Cartesian Plane
MANUSCRIPT

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Chapter 1

Cartesian Plane

1 Graphing

In mathematics, a *graph* is a visual or otherwise sensory representation of data. Data can occur in the forum of numbers, relationships, flowchart components, etc. It follows that 'graphing' mathematical data can mean a variety of things.

1.1 One-Dimensional Graph

The simplest non-trivial graph is facilitated by the standard number line as shown in Figure 1.1. For brevity there is no attempt to draw the entire number line. Any portion of the number line an be used as the basis for a graph.

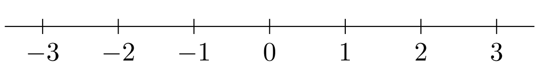

Figure 1.1: Number line.

To make use of the number line as a graph, we can consider the solutions to any equation of one variable. For simplicity, suppose we have

$$x^2 = 4 \,.$$

Knowing the above is satisfied by $x = 2$ and also $x = -2$, these solutions can be marked, or *plotted* at their respective locations on the number line as shown in Figure 1.2.

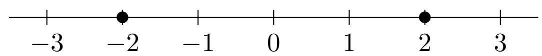

Figure 1.2: Number line with solutions to $x^2 = 4$.

A more informative name for 'one dimensional number line' is the *x-axis*, particularly when a variable named x is to be represented on the line.

Analog Clock Face

A one-dimensional number line that repeats itself is called *periodic*, as exhibited by a standard clock face in Figure 1.3.

Despite being arranged on a two-dimensional surface, the analog clock is a one-dimensional graph of the time being displayed.

When the clock hand sweeps through increasing time, the motion is called *clockwise*. The clock running backwards is going in the *counterclockwise*, or *anticlockwise* direction.

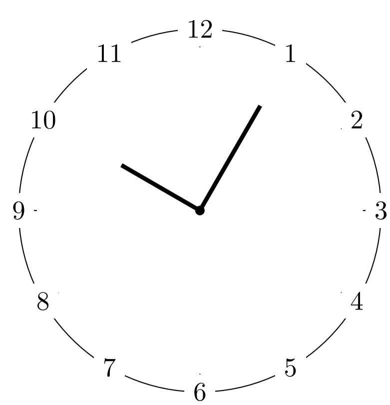

Figure 1.3: Analog clock is a one-dimensional graph.

1.2 Two-Dimensional Graph

We've witnessed that solutions to single-variable systems can be represented on a one-dimensional graph. It should follow that a visual representation of a systems with two independent variables need an extra 'degree of freedom', meaning a two-dimensional graph must b considered.

For example, suppose you're given the information

$$
\begin{aligned}
x &= 2 \\
y &= -2\,.
\end{aligned}
$$

One way to visualize this involves two number lines, one for the x-axis and another for the y-axis, marked accordingly as in Figures 1.4, 1.5.

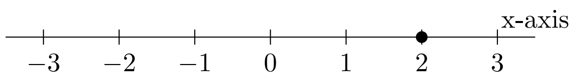

Figure 1.4: $x = 2$.

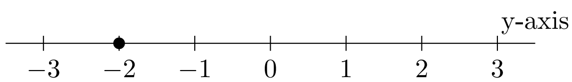

Figure 1.5: $y = -2$.

It would be more efficient, though, if the same solution can be represented in one picture instead of two. Since the x-axis and y-axis are independent, the two can be arranged to form a plane a shown in Figure 1.6.

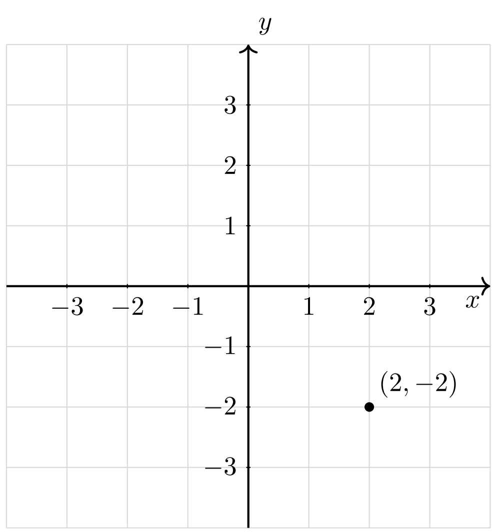

Figure 1.6: A single point representing $x = 2$, $y = -2$.

In constructing a two-dimensional graph, it is by popular convention that the point $x = 0$ overlaps with the point $y = 0$. One could easily imagine laying the y-axis across the x-axis at other locations, but it's not often necessary.

What *is* necessary, however, is that the x-axis and y-axis meet at a right angle. This way, contours of constant x span vertically, and contours of constant y span horizontally.

1.3 Ordered Pairs

If two quantities x, y represent a location on a two-dimensional graph, then the item

$$(x, y)$$

is called an *ordered pair*. The order in which x and y occur *does* matter, meaning (y, x) is different than (x, y).

To demonstrate, suppose there is a need to visualize the equation

$$x^2 + y = 3$$

in two dimensions. Proceed by choosing a sequence of x- values labeled x_1, x_2, etc., such that

$$
\begin{aligned}
x_1 &= -2 \\
x_2 &= -1 \\
x_3 &= 0 \\
x_4 &= 1 \\
x_5 &= 2\,.
\end{aligned}
$$

Of course, the choices −2, −1, etc. are arbitrary. From these, generate the corresponding y-values that satisfy the equation given. Particularly:

$$
\begin{aligned}
y_1 &= 3 - (-2)^2 = -1 \\
y_2 &= 3 - (-1)^2 = 2 \\
y_3 &= 3 - 0 = 3 \\
y_4 &= 3 - (1)^2 = 2 \\
y_5 &= 3 - (2)^2 = -1
\end{aligned}
$$

Matching x_1 to y_1 and so on, we generate five ordered pairs:

$$(-2, -1)\,, (-1, 2)\,, (0, 3)\,, (1, 2)\,, (2, -1)$$

Plotting each of these on the same two-dimensional graph gives rise to Figure 1.7.

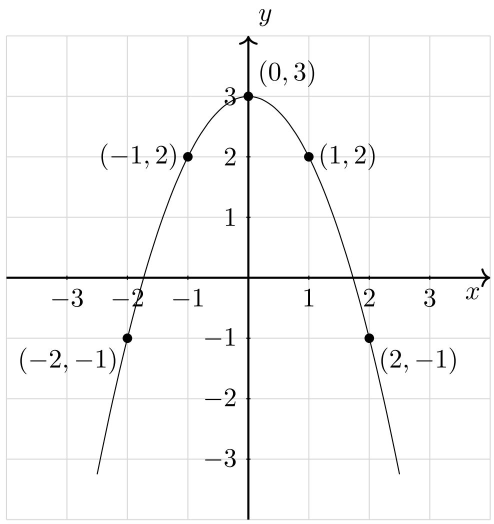

Figure 1.7: Ordered pairs representing $x^2 + y = 3$.

Supposing we sample many more x-values to generate more ordered pairs, it should be possible to fill in enough points on the graph so as to form a convincing 'curve'. Superimposed onto Figure 1.7 is a curve that represents doing such an effort.

Problem 1

Consider the equation

$$4y - 3x + 2 = 0 \; .$$

On a two-dimensional graph, find two ordered-pair solutions (x_1, y_1), (x_2, y_2), draw a straight line through each point on the graph. Find a third solution (x_3, y_3) and check that it occurs along the same line.

1.4 Graphical Solutions

Two-dimensional graphs can be used to hunt for solutions to otherwise very hard problems.

Consider the notorious transcendental equation

$$x^2 = 2^x \; ,$$

and the job is to find solutions for x. To begin we let each side of the equation equal a variable y such that:

$$y_1 = x^2$$
$$y_2 = 2^x$$

Then, choosing a flurry of x-values to the left and right of $x = 0$, we can generate the following table:

x	$y_1 = x^2$	$y_2 = 2^x$
-2	4	0.25
-1.5	2.25	0.3536
-1	1	0.5
-0.5	0.25	0.7071
0	0	1
0.5	0.25	1.4142
1	1	2
1.5	2.25	2.8284
2	4	4
2.5	6.25	5.6569

From the table, we construct the ordered pairs $(-2, 4)$, $(-2, 0.25)$, $(-1.5, 2.25)$, etc. Letting a machine calculate and plot many such ordered pairs gives rise to Figure 1.8.

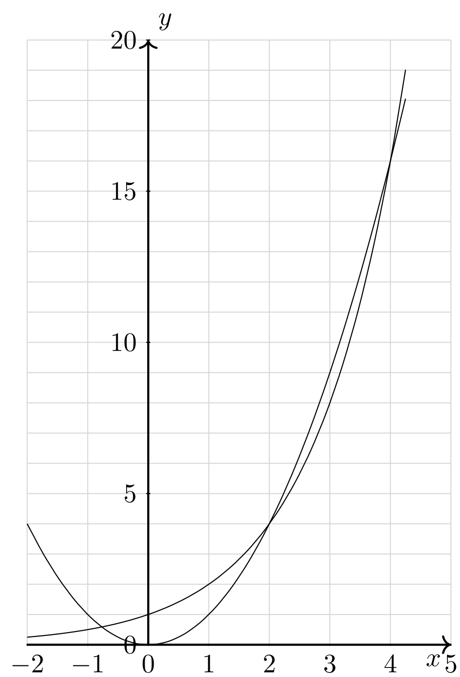

Figure 1.8: Graphical solution to $x^2 = 2^x$.

Right away, we see from the table or the Figure that the solution $x = 2$ readily satisfies $x^2 = 2^x$. Slightly less obvious is the solution at $x = 4$, because $4^2 = 2^4$ is a valid statement.

The least conspicuous solution is implicated somewhere to the left of the y-axis where the two curves intersect. In Figure 1.9 we get a closer look at the window

$$-1 \leq x \leq 0.5$$
$$-0.5 \leq y \leq 1$$

as shown.

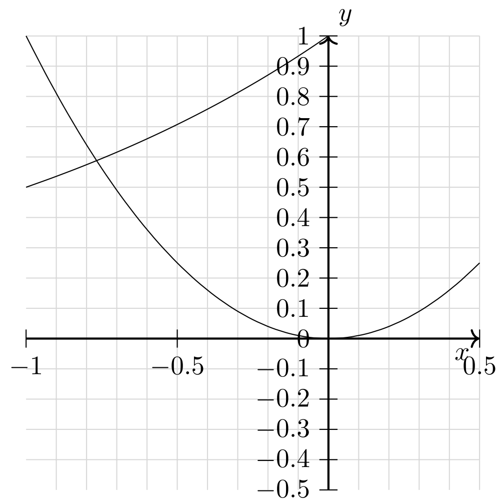

Figure 1.9: Graphical solution to $x^2 = 2^x$, reduced window.

By re-centering and re-zooming the graph, we can 'hone in' on any point we wish, and thus another solution at

$$x \approx -0.766 ,$$

give or take, becomes evident. For a test, we can use a calculator to establish

$$(-0.766)^2 - 2^{(-0.766)} = -0.00129$$

in hopes that the right side comes out as close to zero as possible.

Problem 2

Consider the system of equations

$$x^2 + y = 21$$
$$xy = 20 .$$

By plotting ordered-pair solutions to each equation, show that the two solutions to the system are $(1, 20)$, $(4, 5)$.

Problem 3

For the system

$$x + y - 5 = 0$$
$$y - 2x + 10 = 0 ,$$

use graphical methods to determine the only ordered-pair solution. Answer: $(5, 0)$

Problem 4

Consider the equation

$$x^2 = x + 6 .$$

Convert this to a system of equations

$$y_1 = x^2$$
$$y_2 = x + 6 ,$$

and use graphical methods to find two values for x that satisfy the system. Answer: $x = -2$ and $x = 3$

1.5 Cartesian Plane

Formally, the two-dimensional graphing space in which the x-axis is perpendicular to the y-axis is called the *Cartesian plane*.

Origin

Like the x- and y-number lines, the Cartesian plane spans infinitely in the horizontal and vertical directions so that every possible ordered pair (x, y) of real numbers has a unique 'home' in the plane. The point $(0, 0)$ is called the *origin*.

Quadrants

The rest of the Cartesian plane is divided into four *quadrants*, also known as *quarter-planes*, depending on whether x, y are positive or negative numbers as shown in Figure 1.10.

On a technical note, each quadrant has infinite size. The entire Cartesian plane has infinite size, but it's a 'larger' infinity.

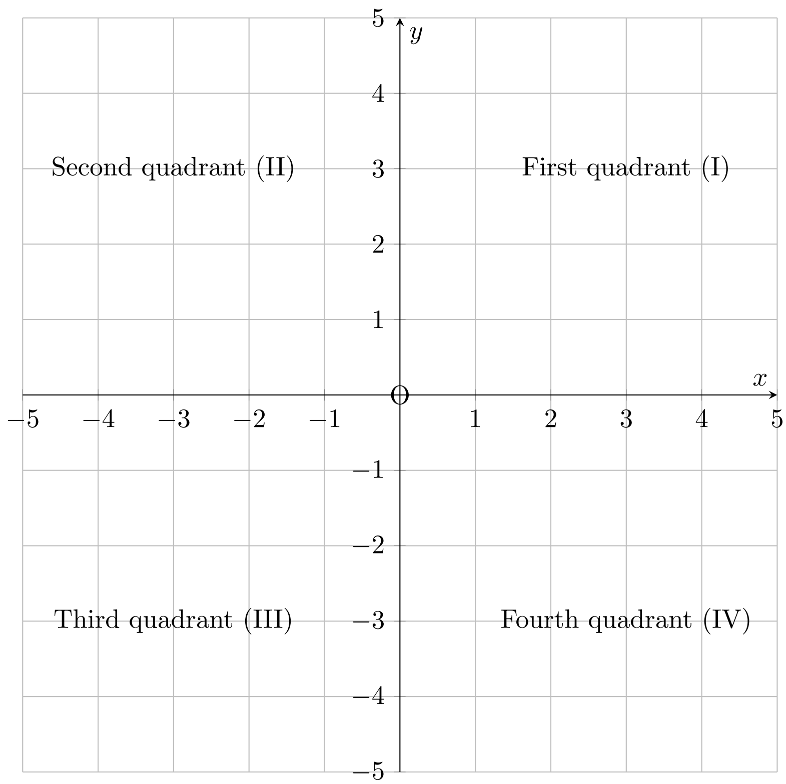

Figure 1.10: Cartesian Plane.

2 Straight Lines

Linear Equation

Any equation of the form

$$Ax + By + C = 0 \tag{1.1}$$

is a *linear equation.* By 'linear', we mean that x and y only occur to the first power. The presence of terms like x^2 or $\sqrt{y}$ are would make the equation nonlinear.

In the Cartesian plane, a linear equation corresponds to a straight line, easily verified by plotting ordered pairs (x, y) that satisfy the equation. Each coefficient A, B, C, each being any real number, dictates the position and orientation of the line.

Intercepts

The special point $(x_{\text{int}}, 0)$ is called the x-intercept, which corresponds to where a straight line crosses the x-axis. An exception occurs when the line is purely horizontal.

Correspondingly, the special point $(0, y_{\text{int}})$ is called the y-intercept, which is where a straight line crosses the y-axis. An exception occurs when the line is purely vertical.

From Equation (1.1), one readily solves for the x- and y-intercepts:

$$x_{\text{int}} = \frac{-C}{A}$$
$$y_{\text{int}} = \frac{-C}{B}$$

Horizontal and Vertical Lines

Looking at each denominator in the above, we see that x_0 only exists if $A \neq 0$. Similarly, y_0 exists only if $B \neq 0$. This makes sense in light of Equation (1.1), for if $A = 0$ then y is always equal to y_0 to make a horizontal line. Likewise, if $B = 0$ then x is always equal to x_0 to make a vertical line.

2.1 Slope

From a linear equation, one may generate two ordered-pair solutions (x_1, y_1), (x_2, y_2) so long as $x_1 \neq x_2$ and $y_1 \neq y_2$. That is, we want a pair of *unique* points in the Cartesian plane as that satisfy Equation (1.1). Doing so gives

$$Ax_1 + By_1 + C = 0$$
$$Ax_2 + By_2 + C = 0\,.$$

Given the points (x_1, y_1), (x_2, y_2), the difference $x_2 - x_1$ is called the *run*, and is written:

$$\Delta x = x_2 - x_1 = \text{run} \tag{1.2}$$

The triangular Greek symbol Δ is called 'delta'. Similarly, the difference $y_2 - y_1$ is called the *rise*:

$$\Delta y = y_2 - y_1 = \text{rise} \tag{1.3}$$

This construction is illustrated in Figure 1.11.

The ratio $\Delta y / \Delta x$ is called the *slope* of the line, denoted m:

$$m = \frac{\Delta y}{\Delta x} = \text{slope} \tag{1.4}$$

In terms of the coefficients in Equation (1.1), the slope comes out to

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-A}{B} \; .$$

The special case $A = 0$ corresponds to a horizontal line, i.e. a line with zero rise, and thus zero slope. On the other hand, the case $B = 0$ causes division by zero, making the slope undefined. This corresponds to a vertical line, which has 'infinite' slope - all rise and no run.

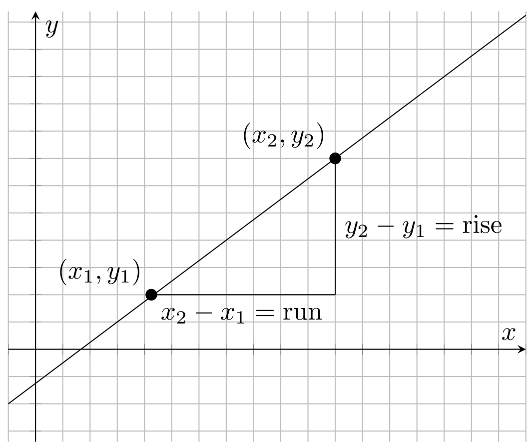

Figure 1.11: Rise and run.

Line through the Origin

In terms of the x- and y-intercepts, it's straightforward to show that the slope can be written

$$m = \frac{-y_{\text{int}}}{x_{\text{int}}}$$

for a striahgt line.

Any line that passes through the origin at $(0, 0)$ always has

$$x_{\text{int}} = y_{\text{int}} = 0 \; ,$$

which causes the slope m to be unreadable from the above. Such a line through the origin can have *any* slope as dictated by the ratio $-A/B$.

Point-Slope Form

To summarize, we've seen that any two points on a line given by Equation (1.1) can be used to calculate the slope.

The same line can be determined a slightly different configuration. Namely, if we start with one given point (x_0, y_0) on the line, along with the slope m, we can write an equation that accounts for the entire line. The run and the rise become

$$\Delta x = x - x_0$$
$$\Delta y = y - y_0 \; ,$$

and similar to Equation (1.4), we can write

$$m = \frac{y - y_0}{x - x_0} \; .$$

Rearranging a little, the *point-slope form* of the line emerges:

$$y - y_0 = m\,(x - x_0) \tag{1.5}$$

Slope-Intercept Form

Another useful equation for a straight line is the so-calld *slope-intercept form*. Starting from Equation (1.1), solve for the y-variable to find

$$y = \frac{-A}{B} x - \frac{C}{B} \; ,$$

or equivalently

$$y = mx + y_{\text{int}} \; .$$

By convention, the y-intercept is given the shorthand label b such that

$$b = \frac{-C}{B} \; ,$$

and the finished equation takes the form:

$$y = mx + b \tag{1.6}$$

The slope-intercept form requires two pieces of information, namely m and b, to determine the entire straight line.

Perpendicular Lines

Suppose we take Figure 1.11 and rotate the page by ninety degrees counterclockwise so that the y-axis now points along $-x$, and similarly the x-axis now points vertically. Any line $y = mx + b$ drawn on the unrotated page snaps to a new orientation with different x- and y-intercepts. Similar comments apply if we rotate the page clockwise instead.

Whether we rotate the page clockwise or counterclockwise, the slope m of a line takes on a new value $m_\perp$, particularly

$$m_\perp = \frac{-\Delta x}{\Delta y} = \frac{-1}{m} \; .$$

Problem 5

Starting from Equation (1.5), derive the form

$$Ax + By = Ax_0 + By_0$$

and show that

$$C = -\left(Ax_0 + By_0\right) \; .$$

Problem 6

Determine the slope of a line connecting the points $(2, 3)$ and $(7, 4)$.

Problem 7

Write the point-slope equation of a line connecting the points $(-1, 10)$ and $(5, 2)$.

Problem 8

Write the slope-intercept equation of a line connecting the points $(-8, -8)$ and $(-7, 9)$.

Problem 9

Find the y-intercept of a line having $m = 2$ that passes through $(2.5, 0)$.

Problem 10

Write the equation of a line with an x-intercept of 3, and a y-intercept of 6.

3 Quadratic Curves

Quadratic Equation

Starting with the point-slope form a straight line in Equation (1.5), one may wonder what happens if $(x - x_0)$ is replaced by $(x - x_0)^2$. Right away we replace the slope term m with a different arbitrary coefficient a to write a *quadratic equation*:

$$y - y_0 = a\left(x - x_0\right)^2 \tag{1.7}$$

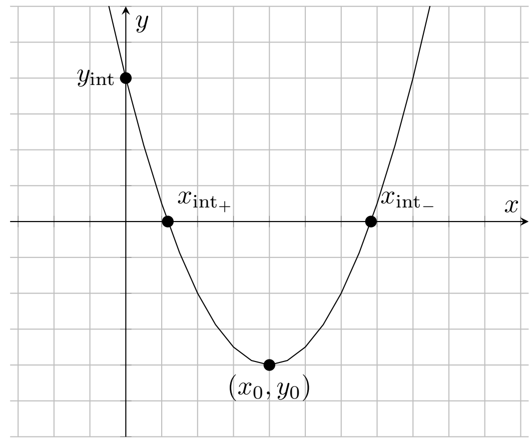

Figure 1.12: Quadratic curve.

Vertex and Symmetry

Figure 1.12 depicts a sketch of Equation (1.7) in the Cartesian plane. The overall U-shape of the curve is called a *parabola*. The point (x_0, y_0) is called the *vertex*, and the vertical line defined by $y = y_0$ is the axis of symmetry.

Concavity

When the parabola is 'facing up' as depicted, the curve is *concave up*. The entire curve flips upside-down (keeping the same vertex) if a is a negative number, and becomes *concave down*.

The magnitude of a determines the wideness or sharpness of the parabola. For smaller a, the distance between the x-intercepts widens and the whole parabola becomes flatter. The limit case $a = 0$ means y is always y_0 for a horizontal line. Conversely, the parabola becomes sharper for increasing a.

Intercepts

As written in Equation (1.7), a parabola always has a y-intercept given by

$$y_{\text{int}} = y_0 + a x_0^2 \; .$$

As for x-intercepts, let $y = 0$ to find

$$x_{\text{int}} = x_0 \pm \sqrt{\frac{-y_0}{a}} \; .$$

Note there are up to two solutions for x_{int}, but no x-intercept exists when y_0/a is a negative number.

General Form

Equation (1.7) can be configured in general form. Multiply out $(x-x_0)^2$ and combine like terms to write

$$y = ax^2 - 2axx_0 + \left(y_0 + ax_0^2\right) \ ,$$

and the parenthesized term is familiar as y_{int}.

Seeking a general form for the quadratic curve, define three coefficients such that

$$\begin{aligned} A &= a \\ B &= -2ax_0 \\ C &= y_0 + ax_0^2 = y_{\text{int}} \ , \end{aligned}$$

so the quadratic curve is written

$$y = Ax^2 + Bx + C \ . \tag{1.8}$$

Starting from Equation (1.8), it's handy to be able to egt back to a, x_0, y_0 in terms of A, B, C. Solving this system is straightforwardly done via:

$$\begin{aligned} a &= A \\ x_0 &= \frac{-B}{2A} \\ y_0 &= C - \frac{B^2}{4A} \end{aligned}$$

3.1 Quadratic Formula

Writing the general form of the parabola in terms of a, x_0, y_0, one has

$$y = \left(C - \frac{B^2}{4A}\right) + A\left(x + \frac{B}{2A}\right)^2 \ .$$

With this, let us repeat the x-intercept calculation by setting $y = 0$. Doing so gives the famed *quadratic formula*:

$$x_{\text{int}_\pm} = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \tag{1.9}$$

From the above, we also see that the sum of intercepts obeys

$$x_{\text{int}_+} + x_{\text{int}_-} = \frac{-B}{A}$$

Product of Quadratic Intercepts

A seldom-utilized identity involves the product of the two x-intercepts x_{int_+}, x_{int_-}. Carrying this out, one finds:

$$x_{\text{int}_+} \cdot x_{\text{int}_-} = \left(\frac{-B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A}\right)\left(\frac{-B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A}\right)$$

This readily simplifies to:

$$x_{\text{int}_+} \cdot x_{\text{int}_-} = \frac{C}{A}$$

Alternate Quadratic Equation

Since each x-intercept is a point where the parabola has $y = 0$, we can also write

$$y = A\left(x - x_{\text{int}_+}\right)\left(x - x_{\text{int}_-}\right) \ . \tag{1.10}$$

Simplify the right side to get

$$\begin{aligned} y &= Ax^2 - A\left(x_{\text{int}_+} + x_{\text{int}_-}\right)x + Ax_{\text{int}_+}x_{\text{int}_-} \\ &= Ax^2 - A\left(\frac{-B}{A}\right) + A\frac{C}{A} \\ &= Ax^2 + Bx + C \ , \end{aligned}$$

recovering the general form.

Discriminant

The quantity $B^2 - 4AC$ is called the *discriminant*, denoted $\mathcal{D}$:

$$\mathcal{D} = B^2 - 4AC$$

For the x-intercepts of a quadratic equation be real-valued, the discriminant must be positive. The special case $B^2 = 4AC$ corresponds to there being one intercept, i.e. $x_{\text{int}_+} = x_{\text{int}_-}$. When the discriminant is negative, there are no x-intercepts.

3.2 Line Intersecting Parabola

Consider the following nonlinear system of two equations and two unknowns:

$$\begin{aligned} 4x - y &= 3 \\ 2x^2 + 3y &= 19 \end{aligned}$$

In the Cartesian plane, this system represents the intersection of a line and a parabola.

To solve for x or y analytically, we first combine the two equations to eliminate one of the variables. Solving for y in the first equations and substituting into the second, we find

$$x^2 + 6x = 14 \ ,$$

which is clearly a quadratic equation, having two solutions:

$$\begin{aligned} x_1 &= -3 + \sqrt{23} \approx 1.796 \\ x_2 &= -3 - \sqrt{23} \approx -7.796 \end{aligned}$$

For a similar exercise, re-combine the equations in the system to eliminate x instead, yielding a quadratic equation

$$y^2 + 30y = 143 \,,$$

which itself has two solutions:

$$y_1 = -15 + \sqrt{368} \approx 4.183$$
$$y_2 = -15 - \sqrt{368} \approx -34.183$$

From the information gained, we have four possible solutions to the system

$$(x_1, y_1)$$
$$(x_1, y_2)$$
$$(x_2, y_1)$$
$$(x_2, y_2) \,,$$

however not all are valid. Each must be checked against the original equations, which is equivalent to checking with a graphical method. Doing so leads to Figure 1.13.

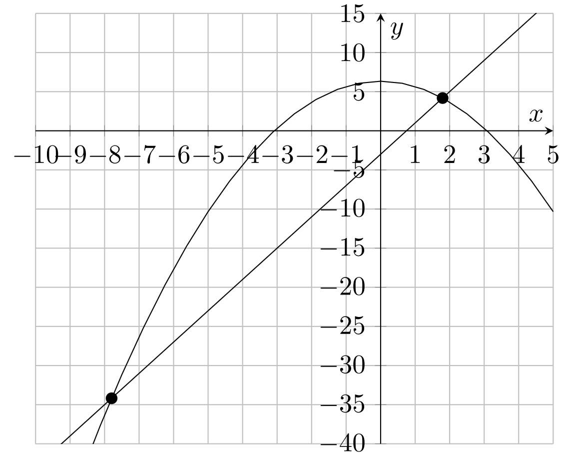

Figure 1.13: Line intersecting Parabola.

Evidently, intersections occur at $(1.7953, 4.1833)$ and $(-7.7958, -34.183)$, thus the valid solutions to the system are (x_1, y_1) and (x_2, y_2).

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