Physical Units

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December 1, 2020

1 Fundamental Dimensions

All physical quantities carry some kind of *dimensionality*, which is a way of tying a number to the physical world. There are only three* fundamental physical dimensions, which are (including abbreviations):

Length (L) Mass (M) Time (T)

The only quantities that need not be expressed in terms of fundamental dimensions are pure mathematical numbers, such as 3 or π.

* It should be pointed out that 'dimension' in this context does not refer to the idea that space has three dimensions (length, width, depth).

1.1 Systems of Units

History has handed down (at least) three different unit systems for keeping track of dimensionality.

SI Systems

The abbreviation *SI* stands for Système International, commonly known as the *metric system*. There are two 'factions' within the metric system: one is called *MKS* and the other is called *CGS*. The letters *M-K-S* tell us to use Meters, Kilograms, and Seconds, respectively, as fundamental units. Meanwhile, the *C-G-S* prefers Centimeters, Grams, and Seconds. That is:

System	**Length**	**Mass**	**Time**
MKS	Meter (m)	Kilogram (kg)	Second (s)
CGS	Centimeter (cm)	Gram (g)	Second (s)

English System

Only three countries besides Great Britain have so far resisted the metric system's campaign to dominate all standards: Burma, Liberia, and the United States. The *English system* offers another set of labels for the fundamental dimensions, those being the Foot, the Slug, and the Second. (Despite its popularity, the English pound is not fundamental.) We add to our table of fundamental units:

System	**Length**	**Mass**	**Time**
English	Foot (ft)	Slug (rarely used)	Second (s)

Expensive Mistake by NASA

Forgetting to properly handle units can be an expensive mistake, namely $328 million when NASA lost the Mars Climate Orbiter. In September 1999, NASA attempted to put a satellite close enough to Mars to study its atmosphere and surface features. However, a ground-based computer in communication with the Orbiter was sending information using the *English* system of units! Specifically, the Orbiter expected a packet of information expressed in Newtons, but it was interpreted as Pounds. The Orbiter didn't steer correctly and slammed into Mars.

1.2 Standard Units

As of yet, there is no pure system of units implied by nature alone - there is always a human element involved in setting standards.

Standard of Length

Length standards can be traced back to ancient history, with one popular unit of measurement being the the elbow-to-fingertip distance on a human arm in an L-shape, known as the 'cubit'. Obviously, the cubit varies form person to person, and more drastically between separate populations, and thus was not at all standard.

From Europe we gained another measure of length, the familiar 'foot', which was once defined by the length of the ruling king's actual foot. It wasn't until the days of Louis XIV that the foot was standardized in France. The name was upgraded to the 'Royal Foot'.

Around 1795, the standard length unit was decidedly the Meter, and was defined as 1/20000000 the length of Earth's prime meridian (this was before orthodox geology was settled on).

By the late 1800's, the standard length unit was determined by the length of one particular (carefully protected) platinum-iridium bar, whose length was simply declared to equal precisely one meter. Precise copies of this bar were produced and shipped off to spread the length standard around the world.

Eventually, experimental physics would demand a more precise definition for the length standard that was robust with respect to special relativity, which inevitably involves the speed of light. *The meter is precisely defined as the distance light travels in* 1/299792458 *sec in vacuum.*

Standard of Mass

The modern definition of the Kilogram is as new as 2019. Before a (relatively) recent decision to uproot tradition, the *kilogramme* was an *actual* chunk of metal (90% platinum, 10% iridium) whose mass was decidedly one kilogram by popular vote. It's likely that the calibration of all science-grade mass scales and standards of the time could be traced back to the *kilogramme*, housed in France for its career.

There was curious problem with the *kilogramme* though: it's mass has slowly *increased* over history, and much effort has been devoted to finding out why this appears so. With or without a perfect explanation for this, the conclusion we must draw is that a chunk of metal is ultimately *not* a good mass standard.

The General Conference on Weights and Measures (CGPM) tried repeatedly to settle a definition of the kilogram that arises from quantum mechanics, specifically involving the Planck mass. They couldn't decide in 2011, and procrastinated again in 2014. Eventually, laboratory measurements of Planck's constant became sufficiently refined (about 13 ppb uncertainty) such that the definition of one kilogram is simply back-read out of the result.

Standard of Time

One reasonable definition for the Second is to slice a 24-hour interval into $60 \cdot 60 \cdot 24$ parts (like an analog watch). This was considered reliable until 1967, until it was decided that the fundamental unit of time had better take on a less 'Earthly' definition. It was then decided that one second would be defined as precisely 9192631770 periods of the light waves emitted from cesium decay.

There was nothing special about cesium except it was an available atom for study. There's nothing terribly special about the number 9192631770, except it seemed good enough for reproducing the 'old' value of one second. Going through the trouble to define the second this way allows scientists to deal more easily with many practical issues, such as adding leap-seconds to the calendar, and calibrating clocks far from Earth. (The latter was on their minds in the 1960's more so than it is today.)

Combinations of Units

There are many combinations of units - some are familiar, some are rarely used.

Combination	**Label**	**MKS**	**SI**	**English**
L^2	Area	m^2	$(cm)^2$	$(ft)^2$
L^3	Volume	m^2	$(cm)^3$	$(ft)^3$
L/T	Speed	m/s	cm/s	ft/s
L/T^2	Acceleration	m/s^2	cm/s^2	ft/s^2
$M \cdot L/T$	Momentum	$kg \cdot m/s$	$g \cdot cm/s$	(rare)
$M \cdot L/T^2$	Force	$kg \cdot m/s^2 = N$	$g \cdot cm/s^2$	Pound
M/L^3	Mass Density	kg/m^3	g/cm^3	(rare)
$M/\left(LT^2\right)$	Pressure	$N/m^2 = Pa$	$.1\ Pa$	PSI
$M \cdot L^2/T^2$	Energy	$N \cdot m = J$	$g \cdot cm^2/s^2$	Calorie

2 Units in Quantum Mechanics

The fundamental constants in quantum theory, expressed in terms of Planck's length $[L]$, mass $[M]$, and time $[T]$ are:

- the speed of light c: $[LT^{-1}]$
- Planck's reduced constant $\hbar = h/2\pi$: $[ML^2T^{-1}]$
- the squared electron charge $e^2/(4\pi\epsilon_0)$: $[ML^3T^{-2}]$
- the electron mass m: $[M]$

Reduced Compton Wavelength

As a useful exercise, we may determine x, y, and z such that

$$\hbar^x c^y m^z = [L] \ ,$$

i.e., has dimensions of length. Using dimensional analysis, are handed three equations

$$x + z = 0 \qquad 2x + y = 1 \qquad x + y = 0 \, ,$$

solved by $x = 1$ and $y = z = -1$, telling us that

$$\lambda_c = \hbar/mc \, ,$$

known as the *reduced Compton wavelength* λ_c, which evaluates to roughly $0.386 \times 10^{-12}\ m$.

Fine Structure Constant

The combination

$$\hbar^x c^y (e^2/4\pi\epsilon_0)^z$$

can yield a dimensionless quantity. In this case, we generate three equations

$$x + z = 0 \qquad 2x - 1 + 3z = 0 \qquad -x + 1 - 2z = 0 \, ,$$

solved by $x = -1$, $z = 1$. Choosing $y = -1$ with the remaining freedom, we get the *fine structure constant*:

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c} \approx \frac{1}{137} \approx 0.00730$$

Bohr Radius

We can recover the *Bohr radius* by taking the ratio of the reduced Compton wavelength to the fine structure constant:

$$a_0 = \frac{\lambda_c}{\alpha} = \frac{\hbar^2}{m\,(e^2/4\pi\epsilon_0)} \approx 137 \times \lambda_c \approx 5.29 \times 10^{-11}\ m$$