{d^2}{dt^2}\vec{r}_1\left(t\right) = m_1\frac{d}{dt}\vec{v}_1\left(t\right) = \vec{F}_{12} \; ,$$

and the force imposed onto particle 2 by particle 1 is given by

$$m_2\frac{d^2}{dt^2}\vec{r}_2\left(t\right) = m_2\frac{d}{dt}\vec{v}_2\left(t\right) = \vec{F}_{21} \; .$$

This setup is called the *two-body problem.*

Center of Mass

In the two-body system, the *center of mass* is defined as a point in space $\vec{R}(t)$ such that

$$\vec{R}(t) = \frac{m_1 \vec{r}_1(t) + m_2 \vec{r}_2(t)}{m_1 + m_2} .$$

The time derivative of the center of mass gives a quantity called the *center of velocity*:

$$\vec{V}(t) = \frac{d}{dt}\vec{R}(t) = \frac{m_1 \vec{v}_1(t) + m_2 \vec{v}_2(t)}{m_1 + m_2} .$$

Taking the time derivative of the center of velocity gives something interesting:

$$\begin{aligned} \frac{d^2}{dt^2}\vec{R}(t) &= \frac{m_1 \left(d\vec{v}_1(t)/dt\right) + m_2 \left(d\vec{v}_2(t)/dt\right)}{m_1 + m_2} \\ &= \frac{\vec{F}_{12} + \vec{F}_{21}}{m_1 + m_2} = \frac{\vec{F}_{12} - \vec{F}_{12}}{m_1 + m_2} = 0 \end{aligned}$$

Evidently, the second derivative of the center of mass is precisely zero because $\vec{F}_{12} = -\vec{F}_{21}$, regardless of how the forces act. This means that two bodies, while free to move individually, are not accelerating anywhere as a group. Moreover, this result proves that the center of velocity $\vec{V}$ is a constant $\vec{V}_0$.

Relative Displacement

If the distance separating the two bodies is r, define a vector

$$\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$$

with $|\vec{r}| = r$, capturing the relative displacement between the two.

Listing this with the center of mass $\vec{R}(t)$, we have a system of two equations that can be solved for $\vec{r}_1(t)$, $\vec{r}_2(t)$ separately: (We know everything is a function of t by now, so drop the extra notation.)

$$\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \end{aligned}$$

Reduced Mass

From the equations above, multiply through by m_1, m_2, respectively, and take two time derivatives:

$$\begin{aligned} m_1 \frac{d^2\vec{r}_1}{dt^2} &= m_1 \cancel{\frac{d^2\vec{R}}{dt^2}} + \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\vec{r}}{dt^2} \\ m_2 \frac{d^2\vec{r}_2}{dt^2} &= m_2 \cancel{\frac{d^2\vec{R}}{dt^2}} - \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\vec{r}}{dt^2} \end{aligned}$$

These results say the same thing, as the left sides are $\vec{F}_{12}$, $\vec{F}_{21}$, respectively, and the right sides differ by the proper negative sign.

Evidently, we have

$$\vec{F}_{12} = \frac{m_1 m_2}{m_1 + m_2}\frac{d^2\vec{r}}{dt^2} .$$

That is, there is only one force equation to worry about, and thus one position to worry about if we work with the relative displacement vector $\vec{r}$ rather than two explicit position vectors $\vec{r}_{1,2}$.

The price we pay is the mass term became a mess. This group of symbols is called the *reduced mass*:

$$m_* = \frac{m_1 m_2}{m_1 + m_2}$$

Representing the effective mass of the total system as m_*, the two-body problem is summarized in one equation:

$$\vec{F}_{12} = m_* \frac{d^2\vec{r}}{dt^2} = m_* \vec{a}$$

A handy identity involving the reduced mass, somewhat reminiscent of resistors in parallel, goes as:

$$\frac{1}{m_*} = \frac{1}{m_1} + \frac{1}{m_2}$$

Linear Momentum

The linear momentum $\vec{p} = m_* \vec{v}$ is not constant in the two-body system. To see this, take a derivative and simplify using Newton's third law:

$$\begin{aligned} \frac{d\vec{p}}{dt} &= m_* \left(\frac{d\vec{v}_1}{dt} - \frac{d\vec{v}_2}{dt}\right) = m_* \left(\frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2}\right) \\ \frac{d\vec{p}}{dt} &= \vec{F}_{12} \end{aligned}$$

1.2 Angular Momentum

Alongside the notion of forces, we'll need to put the ideas of angular momentum to use. In particular, we can show that the angular momentum of the two-body system is constant, and find what it is.

By definition, the angular momentum $\vec{L}$ of the two-body system reads

$$\vec{L} = m_* \vec{r} \times \vec{v} ,$$

where $\vec{r}$ is the relative displacement vector, and $\vec{v}$ is its time derivative. Now calculate the time derivative of $\vec{L}$:

$$\begin{aligned} \frac{d}{dt}\vec{L} &= m_* \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) \\ &= m_* \left(\vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) \\ &= \vec{r} \times \vec{F} \end{aligned}$$

For the remaining cross product to vanish, we go back to Newton's original assumption that

$$\vec{F} = F(r)\,\hat{r}\ ,$$

which means the force vector and the displacement vector are parallel. Using this, we see that the derivative of $\vec{L}$ resolves to zero.

Without knowing the exact motion of the two-body system, we can still write a formula for the angular momentum. For some $r(t)$, $\theta(t)$, we have, in polar coordinates:

$$\begin{aligned}\vec{r} &= r\,\hat{r}\\ \vec{v} &= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\end{aligned}$$

Remembering $\hat{r}\times\hat{r}$ is zero, we then have

$$\vec{L} = m_* r^2 \frac{d\theta}{dt}\left(\hat{r}\times\hat{\theta}\right)\ .$$

The angular momentum is a constant vector that points perpendicular to the plane of motion. We take its magnitude

$$L = m_* r^2 \frac{d\theta}{dt}$$

as a constant of motion in the two-body system.

It's easy to show that the position vector and the angular momentum vector are always perpendicular. Starting with the definition of $\vec{L}$, project $\vec{r}$ into both sides:

$$\vec{r}\cdot\vec{L} = m_*\vec{r}\cdot(\vec{r}\times\vec{v})\ ,$$

and then make use of the triple product:

$$\vec{r}\cdot\vec{L} = m_*\vec{v}\cdot(\vec{r}\times\vec{r}) = 0$$

1.3 Inverse-Square Acceleration

We've made it this far without knowing the magnitude gravitational force $F(r)$, although we have harmlessly assumed that gravity acts in a straight line. Here we will derive the proper gravitational force by using Kepler's first law as a starting point.

In detail, Kepler noticed that the orbit of any planet around the sun takes an elliptical form described by

$$r(\theta) = \frac{r_0}{1+e\cos(\theta)}\ ,$$

where e is the *eccentricity* of the orbit, and r_0 is a positive characteristic length. Notice that $r(\theta)$ as written places the origin (the sun) at the *right* focus of the ellipse. Reverse the sign on the cosine term for the sun at the left focus.

To really get started, take the time derivative of the (constant) angular momentum of the two-body system:

$$0 = \frac{dL}{dt} = m_* r\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)$$

Perhaps you recognize the parenthesized term as being identically the $\hat{\theta}$-component of the acceleration vector in polar coordinates. In terms of L, the acceleration vector is

$$\vec{a} = \left(\frac{d^2r}{dt^2} - \frac{L^2}{m_*^2 r^3}\right)\hat{r} + \frac{1}{m_* r}\cancel{\left(\frac{dL}{dt}\right)}\hat{\theta}\ .$$

We need the polar form of the ellipse to calculate d^2r/dt^2. For this, we find, after simplifying,

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{r_0}{1+e\cos(\theta)}\right) = \frac{L}{m_* r_0}e\sin(\theta)\ ,$$

and keep going to the second derivative:

$$\frac{d^2r}{dt^2} = \frac{L^2}{m_*^2 r^2 r_0}e\cos(\theta) = \frac{L^2}{m_*^2 r^2}\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

The full acceleration vector then reads

$$\vec{a} = \frac{L^2}{m_*^2 r^2}\left(\cancel{\frac{1}{r}} - \frac{1}{r_0} - \cancel{\frac{1}{r}}\right)\hat{r}\ ,$$

which simplifies nicely:

$$\vec{a} = \frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$

This finally reveals the nature of $F(r)$. The r-dependence is present as $-1/r^2$, hence the name inverse-square acceleration.

Going back to the equations that led to the reduced mass, i.e.

$$\begin{aligned}m_1\frac{d^2\vec{r}_1}{dt^2} &= m_*\frac{d^2\vec{r}}{dt^2}\\ m_2\frac{d^2\vec{r}_2}{dt^2} &= -m_*\frac{d^2\vec{r}}{dt^2}\ ,\end{aligned}$$

we can solve for the absolute acceleration of each body:

$$\begin{aligned}\vec{a}_1 &= \frac{m_*}{m_1}\vec{a}\\ \vec{a}_2 &= \frac{-m_*}{m_2}\vec{a}\end{aligned}$$

Eliminate $\vec{a}$ between the two equations to recover Newton's third law:

$$m_1\,\vec{a}_1 + m_2\,\vec{a}_2 = 0$$

1.4 Universal Gravitation

Enough ground work has been done to push toward Newton's universal law of gravitation.

Recall the absolute acceleration of each body $\vec{a}_1$, $\vec{a}_2$, and replace the reduced mass m_* and acceleration $\vec{a}$ with expanded forms:

$$\vec{a}_1 = \left(\frac{m_2}{m_1+m_2}\right)\frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$
$$\vec{a}_2 = \left(\frac{-m_1}{m_1+m_2}\right)\frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$

Multiply each equation through by m_1, m_2, respectively to turn accelerations into forces:

$$\vec{F}_{12} = \left(\frac{m_1 m_2}{m_1+m_2}\right)\frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$
$$\vec{F}_{21} = \left(\frac{-m_1 m_2}{m_1+m_2}\right)\frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$

Newton decided to introduce a new proportionality constant G, named after 'gravity', to wrangle all of the constants inherent to the two-body, system such that

$$G = \left(\frac{1}{m_1+m_2}\right)\frac{L^2}{m_*^2 r_0} \ .$$

Of course, both force equations are saying the same thing due to Newton's third law, thus we write Newton's law of universal gravitation:

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\,\hat{r}$$

Note that the force vector bears the 12-subscript and not the other way around. The subscript is often omitted because the unit vector $\hat{r}$ has an implied 12-subscript that goes back to the definition of $\vec{r}$.

While this calculation was set up in the context of planetary motion, note that the gravitational force is in fact *universal*, which is to say that every pair of particles in the universe obeys the same law.

Problem 1

Show that

$$G = \frac{L^2}{m_* m_1 m_2 r_0} \ .$$

1.5 Equations of Motion

With the law of universal gravitation on hand, we should be able to run the analysis in reverse by starting with $\vec{F}_{12}$ and finishing with the shape of the ellipse, along with all other allowed possibilities.

Acceleration

Use

$$L = m_* r^2 \frac{d\theta}{dt}$$

to eliminate $1/r^2$ in the force vector:

$$\vec{F}_{12} = -G m_1 m_2 \frac{m_*}{L}\frac{d\theta}{dt}\,\hat{r}$$

Also replace $\vec{F}_{12}$ to keep simplifying

$$m_1\,\vec{a}_1 = m_*\,\vec{a} = -G m_1 m_2 \frac{m_*}{L}\frac{d\theta}{dt}\,\hat{r} \ ,$$

and solve for the relative acceleration:

$$\vec{a} = -G\frac{m_1 m_2}{L}\frac{d\theta}{dt}\,\hat{r}$$

Velocity

To proceed, replace the acceleration vector as the derivative of the relative velocity by $\vec{a} = d\vec{v}/dt$. Also replace $\hat{r}$ via $-\hat{r} = d\hat{\theta}/d\theta$ to get

$$\frac{d\vec{v}}{dt} = G\frac{m_1 m_2}{L}\frac{d\theta}{dt}\frac{d\hat{\theta}}{d\theta} \ ,$$

simplifying with the chain rule to:

$$d\vec{v} = G\frac{m_1 m_2}{L}\,d\hat{\theta}$$

Integrate both sides of the above to get a vector equation for the velocity

$$\vec{v}\,(t) = G\frac{m_1 m_2}{L}\,\hat{\theta}\,(t) + \vec{v}_0 \ ,$$

where $\vec{v}_0$ is the integration constant. Letting $\theta = 0$ correspond with the positive x-axis, it must be that $\vec{v}_0 = v_0\,\hat{y}$.

Position

To goal is get hold of a position equation $r\,(\theta)$. To get closer, calculate the full angular momentum vector:

$$\begin{aligned}\vec{L} &= m_*\vec{r}\times\vec{v} \\ &= m_*\vec{r}\times\left(G\frac{m_1 m_2}{L}\,\hat{\theta} + v_0\,\hat{y}\right) \\ &= m_* G\frac{m_1 m_2}{L} r\left(\hat{r}\times\hat{\theta}\right) + m_* v_0 r\,(\hat{r}\times\hat{y})\end{aligned}$$

To handle the cross products, note that

$$\left|\hat{r}\times\hat{\theta}\right| = 1$$
$$\left|\hat{r}\times\hat{y}\right| = \left|\cos\left(\theta\right)\right| \ ,$$

and we can work with just magnitudes:

$$L = m_* G \frac{m_1 m_2}{L} r + m_* v_0 r \cos(\theta)$$

To simplify, use

$$G = \frac{L^2}{m_* m_1 m_2 r_0} ,$$

and work to isolate r, arriving at

$$r_0 = r \left(1 + r_0 \frac{m_* v_0}{L} \cos(\theta)\right) ,$$

and finally get

$$r(\theta) = \frac{r_0}{1 + (m_* r_0 v_0 / L) \cos(\theta)} .$$

With $r(\theta)$ known, the position vector is straightforwardly written:

$$\vec{r} = r(\theta)\, \hat{r}$$

Eccentricity

Comparing the above to the general form of a conic section in polar coordinates, we pick out the eccentricity to be

$$e = \frac{m_* r_0 v_0}{L} .$$

Circular orbits arise from the special case $v_0 = 0$. Another special case is $e = 1$ for a parabolic trajectory. For all $e < 1$, the orbit is strictly an ellipse. For $e > 1$, the path (also technically an orbit) is hyperbolic.

This surely nails the case shut for Kepler's first law. All results reinforce the fact that planetary orbits occur on ellipses with the sun at a focus.

The eccentricity can be expressed by a variety of combinations of terms. For a version without L, one can find

$$e = \frac{\sqrt{r_0}\, v_0}{\sqrt{G(m_1 + m_2)}} ,$$

or, if you need to get rid of r_0:

$$e = \frac{v_0 L}{G m_1 m_2}$$

In terms of the eccentricity, the equations of motion can be simplified. For the position, we simply have

$$r(\theta) = \frac{r_0}{1 + e \cos(\theta)} .$$

For the velocity and acceleration, shuffle the constants around to establish

$$\frac{G m_1 m_2}{L} = \frac{v_0}{e} ,$$

which is only defined for non-circular orbits. With this, we have:

$$\vec{v} = v_0 \left(\frac{\hat{\theta}}{e} + \hat{y}\right)$$

$$\vec{a} = \frac{-v_0}{e} \frac{d\theta}{dt}\, \hat{r}$$

1.6 Runge-Lorenz Vector

The two-body problem exhibits conservation of angular momentum via the constant vector $\vec{L}$. There is, in fact, another constant vector of motion lurking about called the *Runge-Lorenz* vector

$$\vec{Z} = \vec{v} \times \vec{L} - G m_1 m_2\, \hat{r} .$$

Constant of Motion

Take a time derivative to prove $\vec{Z}$ is constant:

$$\begin{aligned} \frac{d}{dt}\vec{Z} &= \frac{d}{dt}\left(\vec{v} \times \vec{L}\right) - G m_1 m_2 \frac{d\hat{r}}{dt} \\ &= \frac{d\vec{v}}{dt} \times \vec{L} + \vec{v} \times \cancel{\frac{d\vec{L}}{dt}} - G m_1 m_2 \frac{d\hat{r}}{dt} \end{aligned}$$

Keep simplifying with

$$\frac{d\vec{v}}{dt} = \frac{1}{m_*}\vec{F} = -G \frac{m_1 m_2}{m_* r^2}\, \hat{r} ,$$

and also with $\vec{L} = m_* \vec{r} \times \vec{v}$, so we have

$$\frac{d}{dt}\vec{Z} = G m_1 m_2 \left(-\frac{\hat{r} \times (\vec{r} \times \vec{v})}{r^2} - \frac{d\hat{r}}{dt}\right) .$$

Replace $\vec{v}$ with its polar expression and note that

$$\vec{r} \times \vec{v} = \vec{r} \times \left(\frac{dr}{dt}\, \hat{r} + r \frac{d\theta}{dt}\, \hat{\theta}\right) = r^2 \frac{d\theta}{dt} \left(\hat{r} \times \hat{\theta}\right) ,$$

and furthermore, using the BAC-CAB formula:

$$\hat{r} \times (\vec{r} \times \vec{v}) = r^2 \frac{d\theta}{dt}\, \hat{r} \times \left(\hat{r} \times \hat{\theta}\right) = -r^2 \frac{d\theta}{dt}\, \hat{\theta}$$

Summarizing, we find

$$\frac{d}{dt}\vec{Z} = G m_1 m_2 \left(\frac{d\theta}{dt}\, \hat{\theta} - \frac{d\hat{r}}{dt}\right) = 0$$

as proposed.

Perigee

With $\vec{Z}$ known to be constant, we're free to evaluate it at any point along the trajectory. Choose a point $\vec{r}_p = r_p\,\hat{x}$ that has $\vec{v}_p \cdot \vec{r}_p = 0$, called a *perigee*:

$$\begin{aligned}\vec{Z} &= \vec{v}_p \times \vec{L} - Gm_1m_2\,\hat{x} \\ &= \vec{v}_p \times (m_* r_p\,\hat{x} \times \vec{v}_p) - Gm_1m_2\,\hat{x} \\ &= \left(m_* r_p v_p^2 - Gm_1m_2\right)\hat{x}\end{aligned}$$

At the perigee, the velocity v_p is momentarily equal to $r_p\,d\theta/dt$, which we'll call

$$v_p = r_p\,\omega_p\;.$$

In the same notation, the angular momentum is

$$L = m_* r_p^2 \omega_p = m_* r_p v_p\;,$$

and the vector $\vec{Z}$ becomes

$$\vec{Z} = \left(\frac{L^2}{m_* r_p} - Gm_1m_2\right)\hat{x}\;.$$

We can keep simplifying. Replace L^2 with the expression involving G:

$$\vec{Z} = Gm_1m_2\left(\frac{r_0}{r_p} - 1\right)\hat{x}\;.$$

The ratio r_0/r_p can be calculated by setting $\theta = 0$ in the polar equation $r(\theta)$ for a conic section:

$$r_p = \frac{r_0}{1 + (r_0 m_* v_0/L)} = \frac{r_0}{1+e}$$

Finally, the simplest form for $\vec{Z}$ is:

$$\vec{Z} = Gm_1m_2\,e\,\hat{x}$$

What $\vec{Z}$ tells us, apart from containing all information about the trajectory, is that all gravitational trajectories contain at least one perigee, defining the x-axis of the coordinate system about which the motion is symmetric.

Apogee

The perigee is also known as the nearest distance attained between the two bodies. For an elliptical orbit or hyperbolic orbit, the perigee is given by $\theta = 0$:

$$r_{\text{perigee}} = \frac{r_0}{1+e}$$

For elliptical orbits, there is also the notion of *apogee*, which is the furthest distance attained between the two bodies. Set $\theta = \pi$ to find

$$r_{\text{apogee}} = \frac{r_0}{1-e}$$

<u>Problem 2</u>

Take derivatives of

$$r(\theta) = \frac{r_0}{1 + e\cos(\theta)}$$

to verify the locations of the perigee and apogee.

<u>Problem 3</u>

Show that:

$$e = \left|\frac{r_p - r_a}{r_p + r_a}\right|$$

Conic Trajectory

The Runge-Lorenz vector

$$\vec{Z} = \vec{v} \times \vec{L} - Gm_1m_2\,\hat{r}\;,$$

together with its particular expression

$$\vec{Z} = Gm_1m_2\,e\,\hat{x}$$

can be used together to quickly recover the polar equation for conic sections by projecting the position vector across the equation and simplifying:

$$\begin{aligned}\vec{r}\cdot\vec{Z} &= \vec{r}\cdot\left(\vec{v}\times\vec{L}\right) - Gm_1m_2\,\vec{r}\cdot\hat{r} \\ rZ\cos(\theta) &= \vec{L}\cdot(\vec{r}\times\vec{v}) - Gm_1m_2 r \\ Gm_1m_2 re\cos(\theta) &= \frac{L^2}{m_*} - Gm_1m_2 r\end{aligned}$$

Now solve for $r(\theta)$ and simplify more:

$$\begin{aligned}r(\theta) &= \left(\frac{L^2}{Gm_1m_2m_*}\right)\frac{1}{1+e\cos(\theta)} \\ &= \frac{r_0}{1+e\cos(\theta)}\end{aligned}$$

Relation to Ellipse

An ellipse is classified by two perpendicular lengths we know as the semi-major and semi-minor axes, denoted a, b, respectively. By studying the ellipse, it's straightforward to show that

$$a = \frac{r_0}{1-e^2}\;,$$

along with

$$b = \frac{r_0}{\sqrt{1-e^2}}\;,$$

so that

$$\frac{b}{a} = \sqrt{1-e^2}$$

and

$$r_0 = \frac{b^2}{a}\;.$$

The a-equation can be derived by taking the difference between $r(0)$ and $r(\pi)$, i.e. the distance between the perigee and apogee. This pair of points defines the distance $2a$.

The b-equation can be derived by finding r_*, θ_* that correspond to $y = b$, the highest point on the ellipse:

$$\begin{aligned} 0 &= \frac{d}{d\theta}\left(y(\theta)\right) = \frac{d}{d\theta}\left(r(\theta)\sin(\theta)\right)\Big|_{r_*,\theta_*} \\ &= \left(\frac{r_0\, e\sin^2(\theta)}{\left(1+e\cos(\theta)\right)^2} + \frac{r_0\cos(\theta)}{1+e\cos(\theta)}\right)\Bigg|_{r_*,\theta_*} \\ &= \frac{r_*^2}{r_0}\left(e+\cos(\theta_*)\right) \end{aligned}$$

Evidently, we have

$$\cos(\theta_*) = -e\ .$$

Taking this with

$$\begin{aligned} b &= r_*\sin(\theta_*) \\ r_* &= \sqrt{e^2a^2+b^2} \end{aligned}$$

is enough to finish the job. Note that similar relationships can be drawn for hyperbolic orbits.

Problem 4

Show that $\vec{r}\cdot\vec{v} = 0$ is true only at the apogee and perigee.

Dimensionless Runge-Lorenz

The Runge-Lorenz vector can be made into a dimensionless vector $\vec{\epsilon}$ by dividing Gm_1m_2 across the whole equation

$$\vec{\epsilon} = \frac{\vec{v}\times\vec{L}}{Gm_1m_2} - \hat{r}\ ,$$

where by the properties of $\vec{Z}$, we also know

$$\vec{\epsilon} = e\,\hat{x}\ .$$

With this setup, write

$$\hat{r} + e\,\hat{x} = \frac{\vec{v}\times\vec{L}}{Gm_1m_2}\ ,$$

and then project $\vec{r}$ into each side to recover the equation of a conic section:

$$r\left(1+e\cos(\theta)\right) = \frac{\vec{r}\cdot\left(\vec{v}\times\vec{L}\right)}{Gm_1m_2} = r_0$$

1.7 Kepler's Laws

We spent a good effort developing the nature of gravitational orbits, and it would be difficult to imagine doing this without all of the modern advantages, particularly calculus and vectors. Somehow, Kepler was able to find enough pattern in sixteenth-century astronomical data to work out three correct laws of planetary motion. The data itself was recorded by astronomer Tycho Brahe over a span of at least thirty years.

Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the sun at a focus.

This law we know very well by now, as did Newton. For the sun at the right focus (reverse the sign for the left focus), a planetary orbit looks like

$$r(t) = \frac{r_0}{1+e\cos(\theta(t))}\ ,$$

where e is the eccentricity.

Law of Equal Areas (1609)

A line drawn between the sun and the planet sweeps out equal areas in equal times.

This is an amazing thing to notice from looking at charts of numbers. It turns out that this law is actually stating the conservation of angular momentum, although Kepler wouldn't have known so.

To derive the law in familiar language, recall the setup for the area integral in polar coordinates, particularly

$$A = \frac{1}{2}\int_{\theta_0}^{\theta_1} r^2\, d\theta\ .$$

In differential form, this same notion reads

$$dA = \frac{1}{2}r^2 d\theta\ .$$

Or, by the chain rule, we can also write

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}\ .$$

Notice, though, that $r^2 d\theta/dt$ is also present in the angular momentum

$$L = m_* r^2\frac{d\theta}{dt}\ ,$$

which can only mean

$$\frac{dA}{dt} = \frac{L}{2m_*}\ ,$$

thus dA/dt is constant. This is the literal mathematical statement of 'equal areas swept in equal times'.

Problem 5

For a body moving on a path $r = f(\theta)$ obeying Kepler's second law, show that the acceleration is:

$$\vec{a} = \frac{L^2}{m_* r^3}\left(\frac{f''(\theta)}{f(\theta)} - 2\left(\frac{f'(\theta)}{f(\theta)}\right)^2 - 1\right)\hat{r}$$

Harmonic Law (1618)

The square of the period of a planet is directly proportional to the cube of the semi-major axis of the orbit.

Years after his first two discoveries, Kepler discerned yet another relationship for linking the time scale of the orbit to its length scale. While Kepler only knew of the proportionality between the period T and the semi-major axis a, we can do better by finding the associated constant.

Integrate the area equation for a full period of the orbit:

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{L}{2m_*}\int_0^T dt = \frac{L}{2m_*}T$$

The area is simply πab, so we find

$$T = \pi ab \frac{2m_*}{L} \; .$$

Replace b using $b = a\sqrt{1-e^2}$, and eliminate L using

$$G = \frac{L^2}{m_* m_1 m_2 r_0} \; .$$

To deal with the r_0 term, recall $a = r_0/\left(1-e^2\right)$ and reason that

$$T = \frac{2\pi a^{3/2}}{\sqrt{G\left(m_1+m_2\right)}} \; .$$

1.8 Energy Considerations

With some fine details of planetary motion finished, it's worth pointing out that the notion of 'energy' was not used at all. To develop some of this now, recall that in one dimension, the force relates to the potential energy by

$$F = -\frac{d}{dx}U(x) \; .$$

Planetary motion, on the other hand, requires three dimensions to express the force, or two dimensions if we already know the plane of the motion. This is why the force is a vector:

$$\vec{F} = -\frac{G m_1 m_2}{r^2}\,\hat{r}$$

Notice, though, that the force is dependent on one spacial quantity, the length, which to say the force is effectively one-dimensional.

Gravitational Potential Energy

Since the gravitational force acts in strictly the radial direction, it stands to reason that the gravitational potential energy $U(r)$ relates to the force by:

$$\vec{F}(\vec{r}) = -\frac{d}{dr}\left(U(r)\right)\hat{r}$$

This is just like the one-dimensional Newton's law $F = -dU/dx$, except the force is a vector, balanced by $\hat{r}$ on the right.

To solve for $U(r)$, project $\hat{r}$ into both sides of the above to get

$$\frac{G m_1 m_2}{r^2} = \frac{d}{dr}\left(U(r)\right) \; ,$$

solved by:

$$U(r) = -\frac{G m_1 m_2}{r}$$

This is the total gravitational potential energy stored between the two masses m_1, m_2.

For a more formal definition, turn Newton's second law into a definite integral in the variable $d\vec{r}$ to get

$$\int_{r_0}^{r_1} \vec{F}(r)\cdot d\vec{r} = -\int_{r_0}^{r_1} \frac{d}{dr}U(r)\,\hat{r}\cdot d\vec{r} \, ,$$

where the integral on the right is redundant to the derivative, leaving $U(r)$ evaluated at the endpoints:

$$\int_{r_0}^{r_1} \vec{F}(r)\cdot d\vec{r} = -\left(U(r_1) - U(r_0)\right)$$

Set r_0 to infinity to recover the previous form.

Kinetic Energy

Containing two objects in total, the kinetic energy energy T of the two-body system is

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \; .$$

What we need, however, is to express the kinetic energy in terms of the relative velocity

$$\vec{v} = \vec{v}_1 - \vec{v}_2 \; .$$

Working out the algebra for this is left as an exercise, but the effort results in

$$T = \frac{1}{2}m_* v^2 + \frac{1}{2}\left(m_1+m_2\right)V_0^2 \; ,$$

where V_0 is the (constant) center of velocity of the whole system. It's harmless to set this term to zero.

Conservation of Energy

The total energy of the two-body system is the sum of the kinetic and the potential contributions:

$$E = T + U = \frac{1}{2} m_* v^2 - \frac{G m_1 m_2}{r}$$

As it turns out, the energy of the system is constant.

To prove this, begin with Newton's second law

$$\vec{F} = -\frac{G m_1 m_2}{r^2} \, \hat{r} \, ,$$

and project the velocity vector into each side:

$$\vec{v} \cdot \vec{F} = -\frac{G m_1 m_2}{r^2} \left(\vec{v} \cdot \hat{r} \right)$$

Replace $\vec{F}$ on the left and $\vec{v}$ on the right

$$m_* \left(\vec{v} \cdot \frac{d\vec{v}}{dt} \right) = -\frac{G m_1 m_2}{r^2} \left(\frac{dr}{dt} \, \hat{r} + r \frac{d\theta}{dt} \, \hat{\theta} \right) \cdot \hat{r} \, ,$$

which simplifies to

$$\frac{1}{2} m_* \frac{d}{dt} \left(\vec{v} \cdot \vec{v} \right) = -\frac{G m_1 m_2}{r^2} \frac{dr}{dt} \, .$$

(Note we didn't really need the polar expression for the velocity. The r-component of the velocity is always dr/dt.) The right side can be undone with the chain rule:

$$\frac{d}{dt} \left(\frac{1}{2} m_* v^2 \right) = \frac{d}{dt} \left(\frac{G m_1 m_2}{r} \right)$$

Finally, we have found

$$\frac{d}{dt} \left(T + U \right) = 0 \, ,$$

as expected.

Apocalypse Problem

Problem 6

If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the sun in a time which is $\sqrt{2}/8$ times the period of the planet's revolution.

Answer: Begin with the total energy of the system

$$-\frac{G m_1 m_2}{a} = \frac{1}{2} m_* \left(\frac{dr}{dt} \right)^2 - \frac{G m_1 m_2}{r\left(t\right)} \, ,$$

where a is the radius of the orbit. Solve for dr/dt to get

$$\frac{dr}{dt} = \sqrt{\frac{2G \left(m_1 + m_2 \right)}{a}} \sqrt{\frac{a}{r} - 1} \, ,$$

which can be separated into two equal integrals:

$$\int_a^0 \frac{dr}{\sqrt{a/r - 1}} = \sqrt{\frac{2G \left(m_1 + m_2 \right)}{a}} \int_0^{t_*} dt \, ,$$

where t_* is the answer we're after.

To solve the r-integral, choose the peculiar substitution

$$\begin{aligned} r &= a \cos^2 \left(\theta \right) \\ dr &= -2a \cos \left(\theta \right) \sin \left(\theta \right) d\theta \, , \end{aligned}$$

and the above reduces to

$$2a \int_{\pi/2}^{\pi} \cos^2 \left(\theta \right) d\theta = t_* \sqrt{\frac{2G \left(m_1 + m_2 \right)}{a}} \, .$$

The remaining θ-integral resolves to $\pi/4$. Solving for t_* gives

$$t_* = \frac{\sqrt{2}}{8} \left(\frac{2\pi a^{3/2}}{\sqrt{G \left(m_1 + m_2 \right)}} \right) = \frac{\sqrt{2}}{8} \, T \, ,$$

as stated. This is about 0.1768 years, or just over two months, supposing there are twelve months per year on that planet.

Inverse Cube Attraction

Problem 7

A particle released from rest a distance D from the origin is attracted by the force

$$F = -\frac{m k^2}{x^3} \, .$$

Show that the time required to fall to the origin is D^2/k. Hint:

$$v = \frac{dx}{dt} = k \sqrt{\frac{1}{x^2} - \frac{1}{D^2}}$$

1.9 Solid Sphere

We've taken on assumption (correctly) the shell theorem, which says a gravitational body with finite size can be treated as a point located at its center of mass.

With the shell theorem, we can calculate the gravitational force inside a uniform sphere of mass M and radius R at any distance $r < R$ from the center. A uniform sphere has the same density throughout, which we'll call λ:

$$\lambda = \frac{M}{4\pi R^3/3}$$

Force Inside Solid Sphere

At a distance $r < R$ from the center, according to the theorem, all of the sphere's mass that is located further from the center than r can be ignored. Only the sphere's mass obeying $r < R$ contributes to the force at distance r. This portion is called the *enclosed mass*. The enclosed mass is written $m(r)$, given by

$$m(r) = \lambda \frac{4}{3}\pi r^3 \, .$$

If the test particle has mass m_0, the magnitude of the force on the test particle is

$$F(r) = -\frac{Gm_0 m(r)}{r^2} = -\frac{Gm_0 M r}{R^3} \, .$$

Due to the r^3 factor that enters the numerator, the usual r^{-2} factor is replaced by r. The gravitational force inside a sphere grows linearly with distance until $r = R$.

As a vector, the force inside the solid sphere reads

$$\vec{F}(r) = -\frac{Gm_0 M}{R^3}\, \vec{r} \, .$$

Energy Inside Solid Sphere

The gravitational potential energy inside a solid sphere is not $U \propto -1/r$. To find the proper answer, first define

$$\lim_{r\to\infty} U(r) = 0$$

which assumes there is no energy when infinitely far from the solid sphere, assumed centered at the origin.

Starting from infinity, let a test particle of mass m_0 approach the solid sphere, eventually penetrating the its surface, stopping at r_1. The energy spent during approach is broken into two integrals:

$$U(r_1) = -\int_\infty^R \vec{F}_{\text{out}} \cdot d\vec{r} - \int_R^{r_1} \vec{F}_{\text{in}} \cdot d\vec{r} \, ,$$

where $\vec{F}_{\text{out}}$, $\vec{F}_{\text{int}}$ are the forces felt by m_0 outside and inside the sphere, respectively.

Carrying out the integrals and simplifying, one finds

$$U(r_1) = \frac{Gm_0 M}{2}\left(\frac{r_1^2 - 3R^2}{R^3}\right) \, .$$

Note that the special point $r_1 = R$ corresponds to being on the sphere's surface, and the potential energy takes a familiar form

$$U(R) = -\frac{Gm_0 M}{R} \, .$$

Self Energy

The *self energy* of a solid object is the energy required to assemble the object from parts initially at infinity. Since the gravitational force is attractive, we can expect the self energy due to gravity to be negative.

Demonstrating on a uniform solid sphere of radius R and mas M, write

$$dU = -\frac{Gm}{r} dm \, ,$$

where dU is the energy added when a particle of mass dm is added to the existing sphere of mass $m = m(r)$. The particle settles at radius r, and each particle does so until $r = R$.

Since the sphere has uniform density λ, we further write

$$dU = -\frac{Gm\lambda v}{r} \lambda dv \, ,$$

where $v = v(r)$ is the volume of the sphere. Turning the crank gives:

$$\begin{aligned} dU &= -G\lambda^2 \frac{4\pi r^3/3}{r} 4\pi r^2 dr \\ U &= -G(4\pi\lambda)^2 \frac{1}{3}\int_0^R r^4 \, dr \\ U &= -\frac{3}{5}\frac{GM^2}{R} \end{aligned}$$

Variable Density

If the gravitational force inside a solid phere is independent of position, determine the density $\lambda(r)$ of the sphere.

While there exist more rigorous ways to solve this problem, assume the density takes the form

$$\lambda(r) = Ar^n \, ,$$

where n is an integer and A is a unit-balancing constant. With this, all mass contained with a radius $r < R$ is given by

$$m(r) = A\frac{4\pi}{3} r^{3+n} \, .$$

Then, the magnitude of the force on the particle is

$$F = -\frac{4\pi G m_0}{3} r^{1+n} \, .$$

If the force is to be independent of r, we must have $n = -1$, or $\lambda(r) = A/r$.

1.10 Gravity Near Earth

Students of classical physics find out early that the force due to gravity near Earth's surface is a vector pointing straight down

$$\vec{F}_g = -mg\,\hat{y}\,,$$

with corresponding potential energy

$$U(y) = mgy\,,$$

where y is the height above the surface (or a location near it), and g is the local gravitation constant:

$$g = \frac{9.8\,\mathrm{m}}{\mathrm{s}^2}$$

On the other hand, we just went through all the pains of showing that the gravitational force is

$$\vec{F}(\vec{r}) = -\frac{Gm_1m_2}{r^2}\,\hat{r}$$

with potential energy

$$U(r) = -\frac{Gm_1m_2}{r}\,.$$

Clearly, these two pictures must be reconciled. To do so, let r be replaced by the quantity $R+y$, where R is a constant distance we'll take to be the radius of the Earth, and y is the effective height, approximately from sea level. What we assume throughout is that $y \ll R$.

Without loss of generality, we can assume all displacements are one dimensional and thus $\hat{r} = \hat{y}$. This identifies m_1 for the mass of the Earth, and m_2 for the mass of a test projectile.

With these restrictions, the force and energy become:

$$\vec{F}(y) = -\frac{Gm_1m_2}{(R+y)^2}\,\hat{y}$$
$$U(y) = -\frac{Gm_1m_2}{(R+y)}$$

Next apply binomial expansion to each denominator, particularly:

$$(R+y)^{-2} \approx \frac{1}{R^2} - \frac{2y}{R^3} + \frac{3y^2}{R^4} - \cdots$$
$$(R+y)^{-1} \approx \frac{1}{R} - \frac{y}{R^2} + \frac{y^2}{R^3} - \cdots$$

To first order, the above equations become

$$\vec{F}(y) \approx -\frac{Gm_1m_2}{R^2}\left(1 - \frac{2y}{R}\right)\hat{y}$$
$$U(y) \approx -\frac{Gm_1m_2}{R}\left(1 - \frac{y}{R}\right)\,.$$

We want the force equation to be constant, thus we see the quantity $2y/R$ must be negligible so the effective force at the surface is

$$\vec{F}_g = -\frac{Gm_1m_2}{R^2}\,\hat{y}\,.$$

The acceleration term is identically g:

$$g = \frac{Gm_{\mathrm{Earth}}}{R_{\mathrm{Earth}}^2}\,.$$

For the potential energy, we have

$$U(y) = U_0 + mgy\,,$$

where U_0 is the potential energy at $y = 0$, often defined to be zero, and the unscripted mass m is that of a test particle (not the Earth).

Note that the first-order potential term is maintained despite y/R being a very small number. The reason for this not only to recover the form mgy, but also that the first derivative must equal a constant, which is what we asked of the force.

Problem 8

A person's weight is F_0 at sea level. To first order in y, what is the person's weight 4km above sea level? Answer: $F_{4\,\mathrm{km}} = F_0\left(1 - 2\,(4\,\mathrm{km})\,/R\right)$

Escape Velocity

In a two-body system with gravity being the only force present, suppose we imparted an initial carefully-chosen *escape velocity* v_e along the line between the bodies such that the kinetic energy goes to zero as the separation becomes infinite.

As a two-body problem, we can apply conservation of energy to write

$$\frac{1}{2}m_* v_0^2 - \frac{Gm_1m_2}{d} = \frac{1}{2}m_* v^2 - \frac{Gm_1m_2}{r} = 0\,,$$

where d is the initial separation between the bodies. The total energy is zero by definition.

From the energy statement, we can easily solve for the escape velocity from a starting separation d:

$$v_e = \sqrt{\frac{2Gm_1m_2}{m_* d}} = \sqrt{\frac{2G(m_1+m_2)}{d}}$$

For the case of a small body escaping Earth, the above becomes

$$v_e \approx \sqrt{2gR_{\mathrm{Earth}}}\,.$$

Hole Through Earth

An object of mass m is dropped through a straight tube connecting two points on Earth's surface. Neglecting rotational effects and friction, what happens to the object?

Unless the hole is drilled through the center, which we will not assume, the gravitational force on the object resolves into two components - one component parallel to the tube, the other component perpendicular. The equilibrium position of the object is in the center of the tube where the parallel component of the force is zero.

Let $\vec{x}$ be the parallel displacement form equilibrium, and let $\vec{r}$ be the position vector from the center. The force on the object parallel to the tube is:

$$F_x = \vec{F}(r) \cdot \hat{x} = -\frac{GMm}{R^3}(\vec{r} \cdot \vec{x}) \ ,$$

or

$$m\frac{d^2x}{dt^2} = -\frac{GMm}{R^3}x = -\frac{gm}{R}x \ .$$

The above is the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

for a simple harmonic oscillator solved by

$$x(t) = A\cos(\omega t + \phi) \ ,$$

where A is the length of the tube and ϕ chooses the initial position of the object. The angular frequency ω is given by

$$\omega = \sqrt{\frac{g}{R}} \ .$$

1.11 Energy and Orbit

Parabolic Orbit

Suppose now that a two-body system has zero total energy

$$E = 0 \ .$$

but the motion is not strictly along the line connecting the two bodies. In this special case, the system is *always* at escape velocity. This does not mean the escape velocity is constant. The distance d is playing the role of r in the v_e equation.

To develop this, recall that the velocity for a parabolic orbit can be written

$$\vec{v} = v_0\left(\hat{\theta} + \hat{y}\right) \ ,$$

which means

$$v^2 = \vec{v} \cdot \vec{v} = 2v_0^2\frac{r_0}{r} \ .$$

Using the escape velocity in place of v allows us to write

$$\frac{2G(m_1+m_2)}{r} = 2v_0^2\frac{r_0}{r} \ ,$$

or

$$v_0^2 = \frac{G(m_1+m_2)}{r_0} \ .$$

Elliptical Orbit

Elliptical orbits are called *bound* orbits, and have negative total energy:

$$E < 0$$

Interestingly, if we take a parabolic orbit with $E = 0$ and subtract a little energy from the total (by some external means), then the parabola becomes an ellipse by having the second focus come in from infinity.

We ought to be able to prove the total energy is negative for an elliptical orbit. Start with the total energy

$$E = \frac{1}{2}m_* v^2 - \frac{Gm_1m_2}{r} \ ,$$

and substitute v^2 using

$$\vec{v} = \frac{v_0}{e}\left(\hat{\theta} + e\,\hat{y}\right) \ ,$$

which excludes the case of circles. Proceeding carefully, find

$$\begin{aligned} v^2 &= \frac{v_0^2}{e^2}\left(\frac{2r_0}{r} - 1 + e^2\right) \\ &= \frac{2Gm_1m_2}{m_*r} - \frac{Gm_1m_2}{m_*r_0}\left(1-e^2\right) \ , \end{aligned}$$

so the kinetic term is

$$E_{\text{kin}} = \frac{Gm_1m_2}{r} - \frac{Gm_1m_2}{2r_0}\left(1-e^2\right) \ .$$

The total energy sums the potential plus the kinetic, which happens to contain equal and opposite $1/r$-like terms, leaving just the constant:

$$E = \frac{-Gm_1m_2}{2r_0}\left(1-e^2\right) = \frac{-Gm_1m_2}{2a} \ ,$$

in terms of v_0,

$$E = -\frac{1}{2}m_* v_0^2\left(\frac{1-e^2}{e^2}\right) \ .$$

Hyperbolic Orbit

Hyperbolic orbits are called *unbound* orbits, and have positive total energy:

$$E > 0$$

The analysis of this situation follows exactly like the elliptical case. For the total energy, you can see $e > 1$ simply flips the sign to make

$$E = \frac{Gm_1 m_2}{2a} = \frac{1}{2} m_* v_0^2 \left(\frac{e^2 - 1}{e^2} \right) .$$

Circular Orbit

For circular orbits, we need to go back to the velocity equation

$$\vec{v} = \frac{Gm_1 m_2}{L} \hat{\theta} ,$$

which has no v_0-term.

The angular momentum is

$$L = m_* R^2 \frac{d\theta}{dt} = m_* a^2 \frac{2\pi}{T} ,$$

where T is the period of the orbit and R is the radius. Simplifying gives

$$L = m_* \sqrt{G \left(m_1 + m_2 \right) R} ,$$

and then the square of the velocity is:

$$v^2 = \frac{G \left(m_1 + m_2 \right)}{R}$$

The time derivative of $\vec{v}$ gives a familiar equation for the acceleration

$$\vec{a} = -\frac{Gm_1 m_2}{L} \frac{d\theta}{dt} \hat{r} ,$$

which for circular orbits simplifies to

$$\vec{a} = \frac{-v^2}{R} \hat{r} ,$$

as expected for circular motion in general.

The energy of a circular orbit is

$$E = \frac{1}{2} \frac{Gm_1 m_2}{R} - \frac{Gm_1 m_2}{R} = \frac{-Gm_1 m_2}{2R} ,$$

thus the kinetic energy is half the potential energy, and the total is negative.

Eccentricity and Orbit

Begin with the Runge-Lorenz vector and replace $\vec{L}$ using its definition:

$$\vec{Z} = \vec{v} \times (m_* \vec{r} \times \vec{v}) - Gm_1 m_2 \, \hat{r} ,$$

and square the whole equation:

$$\begin{aligned} \vec{Z} \cdot \vec{Z} &= \left| \vec{v} \times (m_* \vec{r} \times \hat{v}) \right|^2 \\ &\quad - 2Gm_1 m_2 \, \vec{v} \times (m_* \vec{r} \times \hat{v}) \cdot \hat{r} + G^2 m_1^2 m_2^2 \end{aligned}$$

For the first term on the right, notice $\vec{v}$ is perpendicular to $\vec{r} \times \vec{v}$, so

$$\left| \vec{v} \times (m_* \vec{r} \times \hat{v}) \right| = m_* r v^2 \left| \sin(\phi) \right| ,$$

where ϕ is the angle between $\vec{r}$ and $\vec{v}$.

For the second term, the scalar triple product can be rewritten

$$\vec{v} \times (m_* \vec{r} \times \hat{v}) \cdot \hat{r} = m_* \left(\vec{r} \times \vec{v} \right) \cdot \left(\hat{r} \times \vec{v} \right) .$$

The remaining vectors are parallel and the whole quantity simplifies to

$$\vec{v} \times (m_* \vec{r} \times \hat{v}) \cdot \hat{r} = m_* r v^2 \sin^2(\phi) .$$

Rewriting $\vec{Z} \cdot \vec{Z}$ with this in mind, we have

$$\begin{aligned} Z^2 &= \left(m_* r v^2 \right)^2 \sin^2(\phi) \\ &\quad - 2Gm_1 m_2 m_* r v^2 \sin^2(\phi) + G^2 m_1^2 m_2^2 , \end{aligned}$$

or

$$\frac{Z^2}{G^2 m_1^2 m_2^2} = 1 + \sin^2(\phi) \left(q^2 - 2q \right) ,$$

where

$$q = \frac{m_* r v^2}{Gm_1 m_2} .$$

simplifying this further gives

$$\frac{Z^2}{G^2 m_1^2 m_2^2} = \cos^2(\phi) + \sin^2(\phi) \left(1 - q \right)^2$$

Finally, note that the left size is actually the square of the eccentricity, giving, after restoring q:

$$e^2 = \cos^2(\phi) + \sin^2(\phi) \left(1 - \frac{r v^2}{G \left(m_1 + m_2 \right)} \right)^2$$

This is an enlightening result. For $\phi = 0$ the motion is purely radial and uninteresting. For all other cases, we see the combination of variables being suspiciously like to the escape velocity. Swapping this in gives

$$e^2 = \cos^2(\phi) + \sin^2(\phi) \left(1 - \frac{2v^2}{v_e^2} \right)^2$$

We see if $v = v_e$, then the eccentricity is precisely one, which is consistent with what we know of parabolic orbits. Similarly we see the cases $v < v_e$ and $v > v_e$ give $e < 1$ and $e > 1$ respectively, which is the signature of elliptic and hyperbolic orbits. A circular orbit has $\phi = \pi/2$.

1.12 Shell Theorem

Newton's law of gravitation tells us that every particle in the universe is trying to pull every other particle toward itself with a force proportional to the masses involved and inversely proportional to the square of the separation, and this is duly used to calculate the force onto planets, moons, satellites, and so on.

Using triple integration and spherical coordinates, something Newton didn't have, we finally address an assumption made early in gravitational analysis, namely *why* we're allowed to represent voluminous objects as single points located at the center of mass. This is called the shell theorem, and entails two important proofs.

Outside a Sphere

Consider a solid sphere of radius R, total mass M, and uniform density λ. Also let there be a test particle of mass m somewhere in space. Without loss of generality, place the test particle on the z-axis at the point $\vec{D} = D\,\hat{z}$. The length D is the distance from the test particle to the center of the sphere.

In order to 'properly' calculate the gravitational attraction between the test mass and the sphere, a volume integral over there entire sphere must be calculated. Choose any element of volume dV inside the sphere at location $\vec{r}$, which is located distance r from the center, at an angle θ from the z-axis.

Let vector $\vec{q}$ denote the line connecting $\vec{D}$ to $\vec{r}$ such that

$$\vec{r} + \vec{q} = D\,\hat{z} \,,$$

and also let α be the angle between $\hat{z}$ and $\hat{q}$. From the law of cosines, we can say:

$$q^2 = r^2 + D^2 - 2rD\cos(\theta)$$
$$r^2 = q^2 + D^2 - 2qD\cos(\alpha)$$

The total force on the test particle is the vector $\vec{F}$. However, due to the ϕ-symmetry of this picture, only the z-component of the force will have a net effect on the particle. All xy-components cancel equally and oppositely:

$$F = \int_{\mathcal{D}} d\vec{F}\cdot\hat{z} = \int\int\int_{\text{volume}} dF\cos(\alpha)$$

The differential force is

$$dF = \frac{-Gm}{q^2}dm \,,$$

where dm is the mass of the differential volume element influencing the test particle. The mass term can be replaced using the density

$$\frac{dm}{dV} = \frac{M}{4\pi R^3/3} = \lambda \,,$$

where it is appropriate to replace dV with the volume element in spherical coordinates.

The force integral now is

$$F = -Gm\lambda \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{\cos(\alpha)}{q^2} r^2\sin(\theta)\,dr\,d\theta\,d\phi \,,$$

which, after substituting and simplifying a bit, becomes:

$$F = -Gm\lambda\frac{2\pi}{2D}\int_0^{\pi}\int_0^{R} \left(\frac{1}{q} + \frac{D^2 - r^2}{q^3}\right) r^2\sin(\theta)\,dr\,d\theta$$

Perform implicit differentiation on the q^2 equation to find, remembering r and θ are independent,

$$q\,dq = rD\sin(\theta)\,d\theta \,,$$

and rewrite the integral with the intent of integrating over r last. Make sure you know why the limits are now changed:

$$F = -Gm\lambda\frac{\pi}{D^2}\int_0^{R}\int_{(D-r)}^{(D+r)} \left(1 + \frac{D^2 - r^2}{q^2}\right) r\,dq\,dr$$

The whole q-integral treats r as a constant and resolves to $4r$, so

$$F = -Gm\lambda\frac{\pi}{D^2}\int_0^{R} 4r^2\,dr \,,$$

and the r-integral is elementary. Simplifying everything gives

$$F = -Gm\left(\frac{3M}{4\pi R^3}\right)\frac{\pi}{D^2}\frac{4}{3}R^3 = \frac{-GMm}{D^2} \,.$$

Conveniently, the force acts as if *all* of its mass were concentrated at the center. This result is also true in general, where the notion of 'center' means center of mass, not necessarily the center of the volume.

Inside a Shell

Another interesting question that arises in the course of studying gravity is, what does it feel like inside a hollow uniform shell? To pursue this question, suppose we have a thin spherical shell of radius R and thickness $2a$ that is much less than R, and the test particle is inside anywhere within the shell.

This setup borrows all of the geometry from the previous setup, except this time we have $D < R$, which is the important part. Setting up the same integral and doing the same simplifications, we can jump to

$$F = -Gm\lambda \frac{\pi}{D^2} \int_{R-a}^{R+a} \int_{(r-D)}^{(D+r)} \left(1 + \frac{D^2 - r^2}{q^2}\right) r\, dq\, dr \,.$$

Most notably, the lower integration in the q-integral is swapped to accommodate $D < R$. This causes the q-integral to resolve to zero, and we find

$$F = 0$$

inside the shell.

2 Central Potential

The whole apparatus for studying planetary motion can be grown from the gravitational potential energy

$$U(r) = -\frac{Gm_1 m_2}{r}$$

of a two-body system. The plan now is to develop the theory while assuming as little as possible about $U(r)$.

Two-Body Analysis

As a two-body system, we still deal with the center of mass

$$\vec{R}(t) = \frac{m_1 \vec{r}_1(t) + m_2 \vec{r}_2(t)}{m_1 + m_2} \,,$$

whose time detivative is the center of velocity $\vec{V}(t)$. Two time derivatives of $\vec{R}(t)$ yields the center of acceleration, which is always zero by Newton's third law:

$$\frac{d^2}{dt^2}\vec{R}(t) = 0$$

In terms of $\vec{R}$, the absolute position of each body is

$$\begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{m_1 + m_2}\,\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{m_1 + m_2}\,\vec{r} \,, \end{aligned}$$

and we define the relative displacement

$$\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$$

to deal with one variable instead of two. The time derivative of the relative displacement is the relative velocity $\vec{v}(t)$.

In terms of $\vec{r}(t)$, Newton's second law takes special form

$$\vec{F}_{12} = m_* \frac{d^2\vec{r}}{dt^2} \,,$$

where

$$m_* = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass.

Energy Conservation

The total energy in the two-body system is the sum of a kinetic term and a potential term. For the kinetic energy we have

$$T = \frac{1}{2} m_* v^2 + \frac{1}{2}(m_1 + m_2) V_0^2 \,,$$

where we take $V_0 = 0$ without loss of generality. For the potential energy we're stuck with just $U(r)$.

For the total energy, we write

$$E = T + U = \frac{1}{2} m_* v^2 + U(r) \,.$$

Take a single time derivative to find E to be constant

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} m_* \frac{d}{dt}\left(v^2\right) + \frac{d}{dt} U(r) \\ &= \frac{1}{2} m_* 2\vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{v} \cdot \frac{d}{d\vec{r}}(U(r)) \\ &= \vec{v} \cdot \left(m_* \frac{d\vec{v}}{dt} + \frac{d}{d\vec{r}}(U(r))\right) \,, \end{aligned}$$

because the parenthesized term is identically Newton's second law and resolves to zero.

Angular Momentum

The angular momentum

$$\vec{L} = m_* \vec{r} \times \vec{v}$$

doesn't depend on $U(r)$ at all, thus we recycle the constant of motion from planetary motion analysis:

$$L = m_* r^2 \omega \,,$$

where $\omega = d\theta/dt$. Since L is constant, we know the motion to be planar.

Polar Coordinates

In the polar coordiante system, the relative position, veclocity, and acceleration vectors read

$$\vec{r} = r\,\hat{r}$$
$$\vec{v} = \frac{dr}{dt}\,\hat{r} + r\omega\,\hat{\theta}$$
$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega\right)\hat{\theta}\;.$$

The $\hat{\theta}$-term in the acceleration is proportional to dL/dt and vanishes entirely, leaving a purely radial acceleration vector

$$\vec{a} = \left(\frac{d^2r}{dt^2} - \frac{L^2}{m_*^2 r^3}\right)\hat{r}\;.$$

2.1 Effective Potential

In terms of the angular momentum L, the velocity vector can be written

$$\vec{v} = \frac{dr}{dt}\,\hat{r} + \frac{L}{m_* r}\,\hat{\theta}\;,$$

meaning

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m_*^2 r^2}\;.$$

Feed this v^2 into the energy conservation statement:

$$E = \frac{1}{2}m_*\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2m_* r^2} + U(r)$$

The latter two terms constitute the *effective potential energy*

$$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2m_* r^2}\;,$$

also known as the *centrifugal potential energy*, the gradient of which is the centrifugal force.

In terms of the effective potential, the total energy goes back down to two terms, one with time dependence and one with spatial dependence:

$$E = \frac{1}{2}m_*\left(\frac{dr}{dt}\right)^2 + U_{\text{eff}}(r)$$

Problem 9

For the elliptical orbit of a planet, show for a given radius r_0 that:

$$|U_{\text{eff}}(r_0)| = \frac{-Gm_1m_2}{2r_0}$$

2.2 One-Dimensional Systems

In terms of the effective potential energy, the total energy is reduced to a one-dimensional system in the variable r. Taking this literally, let us study the generic one-dimensional system

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x)$$

to remove complications from planar motion.

Solving the above for the velocity, one writes

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

to establish the euation of motion

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_i}^{x_f}\frac{dx}{\sqrt{E - U(x)}}\;.$$

Time-Reversal Symmetry

The $\pm$ symbol in the equation of motion indicates *time-reversal symmetry* of the problem. Typically in one-dimensional systems, the *solution* to the equation of motion exhibits such symmetry, a stronger constraint than what we have. Supposing $x(t)$ is a solution to the equation of motion, time-reversal symmetry implies that $x_1(t) = x(t_0 - t)$ is also a solution that differs from original $x(t)$ by an integration constant.

For most configurations, there exists at least one turning point t_* at which the velocity goes to zero. We exploit time-reversal symmetry to write an exact time-reversed-and-shifted equation

$$x_1(t) = x(2t_* - t)\;.$$

Next, we note from function- and derivative matching that

$$x_1(t_*) = x(t_*)$$
$$\frac{d}{dt}x_1(x(t_*)) = -\frac{d}{dt}x(x(t_*)) = 0\;,$$

and so on for higher derivatives. We may then drop the 1-subscript to get

$$x(t) = x(2t_* - t)\;.$$

Shifting the above by t_*, the symmetric equation

$$x(t_* + t) = x(t_* - t)$$

emerges. In one dimension, the essence of time-reversal symmetry means that equations of motion are symmetric about turning points t_*.

Trapped Particle

Potential energy functions $U(x)$ that exhibit at least one local minimum can 'trap' a particle into an oscillatory pattern. Supposing x_i and x_f correspond to turning points in the motion, the oscillatory period is given by

$$T = \sqrt{2m} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U(x)}} \, .$$

The quantity $E - U(x)$ is always positive except at the turning points, at which the speed of the particle is instantaneously zero.

Harmonic Oscillations

In the vicinity of a local energy minimum at x_*, the first- and second-derivatives of $U(x_*)$ are

$$\frac{d}{dx} U(x_*) = 0$$
$$\frac{d^2}{dx^2} U(x_*) = \lambda > 0 \, ,$$

which allows $U(x)$ to be approximated by Taylor series:

$$U(x) \approx U(x_*) + \frac{1}{2} \lambda (x - x_*)^2$$

Applying Newton's second law, the corresponding equation of motion is

$$\frac{d^2}{dx^2} x(t) = -\frac{\lambda}{m} (x - x_*) \, ,$$

whose solution is known as the *harmonic oscillator*

$$x(t) = x_* + A \sin\left(\sqrt{\frac{\lambda}{m}} \, t - \phi_0 \right) \, .$$

The amplitude of oscillation is A, and the initial phase is contained in ϕ_0.

Unstable Equilibrium

An equilibrium point x_* exists at any local maximum of $U(x)$, however motion around such a point is unstable (non-oscillatory). To show this, reverse the sign on λ to arrive at the differential equation

$$\frac{d^2}{dx^2} x(t) = \frac{\lambda}{m} (x - x_*) \, ,$$

generally solved by

$$x(t) = x_* + A e^{\lambda t} + B e^{-\lambda t} \, .$$

That is, the particle is pulled away from x_* and rides $U(x)$ downhill.

2.3 Planar Orbits

Returning to the case of planar orbits, we can extrapolate all one-dimensional results to two dimensions by replacing $x \to r$ and acounting for $\theta(t)$ as a dynamic variable.

Equations of Motion

In terms of the total energy E, the time evolution of the two-body system is given by

$$t(r) = \pm \sqrt{\frac{m_*}{2}} \int_{r_i}^{r_f} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} \, .$$

An equation for $\theta(t)$ is attained by integrating the angular momentum $L = m_* r^2 \omega$ to write

$$\theta(t) = \theta_0 + \frac{L}{m_*} \int_0^t \frac{dt'}{(r(t'))^2} \, .$$

Switching to the r domain, the above is also written

$$\theta(r) = \pm \frac{L}{\sqrt{2m_*}} \int_{r_i}^{r_f} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}} \, .$$

Apogee and Perigee

Supposing there exits a time t_* at which the radius r reaches a turning point (i.e. $dr/dt = 0$), the corresponding point (r_*, θ_*) in the plane is called the *apogee* if r is at a maximum, and the *perigee* if r is at a minimum.

Solutions to the θ-equation occur in four explicit branches:

Apogee, $\theta > \theta_*$

$$\theta = \theta_0 + \frac{L}{\sqrt{2m_*}} \int_{r_*}^{r} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}}$$

Apogee, $\theta < \theta_*$

$$\theta = \theta_0 - \frac{L}{\sqrt{2m_*}} \int_{r_*}^{r} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}}$$

Perigee, $\theta > \theta_*$

$$\theta = \theta_0 + \frac{L}{\sqrt{2m_*}} \int_{r}^{r_*} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}}$$

Perigee, $\theta < \theta_*$

$$\theta = \theta_0 - \frac{L}{\sqrt{2m_*}} \int_{r}^{r_*} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}}$$

Evident in the above is the time-refersal symmetry about θ_*, namely

$$r(\theta_* - \theta) = r(\theta_* + \theta) \, .$$

Bounded Orbits

When the potential energy $U(x)$ contains a local minimum, a particle with a sufficiently low energy may become 'trapped' in the so-called *potential well.* Looking at the evolution of θ between two extreme points (perigee to apogee or vice-versa), we have

$$\theta = \frac{L}{\sqrt{2m_*}} \int_{r_p}^{r_a} \frac{dr/r^2}{\sqrt{E - U_{\text{eff}}(r)}} \, ,$$

which is some number that is not generally a rational fraction of π. That is, we find that orbits are bounded aren't necessarily repeated shapes, but may have a any (or an infinite) number of apogees and perigees. We will soon find there are two exceptions to this, where if the energy U has certain dependence on r, closed orbits are possible.

Circular Orbit

The circular orbit is characterized by $dr/dt = 0$, or equivalently $r(t) = r_0$. The equation of motion for $\theta(t)$ becomes trivial

$$\theta(t) = \theta_0 + \frac{Lt}{m_* r_0^2} \, ,$$

and the time T_0 required for $\theta(t) - \theta_0 = 2\pi$ corresponds to the period of the circular orbit:

$$T_0 = \frac{2\pi m_* r_0^2}{L}$$

As for the energy of a circular orbit, we have

$$E = \cancel{\frac{1}{2} m_* \left(\frac{dr}{dt}\right)^2} + \frac{L^2}{2m_* r^2} + U(r) \, ,$$

which remains constant. Taking an r-derivative at r_0 yields

$$0 = -\frac{L^2}{m_* r_0^3} + \frac{d}{dr}\left(U(r)\right)\Big|_{r_0} \, ,$$

or, in shorthand:

$$U'(r_0) = \frac{L^2}{m_* r_0^3}$$

Eliminate L from the above to write

$$U'(r_0) = m_* r_0 \omega^2 = \frac{m_* v_0^2}{r_0} \, ,$$

which is precisely the force 'felt' by an object constrained to uniform circular motion. (The force vector itself points to the center.)

We can eliminate L once more to express the period T of the circular orbit in terms of $U'(r_0)$:

$$T_0 = 2\pi \sqrt{\frac{m_* r_0}{U'(r_0)}}$$

Problem 10

Check that the circular orbit is described by

$$\frac{d}{dr}\left(U_{\text{eff}}(r)\right)\Big|_{r_0} = 0 \, .$$

Problem 11

Show that Kepler's law of equal areas hold for any central force, including straight-line motion.

3 Power Law Potential

Finally we must choose a particular form for the potentuaql energy $U(r)$, thus we'll pose a *central power law potential*

$$U(r) = -\frac{\Lambda}{r^\alpha} \, ,$$

where Λ is an arbitrary constant (positive or negative), and $\alpha = 2$ reproduces the case for planetary motion. In the general case, we use

$$U_{\text{eff}}(r) = -\frac{\Lambda}{r^\alpha} + \frac{L^2}{2m_* r^2}$$

for the effective potential energy.

3.1 Circular Orbit

For orbits that are have a nearly-circular radius r_0, we may approximate the effective potential energy via Taylor expansion in the vicinity $r \approx r_0$:

$$\begin{aligned} U_{\text{eff}}(r) = U_{\text{eff}}(r_0) &+ \frac{d}{dr}\left(U_{\text{eff}}(r)\right)\Big|_{r_0} (r - r_0) \\ &+ \frac{d^2}{dr^2}\left(U_{\text{eff}}(r)\right)\Big|_{r_0} \frac{(r - r_0)^2}{2!} + \cdots \end{aligned}$$

Stability

The first-order derivative term is identically zero by nature of the circular orbit. The second-order term must be done by brute force:

$$\begin{aligned} \frac{d^2}{dr^2} U_{\text{eff}}(r)\Big|_{r_0} &= \frac{d}{dr}\left(\frac{\Lambda\alpha}{r^{\alpha+1}} - \frac{L^2}{m_* r^3}\right)\Big|_{r_0} \\ &= -\frac{\Lambda\alpha(\alpha+1)}{r_0^{\alpha+2}} + \frac{3L^2}{m_* r_0^4} \end{aligned}$$

Now, from the first-order equation we learn

$$\frac{\Lambda\alpha}{r_0^{\alpha+1}} = \frac{L^2}{m_* r_0^3} \, ,$$

where eliminating L^2 in the second-order term now gives

$$\begin{aligned}\frac{d^2}{dr^2}U_{\text{eff}}(r)\bigg|_{r_0} &= -\frac{\Lambda\alpha(\alpha+1)}{r_0^{\alpha+2}} + \frac{3\Lambda\alpha}{r_0^{\alpha+2}} \\ &= \frac{\Lambda\alpha(2-\alpha)}{r_0^{\alpha+2}} \\ &= \frac{L^2}{m_* r_0^4}(2-\alpha) \ .\end{aligned}$$

We can also write the angluar frequency ω_0 in terms of the angular momentum via

$$\omega_0 = \frac{L}{m_* r_0^2} \ ,$$

and thus

$$\frac{d^2}{dr^2}U_{\text{eff}}(r)\bigg|_{r_0} = m_*\omega_0^2(2-\alpha)$$

Rewriting the Taylor expansion, we how have

$$U_{\text{eff}}(r) \approx U_{\text{eff}}(r_0) + m_*\omega_0^2(2-\alpha)\frac{(r-r_0)^2}{2!} \ .$$

For all $\alpha > 2$, near-circular orbits are unstable, meaning particles with high enough energy will slip away to $r \to \infty$, whereas particles with sufficiently low energy or sufficiently low radius will inevitably collapse to $r = 0$. For $\alpha < 2$, the system corresponds to a one-dimensional harmonic oscillator in r, thus near-circular orbits are stable. The angular frequency in the r-variable is given by

$$\omega_r = \omega_0\sqrt{2-\alpha} \ ,$$

implying that periodic closed orbits occur when $\sqrt{2-\alpha}$ is a rational number. Conveniently we'll see that the Coulomb and gravitational potentials ($\alpha = 1$), along with the harmonic oscillator ($\alpha = -2$) each produce closed orbits not limited to circles. The next closed orbit corresponds to $\alpha = -7$.

Period

In terms of ω_0, we can write the period T_0 of a circular orbit. Start with the definition $\omega T = 2\pi$, we have

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi m_* r_0^2}{L}$$

Harmonic Potential

The *harmonic potential* is defined by

$$U(r) = \Lambda r^2 \ .$$

As a central force, all motion is confined to a plane and thus we separate into components as

$$U(r) = \Lambda\left(x^2+y^2\right) \ .$$

Using $F_x = -\partial U/\partial x$ and similar for the y-compoenent, the above implies a pair of independent one-dimensional differential equations

$$\begin{aligned}\frac{d^2}{dt^2}x(t) &= -\omega^2 x(t) \\ \frac{d^2}{dt^2}y(t) &= -\omega^2 y(t) \ ,\end{aligned}$$

where $\omega = \sqrt{2\Lambda/m_*}$.

General solutions to the above are trigonometric, namely

$$\begin{aligned}x(t) &= A_x\cos(\omega t - \phi_x) \\ y(t) &= A_y\cos(\omega t - \phi_y) \ ,\end{aligned}$$

where $A_{x,y}$ and $\phi_{x,y}$ are determined from initial conditions.

We can do away with the ϕ_x-term by placing the particle at A_x at $t = 0$ and defining the x-axis to pass through that point. Then, the y-component of the position must be zero, telling us $\phi_y = \pi/2$. Finally, we find a closed equation for elliptical orbits with the origin at the center:

$$\begin{aligned}x(t) &= A_x\cos(\omega t) \\ y(t) &= A_y\sin(\omega t)\end{aligned}$$

The envelops of positions draws an ellipse:

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$$

3.2 Dimensionless Variables

For a power law potentual $U(r) = -\Lambda/r^\alpha$, we can find a circular orbit characterized by $r = r_0$ that minimizes $U_{\text{eff}}(r)$ at r_0. The period of such an orbit is T_0.

Let us now replace quantities of radius, energy, and time units with dimensionless variables by the following substitutions:

$$\begin{aligned}\rho(t) &= \frac{r(t)}{r_0} \\ \mathcal{E} &= \frac{E}{|U_{\text{eff}}(r_0)|} \\ \tau &= \frac{t}{T_0} = \frac{Lt}{2\pi m_* r_0^2}\end{aligned}$$

We bring Λ into the mix by recalling

$$\frac{L^2}{m_* r_0^2} = \frac{\Lambda\alpha}{r_0^\alpha}$$

for circular orbits, and it further follows that

$$|U_{\text{eff}}(r_0)| = \frac{L^2}{2m_* r_0^2}\left|\frac{2-\alpha}{\alpha}\right| .$$

Problem 12

For a planet on an elliptical orbit with semi-major axis a, use

$$E = \frac{-Gm_1 m_2}{2a}$$

to show that the eccentricity is given by

$$e = \sqrt{1+\mathcal{E}} .$$

Problem 13

For a planet on an elliptical orbit with semi-major axis a and semi-minor axis b, show that

$$\frac{b}{a} = \sqrt{-\mathcal{E}} .$$

If the orbit is hyperbolic, show instead that

$$\frac{b}{a} = \sqrt{\mathcal{E}} .$$

Equations of Motion

The equations of motion

$$t(r) = \pm\sqrt{\frac{m_*}{2}}\int_{r_i}^{r_f}\frac{dr}{\sqrt{E-U_{\text{eff}}(r)}}$$
$$\theta(r) = \pm\frac{L}{\sqrt{2m_*}}\int_{r_i}^{r_f}\frac{dr/r^2}{\sqrt{E-U_{\text{eff}}(r)}}$$

must be recast in dimensionless variables.

Staying in the special case $\Lambda > 0$, $\alpha < 2$, proceed by simplifying the radical term first:

$$\begin{aligned} E - U_{\text{eff}}(r) &= E + \frac{\Lambda}{r^\alpha} - \frac{L^2}{2m_* r^2} \\ &= \frac{\mathcal{E}\Lambda\alpha}{2r_0^\alpha}\left(\frac{2-\alpha}{\alpha}\right) + \frac{\Lambda}{r_0^\alpha\rho^\alpha} - \frac{\Lambda\alpha}{2r_0^\alpha\rho^2} \\ &= \left(\frac{\Lambda\alpha}{2r_0^\alpha}\right)\left((2-\alpha)\mathcal{E}/\alpha + (2/\alpha)/\rho^\alpha - 1/\rho^2\right) \end{aligned}$$

Substituing carefully, one finds

$$\tau = \pm\frac{1}{2\pi}\int_{\rho_i}^{\rho_f}\frac{d\rho}{\sqrt{(2-\alpha)\mathcal{E}/\alpha + (2/\alpha)/\rho^\alpha - 1/\rho^2}}$$
$$\theta = \pm\int_{\rho_i}^{\rho_f}\frac{d\rho/\rho^2}{\sqrt{(2-\alpha)\mathcal{E}/\alpha + (2/\alpha)/\rho^\alpha - 1/\rho^2}} .$$

Note that solutions to

$$0 = (2-\alpha)\mathcal{E}/\alpha + (2/\alpha)/\rho^\alpha - 1/\rho^2$$

indicate all apogees and perigees in the motion.

3.3 Inverse Square Attraction

The attractive power law potential with $\Lambda > 0$ and $\alpha = 1$ corresponds to the gravitational force and the attractive static electric force. Such a potential naturally hosts a circular orbit with:

$$r_0 = \frac{L^2}{\Lambda m_*}$$
$$T_0 = \frac{2\pi m_* r_0^2}{L}$$
$$|U_{\text{eff}}| = \frac{L^2}{2m_* r_0^2} = \frac{\Lambda^2 m_*}{2L^2}$$

Spatial Dynamics

The equations of motion simplify significantly with $\alpha = 1$. For the θ-equation, we have

$$\theta = \pm\int_{\rho_i}^{\rho_f}\frac{d\rho/\rho^2}{\sqrt{\mathcal{E}+2/\rho-1/\rho^2}} .$$

Let $\xi = 1/\rho$ to find

$$\theta = \pm\int\frac{-d\xi}{\sqrt{\mathcal{E}-(\xi-1)^2+1}} ,$$

and then let $\beta = \xi - 1$ to get

$$\theta = \pm\int\frac{-d\beta}{\sqrt{\mathcal{E}-\beta^2+1}} .$$

Factor $\sqrt{1+\mathcal{E}}$ from the denominator and also let $\gamma = \beta/\sqrt{1+\mathcal{E}}$:

$$\begin{aligned} \theta &= \pm\int\frac{-d\beta}{\sqrt{1+\mathcal{E}}\sqrt{1-\beta^2/(1+\mathcal{E})}} \\ &= \pm\int\frac{-d\gamma}{\sqrt{1-\gamma^2}} \end{aligned}$$

Next, let $\gamma = \cos(\psi)$ to find

$$\theta = \pm\int\frac{\sin(\psi)\,d\psi}{\sin(\psi)} = \pm\int d\psi .$$

The remaining integral has a trivial solution

$$\theta = \theta_0 \pm \psi ,$$

and undoing all substitutions gives

$$\theta = \theta_0 \pm \arccos\left(\frac{1/\rho - 1}{\sqrt{1+\mathcal{E}}}\right) ,$$

where the integration constant θ_0 is an ignorable rotation in the plane. Continue solving for ρ to get the equation of a conic section:

$$\rho = \frac{1}{1+\sqrt{1+\mathcal{E}}\cos(\theta)}$$

Eccentricicty

The combination $\sqrt{1+\mathcal{E}}$ is none other than the eccentricity of the orbit:

$$e = \sqrt{1+\mathcal{E}}$$

From what we know of conic sections, recall that $e = 1$ makes a parabola, $e < 1$ makes an ellipse, and $e > 1$ makes a hyperbola. In terms of the dimensionless energy $\mathcal{E}$, this also means:

$$\begin{aligned}
\mathcal{E} &= 0 \rightarrow \text{parabola} \\
\mathcal{E} &< 0 \rightarrow \text{ellipse} \\
\mathcal{E} &> 0 \rightarrow \text{hyperbola}
\end{aligned}$$

Conserved Quantities

As a special case of the two-body central potential system, we've aware that the inverse-square attraction supports conservation of energy E and conservation of angular momentum L.

Also conserved is the Runge-Lorenz vector

$$\vec{Z} = \vec{v} \times \vec{L} - \Lambda\,\hat{r}\,,$$

which fixes the orientation of the total orbit in its plane of motion. Take a time derivative to quickly prove $\vec{Z}$ is constant:

$$\begin{aligned}
\frac{d}{dt}\vec{Z} &= \frac{d\vec{v}}{dt} \times \vec{L} + \cancel{\vec{v} \times \frac{d\vec{L}}{dt}} - \Lambda\frac{d\hat{r}}{dt} \\
&= -\frac{1}{m_*}\frac{\Lambda}{r^2}\,\hat{r} \times \left(\vec{r} \times m_*\,\vec{v}\right) - \Lambda\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \\
&= \Lambda\left(\frac{-\left(\hat{r}\left(\hat{r}\cdot\vec{v}\right) - \vec{v}\left(\hat{r}\cdot\hat{r}\right)\right)}{r} - \frac{\vec{v}}{r} + \frac{\hat{r}}{r}\left(\hat{r}\cdot\vec{v}\right)\right) \\
&= 0
\end{aligned}$$

Being constant, we're free to evaluate $\vec{Z}$ anywhere on the orbit. Choosing a perigee at $\vec{r}_p = r_p\,\hat{x}$ where $\vec{v}_p \cdot \vec{r}_p = 0$, we find:

$$\begin{aligned}
\vec{Z} &= \vec{v}_p \times \left(\vec{r}_p \times m_*\,\vec{v}_p\right) - \Lambda\,\hat{r}_p \\
&= m_*\,\vec{r}_p\left(\vec{v}_p\cdot\vec{v}_p\right) - m_*\,\vec{v}\cancel{\left(\vec{v}_p\cdot\vec{r}_p\right)} - \Lambda\,\hat{r}_p \\
&= \left(\frac{L^2}{m_* r_p} - \Lambda\right)\hat{r}_p = \Lambda\left(\frac{r_0}{r_p} - 1\right)\hat{x} \\
&= \Lambda\left(\frac{1}{\rho_p} - 1\right)\hat{x} = \Lambda\left(1 + \sqrt{1+\mathcal{E}} - 1\right)\hat{x} \\
\vec{Z} &= \Lambda e\,\hat{x}
\end{aligned}$$

Conic Trajectory

Problem 14

Derive the dimensionless Runge-Lorenz vector equation

$$\hat{r} + e\,\hat{x} = \frac{\vec{v}\times\vec{L}}{\Lambda}\,,$$

and then project $\vec{r}$ into each side to recover the equation of a conic section, namely

$$r = \frac{r_0}{1 + e\cos\left(\theta\right)}\,.$$

Temporal Dynamics

The integral for the dimensionless time τ is straightforwardly solved with $\alpha = 1$. Begin with

$$\tau = \pm\frac{1}{2\pi}\int_{\rho_i}^{\rho_f}\frac{d\rho}{\sqrt{\mathcal{E} + 2/\rho - 1/\rho^2}}$$

and assume $\mathcal{E} \neq 0$. Simplify a bit to get

$$\tau = \pm\frac{\sqrt{\mathcal{E}}}{2\pi e}\int_{\rho_i}^{\rho_f}\frac{\rho\,d\rho}{\sqrt{\mathcal{E}\left(\rho\sqrt{\mathcal{E}} + 1/\sqrt{\mathcal{E}}\right)^2/e^2 - 1}}\,.$$

Let $\xi = \sqrt{\mathcal{E}}\left(\rho\sqrt{\mathcal{E}} + 1/\sqrt{\mathcal{E}}\right)/e$ so that $d\xi = d\rho\mathcal{E}/e$ and

$$\rho\,d\rho = e\left(\frac{\xi e - 1}{\mathcal{E}^2}\right)d\xi\,,$$

giving

$$\tau = \pm\frac{1}{2\pi\mathcal{E}^{3/2}}\int\frac{(\xi e - 1)}{\sqrt{\xi^2 - 1}}d\xi\,.$$

So far we've been a bit loose with the sign on $\mathcal{E}$. For $\mathcal{E} > 0$, everything stays as-is. However for $\mathcal{E} < 0$, we have to propagate $\mathcal{E} \rightarrow -\left|\mathcal{E}\right|$ through the calculation. Maintaining both channels, we have:

$$\begin{aligned}
\tau_{\mathcal{E}>0} &= \pm\frac{1}{2\pi\left|\mathcal{E}\right|^{3/2}}\int\frac{(\xi e - 1)}{\sqrt{\xi^2 - 1}}d\xi \\
\tau_{\mathcal{E}<0} &= \pm\frac{1}{2\pi\left|\mathcal{E}\right|^{3/2}}\int\frac{(\xi e - 1)}{\sqrt{1 - \xi^2}}d\xi
\end{aligned}$$

Let

$$\begin{aligned}
\xi_{\mathcal{E}>0} &= \cosh\left(\psi\right) \\
\xi_{\mathcal{E}<0} &= \cos\left(\psi\right)
\end{aligned}$$

and each integral becomes trivial, setting arbitrary integration constants to zero:

$$\begin{aligned}
\tau_{\mathcal{E}>0} &= \pm\frac{1}{2\pi\left|\mathcal{E}\right|^{3/2}}\left(e\sinh\left(\psi\right) - \psi\right) \\
\tau_{\mathcal{E}<0} &= \pm\frac{1}{2\pi\left|\mathcal{E}\right|^{3/2}}\left(\psi - e\sin\left(\psi\right)\right)
\end{aligned}$$

The period of an elliptical orbit corresponds to one full cycle in ψ, i.e. $0 \leq \psi < 2\pi$. For this we have

$$\tau_{\text{period}} = 2\left(\tau_{\pi} - \tau_{0}\right) = \left|\mathcal{E}\right|^{-3/2} .$$

Problem 15

Use

$$\tau = \frac{t}{T_0} = \frac{Lt}{2\pi m_* r_0^2}$$

and let $\tau = \left|\mathcal{E}\right|^{-3/2}$. Solve for $t = T$ to recover Kepler's third law for elliptical orbits.

Radial Component

Using the ρ-substitution from the above integral, namely

$$\rho_{\mathcal{E}>0} = \frac{e\xi - 1}{\mathcal{E}}$$
$$\rho_{\mathcal{E}<0} = \frac{e\xi - 1}{-\left|\mathcal{E}\right|} ,$$

we get solutions for $\rho\left(\psi\right)$:

$$\rho_{\mathcal{E}>0} = \frac{e\cosh\left(\psi\right) - 1}{\mathcal{E}}$$
$$\rho_{\mathcal{E}<0} = \frac{1 - e\cos\left(\psi\right)}{\left|\mathcal{E}\right|}$$

The perigee corresponds to $\psi = 0$. For $\mathcal{E} < 0$, the apogee is at $\psi = \pi$.

Zero-Energy Case

Returning to the equations of motion for τ, θ with $\mathcal{E} = 0$ and $\alpha = 1$, we have

$$\tau = \pm\frac{1}{2\pi}\int_{\rho_i}^{\rho_f} \frac{\rho\, d\rho}{\sqrt{2\rho - 1}}$$
$$\theta = \pm\int_{\rho_i}^{\rho_f} \frac{d\rho/\rho}{\sqrt{2\rho - 1}} ,$$

resolving to, discarding the integration constants:

$$\tau = \pm\frac{1}{6\pi}\sqrt{2\rho - 1}\left(\rho + 1\right)$$
$$\theta = \pm 2\arctan\left(\sqrt{2\rho - 1}\right)$$

3.4 Inverse Square Repulsion

Consider the central potential given by

$$U = \frac{\Lambda}{r} ,$$

as one finds with the repulsive Coulomb force. This is the same as the inverse square attractive case with the modification $\Lambda \rightarrow -\Lambda$, and the recipe for the equations of motion is essentially the same. Some key results are:

$$\rho = \frac{1}{-1 + \sqrt{1 + \mathcal{E}}\cos\left(\theta\right)}$$
$$\tau = \pm\frac{1}{2\pi\left|\mathcal{E}\right|^{3/2}}\left(e\sinh\left(\psi\right) + \psi\right)$$
$$\rho = \frac{1}{\mathcal{E}}\left(e\cosh\left(\psi\right) + 1\right)$$