Classical Field Theory

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1 Review of Classical Mechanics

Classical physics is the subset of all phenomena where quantum mechanics is not needed. Classical branches of physics include Newtonian mechanics, electromagnetism, thermodynamics, and relativity.

1.1 Newtonian Mechanics

Newtonian mechanics begins by defining the velocity and acceleration of a particle in terms of its position vector $\vec{r}$ as a function of time. For a system of N particles, the ith particle obeys

$$\vec{v}_i(t) = \frac{d}{dt}\vec{r}_i(t) \qquad \vec{a}_i(t) = \frac{d}{dt}\vec{v}_i(t) = \frac{d^2}{dt^2}\vec{r}_i(t) \ .$$

Newton's second law tells us that the acceleration of the ith particle is the sum of all forces acting on that particle:

$$m_i\vec{a}_i = \sum_{j=1}^{N}\vec{F}_{ij}$$

The force vector $\vec{F}_{ij}$ is the force exerted by the jth particle onto the ith particle. We also define the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ of the system

$$\vec{P} = \sum_{i=1}^{N} m_i\vec{v}_i \qquad \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times m_i\vec{v}_i \ ,$$

both easily shown to be conserved quantities.

The energy added to the ith particle is the integral of the force along the direction of motion, namely

$$W = m_i \int_{\vec{x}_i}^{\vec{x}_f} \vec{a}_i \cdot d\vec{x}_i = m_i \int_{t_i}^{t_f} \vec{a}_i \cdot \vec{v}_i \, dt \ ,$$

where the differential displacement vector $d\vec{x}_i$ has been replaced by $\vec{v}_i \, dt$. Integrating by parts with

$$W = \left(\vec{U} \cdot \vec{V}\right)\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \vec{V} \cdot d\vec{U} \ .$$

we let

$$\vec{U} = m_i\vec{v}_i \qquad d\vec{U} = m_i\vec{a}_i \, dt \qquad d\vec{V} = \vec{a}_i \, dt \qquad \vec{V} = \vec{v}_i$$

to get

$$W = m_i v_i^2\Big|_{t_i}^{t_f} - W \ .$$

The energy added to the particle is interpreted as kinetic energy T_i. Normalizing the initial velocity to zero, have

$$T_i = \frac{1}{2} m_i v_i^2 \ .$$

Inverting the work equation, the force vector is expressed in terms of the gradient of a scalar energy, as in

$$\vec{F}_{ij} = -\frac{\partial}{\partial \vec{r}_i} U\left(\vec{r}_i, \vec{r}_j\right) \ ,$$

where U is the potential energy between the two particles. By requiring that the potential be symmetric in the indices i, j, we confidently write

$$U\left(\vec{r}_i, \vec{r}_j\right) = U\left(\vec{r}_j, \vec{r}_i\right) = U\left(\left|\vec{r}_i - \vec{r}_j\right|\right) \ ,$$

implying Newton's third law:

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

The total energy of a system of N particles is the sum all kinetic terms plus all potential terms, namely

$$E = T + U = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 + \sum_{i<j}^{N} U\left(\left|\vec{r}_i - \vec{r}_j\right|\right) \ ,$$

where a straightforward calculation of dE/dt shows that energy is conserved.

1.2 Lagrangian Mechanics

Classical mechanics can be re-derived using the calculus of variations and the principle of least action. Begin by considering any function $L\left(\vec{r}, \dot{\vec{r}}, t\right)$, depending on position, velocity ($d\vec{r}/dt = \dot{\vec{r}}$), and a time parameter.

Defining the action

$$S = \int_{t_i}^{t_f} L \, dt \ ,$$

it follows that variations in S are given by

$$\delta S = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial \vec{r}} \cdot \delta \vec{r} + \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \delta \dot{\vec{r}} \right) dt$$

By constraining $\delta \vec{r}$ and $\delta \dot{\vec{r}}$ to be zero on the boundaries t_i, t_f, the above can be integrated by parts to get

$$\delta S = \int_{t_i}^{t_f} \delta \dot{\vec{r}} \cdot \left(\frac{\partial L}{\partial \vec{r}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}} \right) \right) dt + \cancel{\frac{\partial L}{\partial \dot{\vec{r}}} \cdot \delta \dot{\vec{r}} \Big|_{t_i}^{t_f}} \ ,$$

where taking the limit $\delta S \to 0$ tells us the parenthesized quantity is also zero, delivering the Euler-Lagrange equation

$$\frac{\partial L}{\partial \vec{r}_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}_i} \right) = 0 \ ,$$

readily generalizing to handle systems of many parties. In terms of the Lagrangian, the momentum of the ith particle is governed by its partial derivatives:

$$\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} \qquad \qquad \dot{\vec{p}}_i = \frac{\partial L}{\partial \vec{r}_i}$$

To proceed for non-relativistic motion, we define the classical Lagrangian

$$L = T - U = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\dot{\vec{r}_i}\right)^2 - \sum_{i<j}^{N} U\left(\left|\vec{r}_i - \vec{r}_j\right|\right) \ .$$

Insert L into the Euler-Lagrange equation to get, for a single particle,

$$-\sum_{i \neq j}^{N} \vec{\nabla}_i U\left(\left|\vec{r}_i - \vec{r}_j\right|\right) = \frac{d}{dt} \vec{p}_i \ ,$$

none other than Newton's second law.

Relativistic Lagrangian

To account for special relativity, the action is defined in terms of the (rest-frame) proper time τ

$$S = -mc^2 \int_{\tau_i}^{\tau_f} d\tau = \int_{t_i}^{t_f} \sqrt{1 - v^2/c^2} \, dt \ ,$$

which is recast in terms of a non-rest-frame having relative speed v. Tacking on a potential energy term, the quantity to extremize is the relativistic Lagrangian,

$$L = -mc^2 \sqrt{1 - v^2/c^2} - U\left(\vec{r}\right) \ .$$

Applying the Euler-Lagrange equation and proceeding in analogy to the above, we write the relativistic force, momentum, and velocity:

$$-\frac{\partial U}{\partial \vec{r}} = m \frac{d}{dt} \left(\frac{\vec{v}}{\sqrt{1 - v^2/c^2}} \right) = F\left(\vec{r}\right) \ ,$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \qquad \qquad \vec{v} = \frac{\vec{p}/m}{\sqrt{1 - \left(p/mc\right)^2}}$$

1.3 Hamiltonian Formalism

In Hamiltonian mechanics, the system is mapped by generalized coordinates $q\left(t\right)$ and generalized momenta $p\left(t\right)$, where the relation $p = m\dot{q}$ is not taken axiomatically. The Hamiltonian H is defined in terms of the Lagrangian and a Legendre transform:

$$H\left(q_1, q_2, \ldots, p_1, p_2, \ldots\right) = \sum_{i=1}^{N} p_i \dot{q}_i - L\left(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots\right) \ ,$$

where N is the number of degrees of freedom in the system. In a notation that favors spatial dimensions, the above becomes

$$H\left(\vec{q}_1, \vec{q}_2, \ldots, \vec{p}_1, \vec{p}_2\right) = \sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{q}_i} - L\left(\vec{q}_1, \vec{q}_2, \ldots, \vec{p}_1, \vec{p}_2\right) \ ,$$

where N is unambiguously equal to the number of particles in the system.

For a classical Lagrangian

$$L = T - U = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{q}_i^2 - \sum_{i<j}^{N} U\left(|\vec{q}_i - \vec{q}_j|\right) \ ,$$

the Hamiltonian reduces to

$$H = \sum_{i=1}^{N} \left(\vec{p}_i \cdot \dot{\vec{q}}_i - \frac{1}{2} m_i \dot{q}_i^2 \right) + \sum_{i<j}^{N} U\left(|\vec{q}_i - \vec{q}_j|\right) \ .$$

If we finally make the connection $\vec{p} = m\dot{\vec{q}}$, the above becomes

$$H = \sum_{i=1}^{N} \frac{1}{2m_i} p_i^2 + \sum_{i<j}^{N} U\left(|\vec{q}_i - \vec{q}_j|\right) \ ,$$

which looks exactly like the total energy.

Returning to the general case with L unspecified, the action S is written

$$S = \int_{t_i}^{t_f} L \, dt = \int_{t_i}^{t_f} \left(\sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{q}}_i - H\left(\vec{q}_1, \vec{q}_2, \ldots, \vec{p}_1, \vec{p}_2\right) \right) dt \ .$$

Taking the variation in S (and thus all terms inside the integral), we have

$$\delta S = \int_{t_i}^{t_f} \left(\sum_{i=1}^{N} \delta\vec{p}_i \cdot \dot{\vec{q}}_i + \vec{p}_i \cdot \delta\dot{\vec{q}}_i - \frac{\partial H}{\partial \vec{p}_i} \delta\vec{p}_i - \frac{\partial H}{\partial \vec{q}_i} \delta\vec{q}_i \right) dt \ .$$

Collecting like terms and also noting the total derivative

$$\frac{d}{dt}\left(\vec{p}_i \cdot \delta\vec{q}_i\right) = \vec{p}_i \cdot \delta\dot{\vec{q}}_i + \dot{\vec{p}}_i \cdot \delta\vec{q}_i \ ,$$

we find

$$\delta S = \int_{t_i}^{t_f} \left(\sum_{i=1}^{N} \delta\vec{p}_i \cdot \left(\dot{\vec{q}}_i - \frac{\partial H}{\partial \vec{p}_i} \right) + \delta\vec{q}_i \cdot \left(-\dot{\vec{p}}_i - \frac{\partial H}{\partial \vec{q}_i} \right) \right) dt + \cancel{\left(\vec{p}_i \cdot \delta\vec{q}_i\right)\Big|_{t_i}^{t_f}} \ .$$

Taking the limit $\delta S \to 0$, and also noting that all variations vanish on the boundaries, we discover Hamilton's equations of motion:

$$\dot{\vec{q}}_i = \frac{\partial H}{\partial \vec{p}_i} \qquad\qquad \dot{\vec{p}}_i = -\frac{\partial H}{\partial \vec{q}_i}$$

1.4 Poisson Brackets

The Poisson bracket is an operation on two differentiable functions $A(\vec{q},\vec{p})$ and $B(\vec{q},\vec{p})$ such that

$$\{A,B\}_{qp} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right) ,$$

where N is the total number of degrees of freedom. Right away, we can see two properties that always hold:

$$\{A,B\}_{qp} = -\{B,A\}_{qp} \qquad\qquad \{A,A\}_{qp} = 0$$

Next, consider some function $F(q_i,p_i,t)$. By the chain rule, we know

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \right) ,$$

which, after substituting Hamilton's equations of motion, becomes

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\}_{qp} .$$

If the system is isolated from its environment, the total energy E must be conserved, making $\partial H/\partial t = 0$. Poisson brackets immediately hand us the stronger statement that $dH/dt = 0$ implying $E = H$. It follows that the Hamiltonian can be used as the generator of infinitesimal time translations. Using the above relations, it's straightforward to show that

$$\dot{q}_k = \{q_k,H\}_{qp} \qquad \dot{p}_k = \{p_k,H\}_{qp} \qquad \{q_j,p_k\}_{qp} = \delta_{jk}$$

2 Gravitational and Electric Fields

2.1 Analogy between Gravity and Electricity

Force and Field

The gravitational force and the electric force are similar, being purely radial and depending on an inverse square law. For two particles labeled 1 and 2 having mass $m_{1,2}$ and charge $q_{1,2}$ the gravitational and electric forces are written

$$\vec{F}_{12}^{\text{grav}} = -G\frac{m_1 m_2}{r^2}\,\hat{r} \qquad\qquad \vec{F}_{12}^{\text{elec}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,\hat{r}\;.$$

The unit vector $\hat{r}$ extends from particle 1 to particle 2. Dividing out the 'test' particle's mass and charge, respectively, deliver equations for the gravitational field and the electric field

$$\vec{g} = \frac{1}{m_2}\vec{F}_{12}^{\text{grav}} = -G\frac{m}{r^2}\,\hat{r} \qquad\qquad \vec{E} = \frac{1}{q_2}\vec{F}_{12}^{\text{elec}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r}\;,$$

where the 1-subscript has been omitted.

Flux and Gauss's Law

Next, suppose the 'source' particle is enclosed by a Gaussian surface $\mathcal{S}$. The gravitational flux Φ_G and the electric flux Φ_E is the integral of the field projected onto the surface normal, as in

$$\Phi_G = \int_{\mathcal{S}} \vec{g}\cdot d\vec{A} = -Gm\int_{\mathcal{S}}\frac{\cos\theta}{r^2}\,dA \qquad\qquad \Phi_E = \int_{\mathcal{S}} \vec{E}\cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int_{\mathcal{S}}\frac{\cos\theta}{r^2}\,dA\;,$$

where $\cos\theta$ is the angle between the unit vector $\hat{r}$ and the normal vector to the surface. Without loss of generality, assume the Gaussian surface to be spherical, reducing the integral term as

$$\int_{\mathcal{S}}\frac{\cos\theta}{r^2}\,dA = \frac{\cancel{\cos\theta}}{R^2}R^2\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi = 4\pi\;,$$

finishing each flux calculation:

$$\Phi_G = -4\pi G m \qquad\qquad \Phi_E = \frac{q}{\epsilon_0}$$

Density and Divergence

Note that the mass m and charge q can be cast as the integral of a density times a volume element, as in

$$\Phi_G = -4\pi G\int_{\mathcal{V}}\rho_m\,d^3x \qquad\qquad \Phi_E = \frac{1}{\epsilon_0}\int_{\mathcal{V}}\rho_q\,d^3x\;.$$

Meanwhile we may set the Gaussian surface just outside the particle's edge and apply the divergence theorem to give

$$\Phi_G = \int_{\mathcal{V}}\vec{\nabla}\cdot\vec{g}\,d^3x \qquad\qquad \Phi_E = \int_{\mathcal{V}}\vec{\nabla}\cdot\vec{E}\,d^3x\;.$$

Comparing each representation of Φ_G and Φ_E, the volume integrals can be eliminated, leaving us with the differential version of Gauss's law:

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \rho_m \qquad\qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho_q}{\epsilon_0}$$

Scalar Potential and Poisson's Equation

Since the gravitational and electric fields are each purely radial, each has zero curl, i.e. $\vec{\nabla} \times \vec{g} = 0$ and $\vec{\nabla} \times \vec{E} = 0$. It follows that each vector field can be expressed in terms of the gradient of a scalar field according to

$$\vec{g} = -\vec{\nabla} V(\vec{r}) \qquad\qquad \vec{E} = -\vec{\nabla} \phi(\vec{r}) \ ,$$

where each scalar field $V(\vec{r})$ and $\phi(\vec{r})$ is a function of position. In terms of the scalar potential, each instance of Gauss's law becomes Poisson's equation:

$$\nabla^2 V(\vec{r}) = 4\pi G \rho_m \qquad\qquad \nabla^2 \phi(\vec{r}) = -\frac{\rho_q}{\epsilon_0}$$

The scalar potential is calculated from the line integral against the field, starting from the zero-energy state:

$$V(\vec{r}) = \int_\infty^{\vec{r}} \vec{g}(\vec{r}') \cdot (-\hat{r})\, dr' = -\frac{Gm}{r} \qquad\qquad \phi(\vec{r}) = \int_\infty^{\vec{r}} \vec{E}(\vec{r}') \cdot (-\hat{r})\, dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Current and Continuity

The notion of 'current' I_q is ubiquitous in electricity, but also has an analogy in neutral matter with mass playing the role of charge. The current density J_q is the current divided per area perpendicular to the direction of travel. By vector calculus and conservation arguments, it follows that the closed surface integral of the current density must equal the quantity leaving the enclosed volume:

$$\int_{\mathcal{S}} \vec{J}_m \cdot d\vec{A} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_m \, d^3x \qquad\qquad \int_{\mathcal{S}} \vec{J}_q \cdot d\vec{A} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_q \, d^3x$$

Applying the divergence theorem and shuffling the time derivative onto the ρ-terms, we have

$$\int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{J}_m + \frac{\partial \rho_m}{\partial t} \right) d^3x = 0 \qquad\qquad \int_{\mathcal{V}} \left(\vec{\nabla} \cdot \vec{J}_q + \frac{\partial \rho_q}{\partial t} \right) d^3x = 0 \ ,$$

which is true at any volume, implying the pair of continuity equations:

$$\vec{\nabla} \cdot \vec{J}_m = -\frac{\partial \rho_m}{\partial t} \qquad\qquad \vec{\nabla} \cdot \vec{J}_q = -\frac{\partial \rho_q}{\partial t}$$

Needless to mention, these are non-relativistic equations.

2.2 Breakdown of the Analogy

While it appears that gravitational fields and electric fields are analogous, the similarities stop here. Observe first that the gravitational force is purely attractive, and the masses $m_{1,2}$ are always positive. By contrast, the overall sign on the electric force can be positive or negative, depending on the product $q_1 q_2$. Moreover, the phenomenon of magnetism is understood entirely in terms of electric currents, whereas there is no analogy to magnetism in the domain of gravity.

A more subtle difference between gravity and electromagnetism arises from relativity. It's straightforward to show that the energy stored in a gravitational field adds to the effective mass via $E = mc^2$, which in turn adds to the effective gravity, and so on. As a result, the total gravitational field compounds nonlinearly. Electric charge, on the other hand, is not subject to the same effect.

3 Index Notation and Coordinates

In order to refine the analogy between the equations of gravity and electricity, we must shed the old vector-based notation in favor of tensors. This is largely encouraged by the success of tensors in special/general relativity, in where space and time are regarded equally as the so-called spacetime fabric.

3.1 Four-Vectors and Tensors

Let us begin be defining the *contravariant position four-vector* q^μ in flat three-dimensional space with a spatially-normalized time component, where c is the (invariant) speed of light:

$$q^\mu = (ct, x, y, z)$$

The up-index position vector has a down-index counterpart q_μ, called the *covariant position vector*, namely

$$q_\mu = (-ct, x, y, z) \ .$$

Four-vectors fit under a more general classification called *tensors*, and are classified by their *type*, which is an ordered pair that denotes the number of indices in the up- and down-position. For instance, a contravariant (up-index) four-vector is a type $(1, 0)$ tensor, whereas as covariant (down-index four-vector is a type $(0, 1)$ tensor. Tensors can contain any number of indices. A two-index tensor Λ has three possible types $(2, 0)$, $(1, 1)$, $(0, 2)$, represented by $\Lambda^{\mu\nu}$, Λ^{ν}_{μ}, $\Lambda_{\mu\nu}$, respectively.

3.2 Contraction

The act of equating an up-index and a down-index is called *contraction*, and triggers a sum over that index,. Summation symbols are generally omitted according to the Einstein summation convention. For example, consider a type $(1, 1)$ tensor tensor product $x^\mu x_\nu$. Setting $\mu = \nu$ implies:

$$x^\mu x_\mu = x^1 x_1 + x^2 x_2 + \cdots + x^N x_N = S^2$$

The scalar result is a real or complex tensor of type $(0, 0)$ loosely represented as S^2, formally called the *norm*. For ordinary vectors, this is equivalent to the dot product.

3.3 Metric Tensor

One question naturally implied by four-vectors concerns whether there exists an object or operation that converts a contravariant four-vector to a covariant one, or vice-versa. It turns out that contraction with the *metric tensor* does just this. One particularly special tensor is called the *flat space metric*, also known as the *Minkowski space metric*, denoted $\eta_{\mu\nu}$, defined as

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Note $\eta_{\mu\nu}$ is *not* a matrix.

Raising and Lowering Indices

To see $\eta_{\mu\nu}$ at work, consider the contraction $\eta_{\mu\nu}A^\mu$, where A^μ is any four-vector. Note the μ-index occurs twice, both in the up- and down- positions. When any index is repeated, contraction triggers a sum over that index. This means

$$\eta_{\mu\nu}A^\mu = \sum_{\mu=0}^{3} \eta_{\mu\nu}A^\mu \ ,$$

however the summation symbol is almost always ignored, a shortcut called the *Einstein summation convention*. Carrying out the calculation, we find:

$$\begin{aligned}
\eta_{\mu\nu}A^\mu &= \eta_{00}A^0 + \eta_{11}A^1 + \eta_{22}A^2 + \eta_{33}A^3 \\
\eta_{\mu\nu}A^\mu &= -A^0 + A^1 + A^2 + A^3 \\
\eta_{\mu\nu}A^\mu &= A_\nu
\end{aligned}$$

Similarly, the *inverse metric tensor*, has both indices in the up-position, namely $\eta^{\mu\nu}$. In flat space (but not generally for curved space), the inverse metric components are the same as the ordinary metric:

$$\eta^{\mu\nu} = \eta_{\mu\nu}$$

By identical arguments, it follows that the inverse metric can raise the index on a four-vector, namely

$$\eta^{\mu\nu}A_\nu = A^\mu \ .$$

Note that the definition of the inverse metric tensor tells us the contraction between an up- and down-index yields a Kronecker delta function:

$$\eta_{\nu\rho}\eta^{\rho\mu} = \delta_\nu^\mu \ ,$$

where δ_μ^ν resolves to 1 if $\mu = \rho$, and equals zero otherwise.

3.4 Coordinate Transformations

As part of a the formal definition of a tensor, let us demand that under a general set of coordinate transformations, a tensor must obey an analog to the $A\vec{x} = \vec{b}$ calculation from linear algebra. In the most general case possible we would have a tensor A of type (N, M) undergoing $N + M$ coordinate changes:

$$A^{\mu_1' \cdots \mu_N'}_{\nu_1' \cdots \nu_M'} = \frac{\partial q^{\mu_1'}}{\partial q^{\mu_1}} \cdots \frac{\partial q^{\mu_N'}}{\partial q^{\mu_N}} \frac{\partial q^{\nu_1}}{\partial q^{\nu_1'}} \cdots \frac{\partial q^{\nu_M}}{\partial q^{\nu_M'}} A^{\mu_1 \cdots \mu_N}_{\nu_1 \cdots \nu_M}$$

Strictly, any object not obeying the above is not a tensor. For the simple case of one-index vectors V^μ and V_μ, the transformation law reads

$$V^{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu \qquad\qquad V_{\mu'} = \frac{\partial q^\mu}{\partial q^{\mu'}} V_\mu \ ,$$

where the partial derivative terms are analogous to the 'Jacobian' matrix from vector calculus.

3.5 Lorentz Transformation

Coordinate systems q^μ and $q^{\mu'}$ that are inertial, meaning not unjustly accelerated, adhere to the Lorentz transformation:

$$q^{\mu'} = \Lambda^{\mu'}_{\nu} q^\nu$$

For a so-called 'Lorentz-boost' along the x-direction at speed v, the coordinates q^μ transform via

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} ,$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad\qquad \eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta \, .$$

The inverse Lorentz transformation tensor (Λ^{-1}) has the signs on v reversed. The combination $(\Lambda^{-1})^\mu_\rho \Lambda^\rho_\nu$ must resolve to a Kronecker delta function with μ as the up-index and ρ as the down-index:

$$\left(\Lambda^{-1}\right)^\mu_\rho \Lambda^\rho_\nu = \delta^\mu_\nu$$

Covariant Lorentz Transformation

The Lorentz transformation of a covariant four-vector reads:

$$q_{\mu'} = \Lambda^\nu_{\mu'} q_\nu$$

We prove this by starting with $q^\alpha = \Lambda^\alpha_\beta q^\beta$, and then proceeding as follows:

$$\eta_{\alpha\mu'} q^\alpha = \Lambda^\alpha_\beta q^\beta \eta_{\alpha\mu'} = \Lambda^\alpha_\beta q^\beta \eta_{\rho\nu} \Lambda^\rho_\alpha \Lambda^\nu_{\mu'} = \delta^\rho_\beta q^\beta \eta_{\rho\nu} \Lambda^\nu_{\mu'} = \Lambda^\nu_{\mu'} q_\nu$$

3.6 Proper Time

The notion of time becomes slippery when comparing inertial reference frames in relative motion. Using the Lorentz transformation, the time interval in a boosted reference frame is

$$c\Delta t' = \gamma \left(c\Delta t - v\Delta x/c^2 \right) \, ,$$

and setting $\Delta x = 0$ delivers the an equation for time dilation,

$$\Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t = \gamma \Delta \tau \, ,$$

where the rest-frame time (the unprimed t-variable) is known as the proper time τ.

3.7 Four-Velocity

The derivative of the position vector q^μ with respect to the proper time τ is the four-velocity:

$$U^\mu = \frac{d}{d\tau} q^\mu (\tau) = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$$

Using the Minkowski metric $\eta_{\mu\nu}$, it's straightforwardly shown that the norm of the velocity vector is invariant:

$$U^\mu U_\mu = -\gamma^2 c^2 + \gamma^2 v^2 = -c^2$$

4 Fields and Derivatives

A *field*, in the broadest sense, is any function that depends on the generalized position q^μ (including time), and covers the entire coordinate system, A *scalar field* $\phi\left(q^\mu\right)$ is a single function, whereas a *vector field* $V^\mu\left(q^\mu\right)$ has multiple components tracked by an index.

4.1 Scalar Fields

Scalar fields $\phi\left(q^\mu\right)$ are always invariant under coordinate transformations. For a simple example, the temperature $T\left(q^\mu\right)$ at a point in a room must be invariant with respect to the coordinate system used to map the room.

To prove this, note first that the contraction of two four-vectors V^μ and U^μ always yields a scalar, as in

$$V^\mu U_\mu = V^0 U_0 + V^1 U_1 + V^2 U_2 + V^3 U_3 = S^2 \ .$$

Next, consider a primed reference frame (i.e. different coordinate system) where the two four-vectors manifest as $V^{\mu'}$, $U^{\mu'}$. In terms of coordinate transformations, the primed and un-primed vectors relate by

$$V^{\mu'} U_{\mu'} = \frac{\partial q^{\mu'}}{\partial q^\mu} \frac{\partial q^\nu}{\partial q^{\mu'}} V^\mu U_\nu = \delta^\nu_\mu V^\mu U_\nu = V^\nu U_\nu = S^2 \ ,$$

producing the same scalar as the unprimed case.

Example

Consider the scalar field

$$\phi\left(q^\mu\right) = ct - 2y \ .$$

According to a primed reference frame moving with velocity v along the x-direction, the field has each coordinate Lorentz-transformed:

$$\phi\left(q^{\mu'}\right) = \frac{ct' + vx'/c^2}{\sqrt{1 - v^2/c^2}} - 2y'$$

4.2 Vector Fields

Vector fields have a tighter restriction than scalar fields. Recall that under a general change of coordinates, a multi-component vector field $V^\mu\left(q^\nu\right)$ transform via

$$V^\mu\left(q^\nu\right) \quad \rightarrow \quad V^{\mu'}\left(q^{\nu'}\right) = \frac{\partial q^{\mu'}}{\partial q^\mu} V^\mu\left(q^\nu\left(q^{\nu'}\right)\right)$$

$$V_\mu\left(q^\nu\right) \quad \rightarrow \quad V_{\mu'}\left(q^{\nu'}\right) = \frac{\partial q^\mu}{\partial q^{\mu'}} V_\mu\left(q^\nu\left(q^{\nu'}\right)\right) \ ,$$

which we take as part of the definition of a vector field. The transformation occurs in the components of the vector field *and* the coordinates.

Example

Consider the vector field

$$V^{\mu}(q^{\nu}) = (2x, 0, ct, 0) \, .$$

According to a primed reference frame moving with velocity v along the x-direction, the field components simplify to

$$\begin{aligned}
(V')^0 &= \left(\Lambda^{-1}\right)^0_0 V^0 = 2\gamma x = 2\gamma^2 \left(x' + vt'\right) \\
(V')^1 &= \left(\Lambda^{-1}\right)^1_0 V^0 = -2\gamma\beta x = -2\gamma^2\beta \left(x' + vt'\right) \\
(V')^2 &= \left(\Lambda^{-1}\right)^2_2 V^2 = ct = \gamma \left(ct' + vx'/c\right) \\
(V')^3 &= \left(\Lambda^{-1}\right)^3_3 V^3 = 0 \, ,
\end{aligned}$$

so the result is written

$$(V')^{\mu} = \left(2\gamma^2 \left(x' + vt'\right), -2\gamma^2\beta \left(x' + vt'\right), \gamma \left(ct' + vx'/c\right), 0\right) \, .$$

4.3 Tensor Fields

Tensor fields are the multi-index generalization of vector fields. The number and position of indices (i.e. type) indicate how many partial derivative terms are needed for one coordinate transformation. For a two-index tensor, we have

$$A^{\mu'\nu'} = \frac{\partial q^{\mu'}}{\partial q^{\mu}} \frac{\partial q^{\nu'}}{\partial q^{\nu}} A^{\mu\nu} \qquad A^{\mu'}_{\nu'} = \frac{\partial q^{\nu}}{\partial q^{\nu'}} \frac{\partial q^{\mu'}}{\partial q^{\mu}} A^{\mu}_{\nu} \qquad A_{\mu'\nu'} = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial q^{\nu}}{\partial q^{\nu'}} A_{\mu\nu} \, ,$$

which, as for vector fields, *must* uphold if A qualifies as a tensor.

4.4 Gradient

A scalar field, classified as a type $(0, 0)$ tensor, can be converted into a vector field by taking the 'spatial' derivative with respect to the position four-vector, resulting in the four-gradient:

$$\phi(q^{\mu}) \qquad \rightarrow \qquad \frac{\partial}{\partial q^{\mu}} \phi(q^{\mu}) = \partial_{\mu}\phi = \left(\frac{\partial \phi}{\partial q^0}, \frac{\partial \phi}{\partial q^1}, \frac{\partial \phi}{\partial q^2}, \frac{\partial \phi}{\partial q^3}\right)$$

Similarly, the derivative can be taken with respect to the contravariant version of the position vector:

$$\frac{\partial}{\partial q_{\mu}} \phi(q^{\mu}) = \partial^{\mu}\phi = \eta^{\mu\nu}\partial_{\nu}\phi \qquad \rightarrow \qquad \partial^{\mu}\phi = \left(\frac{\partial \phi}{\partial q_0}, \frac{\partial \phi}{\partial q_1}, \frac{\partial \phi}{\partial q_2}, \frac{\partial \phi}{\partial q_3}\right)$$

Two Gradients

Unfortunately, higher-order derivatives of a scalar field $\phi(q^{\mu})$ don't generally result in tensors, as demonstrated by calculating two gradients $\partial_{\mu}\partial_{\nu}\phi$ with respect to a primed coordinate system, as

$$\partial_{\mu'}\partial_{\nu'}\phi = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial}{\partial q^{\mu}} \left(\frac{\partial q^{\nu}}{\partial q^{\nu'}} \frac{\partial \phi}{\partial q^{\nu}}\right) = \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial q^{\nu}}{\partial q^{\nu'}} \partial_{\mu}\partial_{\nu}\phi + \frac{\partial q^{\mu}}{\partial q^{\mu'}} \frac{\partial^2 q^{\nu}}{\partial q^{\mu}\partial^{\nu'}} \frac{\partial \phi}{\partial q^{\nu}}$$

clearly violates the tensor transformation rule. The last term shouldn't be there.

4.5 Divergence

The four-divergence of a vector field $A^\mu(q^\mu)$ is a contraction across the derivative of each component

$$\frac{\partial A^\mu}{\partial q^\mu} = \partial_\mu A^\mu \,,$$

which resolves to a scalar.

5 Field Theory

Classical field theory shares the same philosophical starting point as Lagrangian mechanics, that is, the principle of least action. To generalize Lagrange and Hamiltonian formalism for fields, the bold innovation is that generalized coordinates $q^\mu(t)$ are replaced by spacetime-dependent vector fields $\phi^\mu(q^\mu)$, along with their derivatives.

Begin by defining the action S as the integral of the Lagrangian L between two times t_i and t_f:

$$S = \int_{c\,t_i}^{c\,t_f} L\left(\vec{q}, \partial_0\vec{q}, q^0\right)\ c\,dt$$

As a function of space and time, the Lagrangian can be recast as the spatial integral of the *Lagrangian density*, denoted $\mathcal{L}$. Let us also replace the spatial arguments q^μ with the field ϕ^μ, as in

$$L = \int_{\vec{q}_i}^{\vec{q}_f} \mathcal{L}\left(\phi^\mu, \partial_\nu\phi^\mu\right) d^3\vec{q}\,.$$

The action becomes a four-dimensional integral

$$S = \int_{q_i^\mu}^{q_f^\mu} \mathcal{L}\left(\phi^\mu, \partial_\nu\phi^\mu\right) d^4q\,.$$

5.1 Euler-Lagrange Equation for Fields

To derive an analog to the Euler-Lagrange equation, we extremize the action subject to small variations in the field and its derivative

$$\phi^\mu \to \phi^\mu + \delta\phi^\mu$$
$$\partial_\nu\phi^\mu \to \partial_\nu\phi^\mu + \partial_\nu\left(\delta\phi^\mu\right)$$

such that first-order expansion of the Lagrangian density becomes

$$\mathcal{L}\left(\phi^\mu, \partial_\nu\phi^\mu\right) \to \mathcal{L}\left(\phi^\mu, \partial_\nu\phi^\mu\right) + \frac{\partial\mathcal{L}}{\partial\phi^\mu}\delta\phi^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^\mu\right)}\partial_\nu\left(\delta\phi^\mu\right)\,.$$

Then, the variation in the action is

$$\delta S = \int_{q_i^\mu}^{q_f^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^\mu}\delta\phi^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^\mu\right)}\partial_\nu\left(\delta\phi^\mu\right)\right) d^4q\,,$$

where the equality $\delta\left(\partial_\nu\phi^\mu\right) = \partial_\nu\left(\delta\phi^\mu\right)$ has been used. The second term can be integrated by parts by setting

$$u = \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^\mu\right)} \qquad du = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^\mu\right)}\right) d^4q$$

$$dv = \partial_\nu\left(\delta\phi^\mu\right) d^4q \qquad v = \delta\phi^\mu$$

such that

$$\delta S = \int_{q_i^\mu}^{q_f^\mu} \delta\phi^\mu \left(\frac{\partial \mathcal{L}}{\partial \phi^\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^\mu\right)} \right) \right) d^4 q + \left. \cancel{\left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^\mu\right)} \delta\phi^\mu \right)} \right|_{q_i^\mu}^{q_f^\mu} ,$$

where all variations are set to zero on the integration boundaries. Letting $\delta S \to 0$, we pick out the Euler-Lagrange equation for fields:

$$\frac{\partial \mathcal{L}}{\partial \phi^\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^\mu\right)} \right) = 0$$

5.2 Real Scalar Fields

In order to produce a kind of 'mechanics' for fields, the Lagrangian density $\mathcal{L}$ must be expressed in terms of some Lorentz-invariant combination of kinetic and potential terms. As scalar fields, energy potentials $U(\phi)$ already satisfy this - the trouble is choosing a kinetic term to fit into $\mathcal{L}$. Starting with something that we *know* will work, recall that spacetime interval invariant

$$dS^2 = dq^\mu dq_\mu = \eta_{\mu\nu} dq^\mu dq^\nu$$

is the same in all reference frames. A similar invariant $\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$ exists for a real scalar field ϕ (with no index). Exploiting this, we take the Lagrangian density as

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi) = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - U(\phi) \ .$$

Applying the Euler-Lagrange equation for fields to $\mathcal{L}$, we find

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dU}{d\phi} \qquad\qquad \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} = \frac{\partial}{\partial \left(\partial_\nu \phi\right)} \left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \right) ,$$

where the second term simplifies as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} &= -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial\left(\partial_\alpha\phi\right)}{\partial \left(\partial_\nu \phi\right)}\partial_\beta\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\frac{\partial\left(\partial_\beta\phi\right)}{\partial \left(\partial_\nu \phi\right)} \\ &= -\frac{1}{2}\eta^{\alpha\beta}\delta_\alpha^\nu\partial_\beta\phi - \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\delta_\beta^\nu \\ &= -\eta^{\mu\nu}\partial_\mu\phi \ . \end{aligned}$$

The Euler-Lagrange equation for scalar fields becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{dU}{d\phi} = 0 \ .$$

d'Alembert Operator

The combination $\eta^{\mu\nu}\partial_\mu\partial_\nu$ has a special name called the *d'Alembert operator*, which is essentially the Minkowski generalization of the Laplacian operator $\nabla^2 = \Delta$. Because the d'Alembert operator deals with four dimensions, it's symbol is the square, particularly

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \ .$$

The Euler-Lagrange equation for for scalar fields is thus written

$$\Box\phi - \frac{dU}{d\phi} = 0\,.$$

Klein-Gordon Equation

Perhaps the most common form for $U(\phi)$ is the simple harmonic oscillator, namely

$$U\left(\phi\right) = \frac{1}{2}\tilde{m}^2\phi^2\,,$$

where $\tilde{m} = mc/\hbar$ is a constant that has units of inverse length, but is referred to as the 'mass' of the field. (Note that the combination h/mc is the Compton wavelength.) The Euler-Lagrange equation becomes the *Klein-Gordon equation*

$$\Box\phi - \tilde{m}^2\phi = 0\,,$$

which is easily solved by plane waves, namely

$$\phi = \phi_0\, e^{i(Et - \vec{p}\cdot\vec{q})/\hbar}\,,$$

where the momentum $\vec{p}$ is related to the relativistic energy E via

$$E^2 = \left(\vec{p}\right)^2 c^2 + m^2c^4\,.$$

Hamiltonian Formalism for Scalar Fields

By analogy to the definition of the momentum vector in Lagrangian mechanics, the *momentum density* Π can be defined for fields:

$$\vec{p}_i = \frac{\partial L}{\partial\dot{\vec{r}}_i} \qquad \rightarrow \qquad \Pi\left(q^\mu\right) = \frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}$$

Then, by analogy to $H = \sum_i p_i\dot{q}_i - L$, the *Hamiltonian density* is defined as

$$\mathcal{H}\left(\phi,\Pi\right) = \left(\Pi\left(q^\mu\right)\right)\left(\partial_0\phi\left(q^\mu\right)\right) - \mathcal{L}\left(\phi,\partial_\mu\phi\right)\,.$$

Substituting the Lagrangian density for real scalar fields into the above, the momentum density simplifies to $\partial_0\phi$, and the Hamiltonian density readily becomes

$$\mathcal{H}\left(\phi,\Pi\right) = \frac{1}{2}\left(\Pi\left(q^\mu\right)\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + U\left(\phi\right)\,.$$

Poisson Brackets for Scalar Fields

The notion of Poisson brackets extends to fields. For two differentiable functions $A\left(\phi,\Pi\right)$, $B\left(\phi,\Pi\right)$, we have

$$\left\{A,B\right\}_{\phi\Pi} = \int\left(\frac{\partial A}{\partial\phi\left(q^\mu\right)}\frac{\partial B}{\partial\Pi\left(q^\mu\right)} - \frac{\partial A}{\partial\Pi\left(q^\mu\right)}\frac{\partial B}{\partial\phi\left(q^\mu\right)}\right)d^3\vec{q}\,,$$

which differs from the 'traditional' version by integrating over space. Letting $A = \phi$, $B = \Pi$, we recover an analog to Hamilton's equation of motion,

$$\left\{\phi\left(q^0,\vec{q}\right),\Pi\left(q^0,\vec{r}\right)\right\}_{\phi\Pi} = \delta\left(\vec{q}-\vec{r}\right)\,,$$

called the *equal-time Poisson bracket relation.*

Connection to Statistical Mechanics

Field theory notation can be used to construct the main object of statistical mechanics, known as the the partition function Z. We shall employ the notion of *imaginary time* by letting $t \to -it$, or in four-vector notation, $q^0 \to -iq^0$. The operator ∂_0 is replaced with $i\partial_0$.

Under this transformation, the action becomes

$$S_i = -i \int \mathcal{L}_i \left(\phi, i\partial_0\phi, \partial_j\phi\right) d^4q \ ,$$

where $\mathcal{L}_i$ is the Lagrangian density, and operator ∂_j excludes the time component altogether. If the Lagrangian density represents that of a scalar field, the imaginary-time version reads

$$\mathcal{L}_i = -\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - U\left(\phi\right) \ .$$

The action is therefore

$$S_i = i \int \left(\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + U\left(\phi\right)\right) d^4q \ .$$

The action now extremizes an energy quantity that lives in $0+4$ dimensions instead of $1+3$. With time removed form the picture, a system's evolution can be understood as a walk through accessible states. Defining $\mathcal{D}\phi$ as the density of states, the partition function of a classical system reads

$$Z = \int \left(\mathcal{D}\phi\right) \exp\left[iS\left(\phi, i\partial_0\phi, \partial_j\phi\right)/\hbar\right] \ .$$

6 Symmetry and Conservation

6.1 Conservation Laws

The notions of energy conservation, linear/angular momentum conservation, and Galilean invariance are all derived from Lagrangian mechanics. Using the form

$$L = T - U = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\dot{\vec{r}_i}\right)^2 - \sum_{i<j}^{N} U\left(|\vec{r}_i - \vec{r}_j|\right) \ ,$$

it's possible to show that a constant of motion arises for every transformation that leaves the Lagrangian unchanged, also called a symmetry.

Linear/Angular Momentum

Leaving the full derivations to a full study of Lagrangian mechanics, it turns out that a constant translation $\vec{\epsilon}$ of every position vector implies conservation of linear momentum:

$$\vec{r}_i \to \vec{r}_i + \vec{\epsilon} \qquad \to \qquad \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{\vec{r}_i}} = \vec{P} = \text{constant}$$

Meanwhile, a constant angular rotation of every position vector and velocity vector such that

$$\vec{r}_i \to \vec{r}_i + \delta\vec{r}_i \qquad \vec{v}_i \to \vec{v}_i + \delta\vec{v}_i \ ,$$

generates the conservation of momentum:

$$\sum_{i=1}^{N} \vec{r}_i \times \frac{\partial L}{\partial \vec{v}_i} = \sum_{i=1}^{N} \vec{r}_i \times m_i \vec{v}_i = \text{constant}$$

Energy

Conservation of energy arises from Lagrangian invariance with respect to variation in time. Using Poisson bracket notation, recall that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}_{qp} \ ,$$

and let $F = H$ to immediately find

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \ .$$

The Hamiltonian is simply $H = T + U$, which has implicit t-dependence, but no explicit t-dependence. Thus, $\partial H/\partial t = 0$ and $H = E = \text{constant}$.

Alternatively, take the Lagrangian

$$L = T - U = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\vec{v}_i\right)^2 - \sum_{i<j}^{N} U\left(|\vec{r}_i - \vec{r}_j|\right) \ ,$$

and add a small variation to all position and velocity vectors such that

$$\vec{r}_i \rightarrow \vec{r}_i + \delta\vec{r}_i \qquad\qquad \vec{v}_i \rightarrow \vec{v}_i + \delta\vec{v}_i \ ,$$

where each differential vector can be recast in terms of its derivative, as in

$$\delta\vec{r}_i = \vec{v}_i \, \delta t \qquad\qquad \delta\vec{v}_i = \vec{a}_i \, \delta t \ ,$$

with $\vec{a}_i$ being the acceleration vector. To first order in δ, the variation in the Lagrangian is

$$\delta L = \delta t \left(\sum_{i=1}^{N} \vec{v}_i \cdot \left(m_i \vec{a}_i - \sum_{j \neq i}^{N} \frac{\partial}{\partial \vec{r}_i} U \left(|\vec{r}_i - \vec{r}_j| \right) \right) \right) \ .$$

The inner parenthesized term is none other than Newton's second law applied to a single particle, and drops out, showing $\delta L \rightarrow 0$ for finite δt.

6.2 Noether Theorem

To crystallize the connection between symmetries and the Lagrangian, *Noether's theorem* states that any differentiable symmetry of the action S of a physical system has a corresponding conservation law.

6.3 Noether Current

Consider an infinitesimal change in a scalar field $\phi(q^\mu)$ such that

$$\phi \left(q^\mu \right) \rightarrow \phi \left(q^\mu \right) + \delta\phi \left(q^\mu \right) \ .$$

In response to the change $\delta\phi$, the Lagrangian density behaves as $\mathcal{L} \rightarrow \mathcal{L} + \delta\mathcal{L}$, where the change $\delta\mathcal{L}$ shall be expressed as a four-divergence $\partial_\mu \mathcal{J}^\mu$ via

$$\delta\mathcal{L} = \partial_\mu \mathcal{J}^\mu \ .$$

In terms of $\delta\mathcal{L}$, the variation in the action is simply

$$\delta S = \int_R \left(\delta\mathcal{L} \right) d^4q = \int_R \partial_\mu \mathcal{J}^\mu \, d^4q \ .$$

Now take the variation of the action $S = \int \mathcal{L}(\phi, \partial_\mu\phi) \, d^4q$ in the region R. To first-order approximation, the variation in the action is

$$\delta S = \int_R \left(\frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \partial_\mu \delta\phi \right) d^4q \ ,$$

where the equality $\delta\partial_\mu\phi = \partial_\mu\delta\phi$ has been used. Examining the second term, the chain rule tells us

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi \right) = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) \delta\phi + \frac{\partial\mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \partial_\mu \delta\phi \ ,$$

so δS becomes

$$\delta S = \int_R \left(\delta\phi \cancel{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right) \right)} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi \right) \right) d^4 q \; ,$$

where an instance of the Euler-Lagrange equation drops out.

Comparing the two versions of δS on hand, we must have

$$\delta S = \int_R \partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi \right) - \mathcal{J}^\mu \right) d^4 q \; ,$$

where taking the limit $\delta S \to 0$ we discover the *Noether current*:

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \delta\phi \right) - \mathcal{J}^\mu \; ,$$

clearly having zero divergence:

$$\partial_\mu j^\mu \left(\phi \left(q^\mu \right) \right) = 0$$

Noether Charge

Define a quantity called 'charge' that is the volume integral over the 0-component of the Noether current:

$$Q = \int_R j^0 \; d^3 q$$

Interpreting j^0 as a charge density implies the existence of a current vector $\vec{J}$ obeying the continuity equation

$$\partial_0 j^0 = -\vec{\nabla} \cdot \vec{J} \; .$$

Applying the divergence theorem to both sides, we find

$$\int_R \partial_0 j^0 \; d^3 q = -\int_R \vec{\nabla} \cdot \vec{J} \, d^3 q = \int_{\partial R} \vec{J} \cdot d\vec{A}$$

where the last integral vanishes by arguing that R may be sufficiently large such that ∂_R far out-paces growth in $\vec{J}$. Since the time-derivative of j^0 is zero under the integral, we have evidently found Q is conserved:

$$\frac{dQ}{dt} = \int_R \partial_0 j^0 \; d^3 q = 0$$

6.4 Rotations

Consider three scalar fields ϕ_i in three-dimensional space. To examine $SO\left(3\right)$ symmetry we let each field infinitesimally rotate about a unit vector n as

$$\phi_i \to \phi_i + \Delta\theta \epsilon_{ijk} n_j \phi_k \; .$$

Lagrangian Invariance

It's straightforward to show by substitution that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}\tilde{m}^2\phi_i\phi_i$$

remains invariant under infinitesimal rotations:

$$\begin{aligned}
\mathcal{L}' &= \frac{1}{2}\partial_\mu\left(\phi_i + \Delta\theta\epsilon_{ijk}n_j\phi_k\right)\partial^\mu\left(\phi_i + \Delta\theta\epsilon_{ijk}n_j\phi_k\right) \\
&\quad - \frac{1}{2}\tilde{m}^2\left(\phi_i + \Delta\theta\epsilon_{ijk}n_j\phi_k\right)\left(\phi_i + \Delta\theta\epsilon_{ijk}n_j\phi_k\right) \\
\mathcal{L}' &= \mathcal{L} + \frac{1}{2}\left(\partial_\mu\phi_i\partial^\mu\phi_k - \tilde{m}^2\phi_i\phi_k\right)\Delta\theta\cancel{\left(\epsilon_{ijk} + \epsilon_{kji}\right)}n_j + \cancel{O\left(\Delta\theta^2\right)} \\
\mathcal{L}' &= \mathcal{L}
\end{aligned}$$

Current

Evidently we have $\delta\mathcal{L} = 0$, and the corresponding Noether current is

$$j^\mu = \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi\right) - \cancel{\mathcal{J}^\mu} = \frac{1}{2}\left(\partial^\mu\phi_i\right)\Delta\theta\epsilon_{ijk}n_j\phi_k$$

7 Stress-Energy Tensor

Armed with the notions of Noether's theorem and the Noether current, we may explore symmetries in the Lagrangian density to uncover new statements of conservation. In this section we show that the conserved Noether current associated with spacetime translations is the *stress-energy tensor*, also known as the *energy-momentum tensor*. The stress-energy tensor is defined as a two-index object written as $T^{\mu\nu}$, and contains (you guessed it) all information about the energy, momentum, pressure, etc., associated with a field.

7.1 Derivation from Noether Current

Begin by shifting all position four-vectors by an infinitesimal constant four-vector a^μ such that

$$q^\mu \to q^\mu - a^\mu \ ,$$

causing the field and the variation in $\mathcal{L}$ to respond by

$$\delta\phi = -a^\nu \partial_\nu \phi \qquad\qquad \mathcal{J}^\mu = -a^\mu \mathcal{L} \ .$$

Inserting these into the Noether current gives

$$j^\mu = a^\nu \left(-\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi + \delta^\mu_\nu \mathcal{L} \right) = a_\nu \left(-\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L} \right)$$

The parenthesized term is identified as the stress-energy tensor $\hat{T}^{\mu\nu}$. Explicitly, we have found

$$j^\mu = a_\nu \hat{T}^{\mu\nu} \qquad\qquad \hat{T}^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L} \ ,$$

and because we know the Noether current is conserved, it follows that

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \ .$$

Note the 'hat' symbol ($\hat{}$) above each instance of T foreshadows a generalization of $\hat{T}^{\mu\nu}$ based on symmetry arguments (see below).

7.2 Derivation from Action

An equivalent derivation starts with the Lagrangian density $\mathcal{L}\left(q^\mu, \phi, \partial_\nu \phi\right)$, leading to variation in the action

$$\delta S = \int_R \left(\frac{\partial \mathcal{L}}{\partial q^\mu} \delta q^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\mu \delta \phi \right) d^4 q \ ,$$

where the equality $\delta \partial_\mu \phi = \partial_\mu \delta \phi$ has been used. Note also that δx^ν is equal to the infinitesimal shift a^ν, and (as we required in the previous derivation) $\delta\phi = -a^\nu \partial_\nu \phi$. Integrating the last term by parts (identically as before), we get

$$\delta S = \int_R \delta\phi \cancel{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)} \right) \right)} d^4 q + \int_R a^\nu \left(\frac{\partial \mathcal{L}}{\partial q^\nu} \right) d^4 q - \int_{\partial R} a^\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi \, d_\mu \sigma \ ,$$

where an instance of the Euler-Lagrange equation drops out. Note the boundary terms are *not* set to zero, but are instead retained in a surface integral bordering the region R. The final R-integral contains a derivative of $\mathcal{L}$, which can be recast as a surface integral on the same domain ∂R:

$$\delta S = \int_{\partial R} a^\nu \left(-\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\nu \phi + \delta^\mu_\nu \mathcal{L} \right) d_\mu \sigma = \int_{\partial R} a^\nu \hat{T}^\mu_\nu \, d_\mu \sigma = \int_{\partial R} a_\nu \hat{T}^{\mu\nu} \, d_\mu \sigma$$

Finally, we take the limit $\delta S \to 0$ to simultaneously (re)discover

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \qquad\qquad \hat{T}^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial^\nu \phi + \eta^{\mu\nu} \mathcal{L} \, .$$

7.3 Stress-Energy Tensor for Real Scalar Field

Consider a region R enclosed by ∂R, on which we define the action

$$S = \int_{\partial R} \sqrt{-\eta} \, \mathcal{L} \, d\sigma \, .$$

The Lagrangian density $\mathcal{L}$ shall be replaced by the form used for real scalar fields, i.e.

$$\mathcal{L} = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U\left(\phi\right) \, ,$$

and the index-free symbol η is the determinant of the metric tensor. (Including this term handles Lorentz transformations among coordinates q^μ.) The variation in S is thus

$$\delta S = \int_{\partial R} \left(\delta \left(\sqrt{-\eta} \right) \mathcal{L} + \sqrt{-\eta} \, \delta \mathcal{L} \right) d\sigma \, ,$$

subject to the variations

$$\eta^{\mu\nu} \to \eta^{\mu\nu} + \delta \eta^{\mu\nu} \qquad\qquad \delta \mathcal{L} = -\frac{1}{2} \delta \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, .$$

To proceed, take the identity

$$\delta^\mu_\nu = \left(\eta^{\mu\rho} + \delta \eta^{\mu\rho} \right) \left(\eta_{\rho\nu} + \delta \eta_{\rho\nu} \right)$$

and solve for $\delta \eta_{\mu\nu}$ to get, to first order,

$$\delta \eta_{\mu\nu} = -\eta_{\mu\alpha} \eta_{\beta\nu} \delta \eta^{\alpha\beta} \, .$$

Meanwhile, the variation $\delta \left(\sqrt{-\eta} \right)$ expands as

$$\delta \left(\sqrt{-\eta} \right) = -\frac{\delta \eta}{2\sqrt{-\eta}} = -\frac{1}{2} \sqrt{-\eta} \left(\eta_{\mu\nu} \delta \eta^{\mu\nu} \right) \, .$$

Putting the pieces together, the variation in S becomes

$$\delta S = \int_{\partial R} \left(-\frac{\sqrt{-\eta}}{2} \right) \delta \eta^{\mu\nu} \left(\partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \left(\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + U\left(\phi\right) \right) \right) d\sigma \, .$$

Taking the limit $\delta S \to 0$, we are handed the stress-energy tensor $\hat{T}_{\mu\nu}$ with low indices. Raising the each index, we finally find, for real scalar fields:

$$\hat{T}^{\mu\nu} = \eta^{\mu\alpha} \eta^{\beta\nu} \partial_\alpha \phi \partial_\beta \phi - \eta^{\mu\nu} \left(\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + U\left(\phi\right) \right)$$

Symmetry and Trace

The stress-energy tensor $\hat{T}^{\mu\nu}$ derived above is clearly symmetric, as swapping $\mu \leftrightarrow \nu$ leaves the tensor unchanged:

$$\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$$

Meanwhile, the trace

$$\hat{T}^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \delta^{\mu}_{\nu}\left(\frac{1}{2}\partial^{\rho}\phi\partial_{\rho}\phi + U\left(\phi\right)\right)$$

is readily shown not to vanish.

7.4 General Properties

The stress-energy tensor $\hat{T}^{\mu\nu}$ derived above must be repackaged for general use, namely because $\hat{T}^{\mu\nu}$ is only symmetric for real scalar fields. To remedy this, we introduce a new tensor $\mathcal{K}$ such that

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \partial_{\rho}\mathcal{K}^{[\rho\mu]\nu}\,,$$

where $\mathcal{K}$ is antisymmetric in the indices ρ, μ, and is also a total divergence:

$$\mathcal{K}^{[\rho\mu]\nu} = \frac{1}{2}\left(\mathcal{K}^{\rho\mu\nu} - \mathcal{K}^{\mu\rho\nu}\right) \qquad\qquad \partial_{\mu}\partial_{\rho}\mathcal{K}^{[\rho\mu]\nu} = 0$$

The general stress-energy tensor $T^{\mu\nu}$ obeys the following:

- $T^{\mu\nu}$ is symmetric:
$$T^{\mu\nu} = T^{\nu\mu}$$
- $T^{\mu\nu}$ is traceless:
$$T^{\mu}_{\mu} = 0$$
- The 00-component is positive-definite:
$$T^{00} > 0$$

8 Electromagnetism

8.1 Maxwell's Four Equations

Classical electrodynamics is neatly contained in Maxwell's four equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \, ,$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields in vacuum. The charge density ρ and current density $\vec{J}$ are subject to the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \, .$$

Electromagnetic Radiation

It's readily shown that electromagnetic wave equations emerge from Maxwell's equations plus the vector calculus identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{F} \right) - \nabla^2 \vec{F} \, .$$

In a region free of charges and currents, the time-derivative of each $\vec{\nabla}\times$-equation develops as follows:

$$\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 \vec{B}}{\partial t^2} \qquad \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

The final pair of relations are simultaneous wave equations having propagation speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \, .$$

8.2 Scalar Potential and Vector Potential

Insight from vector calculus tells us that the electric and magnetic fields $\vec{E}$, $\vec{B}$ can be recast as derivatives of other functions

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \, ,$$

where $\phi\left(\vec{x}, t\right)$ is the scalar potential, and $\vec{B}\left(\vec{x}, t\right)$ is the vector potential.

Equations of Motion

In terms of scalar and vector potentials, Maxwell's equations readily yield two new equations of motion:

$$\nabla^2\phi + \frac{\partial\left(\vec{\nabla}\cdot\vec{A}\right)}{\partial t} = -\frac{\rho}{\epsilon_0}$$

$$-\mu_0\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \nabla^2\vec{A} = \Box\vec{A}$$

Calculating Potentials

For a given charge distribution $\rho(\vec{x},t)$ and current density $\vec{J}(\vec{x},t)$, the scalar and vector potentials are given by

$$\phi\left(\vec{x},t\right) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho\left(\vec{x}',t_r\right)}{|\vec{x}-\vec{x}'|} \qquad \vec{A}\left(\vec{x},t\right) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}\left(\vec{x}',t_r\right)}{|\vec{x}-\vec{x}'|}\,,$$

where

$$t_r = t - \frac{|\vec{x}-\vec{x}'|}{c}$$

is the *retarded time*.

8.3 Electromagnetic Field Strength Tensor

Four-Potential

To begin updating the full apparatus of electromagnetism with index-notation, merge the electric scalar potential with the magnetic vector potential into the *electromagnetic four-potential*:

$$A^\mu(x^\mu) = \left(\frac{1}{c}\phi\left(\vec{x},t\right), \vec{A}\left(\vec{x},t\right)\right)$$

Next, we construct a Lagrangian L for a particle of mass m and charge q that is the sum of a kinetic term plus a 'minimal coupling' to the covariant version of the four-potential as

$$L = \frac{m}{2}U^\mu U_\mu + \frac{q}{2}A_\mu U^\mu\,,$$

where U^μ is four-velocity, that is, the proper time derivative of the position four-vector x^μ. The Euler-Lagrange equation

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial L}{\partial U^\mu}$$

readily tells us

$$m\frac{dU_\mu}{d\tau} = qU^\nu\left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right) = qU^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,.$$

The parenthesized quantity is known as the *electromagnetic field strength tensor*:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Derivation

Observe that $F_{\mu\nu}$ is an antisymmetric tensor, which means

$$F_{\nu\mu} = -F_{\mu\nu} \ ,$$

indicating that the 'diagonal' entries $F_{\mu\mu}$ are identically zero, thus there are six independent components in F. The nonzero components of $F_{\mu\nu}$ are straightforward to evaluate using the four-potential $A_\mu = (-\phi/c, \vec{A})$. Begin with F_{0i}, with $i = (1, 2, 3)$, giving the i-th component of the electric field:

$$F_{0i} = \left(\partial_t \vec{A}/c + \vec{\nabla}\phi/c\right)_i = -\frac{E_i}{c}$$

The magnetic field components are contained in terms F_{ij}, namely:

$$F_{23} = \left(\vec{\nabla} \times \vec{A}\right)_x \qquad F_{13} = -\left(\vec{\nabla} \times \vec{A}\right)_y \qquad F_{12} = \left(\vec{\nabla} \times \vec{A}\right)_z$$

Altogether we have, in block form (not a matrix):

$$F_{\mu\nu} = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Note the components F_{ik} can be written more clearly in terms of $\vec{B}$ by using the Levi-Civita symbol:

$$F_{ik} = \sum_{k=1}^{3} \epsilon_{ijk} B_k$$

Using the Minkowski metric $\eta_{\mu\nu}$, the $(2, 0)$ and $(1, 1)$ forms of F work out to:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \qquad F^{\mu}_{\ \nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Invariants

The inner product of the electromagnetic field tensor is equal to a Lorentz invariant:

$$F_{\mu\nu} F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

A pseudo-scalar invariant involves a contraction with the Levi-Civita symbol:

$$\frac{1}{2}\epsilon_{\alpha\beta\gamma\rho} F^{\alpha\beta} F^{\gamma\rho} = -\frac{4}{c}\vec{E} \cdot \vec{B}$$

The determinant of F yields yet another Lorentz invariant:

$$\det(F) = \frac{c}{c^2}\left(\vec{E} \cdot \vec{B}\right)^2$$

8.4 Equation of Motion

In terms of the electromagnetic field tensor, we have found (after adjusting the indices for aesthetics), the relativistic equation of motion of a particle in mixed fields in flat space:

$$\frac{d^2x^\mu}{d\tau^2} = \frac{q}{m} F^\mu_{\ \nu} \frac{dx^\nu}{d\tau}$$

Lorentz Force Law

The Lorentz force law $\vec{F} = q\vec{v} \times \vec{B}$ is only correct in the non-relativistic limit. The true force law is derived from the above, giving:

$$\frac{d\left(E - E_0\right)}{dt} = \frac{dT}{dt} = \frac{q\vec{E} \cdot \vec{v}}{\gamma^2} \qquad\qquad m\left(\frac{d^2\vec{x}}{d\tau^2}\right) = q\gamma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Boosted E and B Fields

Consider two inertial reference frames in relative motion at speed v in the x-direction. Using the Lorentz transformation $(F')^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$ the components of (F') in terms of the unprimed fields are straightforward to compute:

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - vB_z\right) & B'_y &= \gamma\left(B_y + vE_z/c^2\right) \\ E'_z &= \gamma\left(E_z + vB_y\right) & B'_z &= \gamma\left(B_z - vE_y/c^2\right) \end{aligned}$$

(Note this is not the *whole* story, because the components of x^μ must also be Lorentz-transformed.) We can extend the above to handle boosts in any direction $\vec{v}$ as:

$$\begin{aligned} E'_\parallel &= E_\parallel & B'_\parallel &= B_\parallel \\ \vec{E}'_\perp &= \gamma\left(\vec{E}_\perp - \vec{v} \times \vec{B}\right) & \vec{B}'_\perp &= \gamma\left(\vec{B}_\perp + \vec{v} \times \vec{E}/c^2\right) \end{aligned}$$

Particle in Electric Field

Consider a particle of mass m and charge q initially at rest at the origin in a constant electric field $\vec{E} = E\,\hat{x}$. The relativistic equation of readily tells us

$$\frac{dU^t}{d\tau} = \frac{q}{m}\frac{E}{c}U^x \qquad\qquad \frac{dU^x}{d\tau} = \frac{q}{m}\frac{E}{c}U^t\ ,$$

where cross substituting the τ-derivative of each equation gives

$$\frac{d^2U^t}{d\tau} = \left(\frac{q}{m}\frac{E}{c}\right)^2 U^t \qquad\qquad \frac{d^2U^x}{d\tau^2} = \left(\frac{q}{m}\frac{E}{c}\right)^2 U^x\ ,$$

indicating hyperbolic solutions

$$U^t\left(\tau\right) = c\cosh\left(\frac{qE}{mc}\tau\right) \qquad\qquad U^x\left(\tau\right) = c\sinh\left(\frac{qE}{mc}\tau\right)\ .$$

Recalling that $U^t = d\left(ct\right)/d\tau$ and $U^x = dx/d\tau$, the above can be integrated:

$$t\left(\tau\right) = \left(\frac{mc}{qE}\right)\sinh\left(\frac{qE}{mc}\tau\right) \qquad\qquad x\left(\tau\right) = \left(\frac{mc^2}{qE}\right)\cosh\left(\frac{qE}{mc}\tau\right) - \frac{mc^2}{qE}$$

Using the identity

$$\cosh\left(\sinh^{-1}\left(x\right)\right) = \sqrt{1+x^2}\,,$$

the results are combined to solve for $x\left(t\right)$:

$$x\left(t\right) = \frac{c}{qE}\sqrt{\left(qEt\right)^2 + \left(mc\right)^2} - \frac{mc^2}{qE}$$

Particle in Magnetic Field

Consider a particle of mass m and charge q initially at rest at in a constant magnetic field $\vec{B} = B\,\hat{x}$. The relativistic equation of readily tells us

$$\frac{dU^y}{d\tau} = \frac{qB}{m}U^z \qquad\qquad \frac{dU^z}{d\tau} = -\frac{qB}{m}U^y\,,$$

where cross substituting the τ-derivative of each equation gives

$$\frac{d^2U^y}{d\tau} = -\left(\frac{qB}{m}\right)^2 U^y \qquad\qquad \frac{d^2U^z}{d\tau^2} = -\left(\frac{qB}{m}\right)^2 U^z\,,$$

indicating sinusoidal solutions

$$U^y\left(\tau\right) = A\sin\left(\frac{qB}{m}\tau + \phi_0\right) \qquad\qquad U^z\left(\tau\right) = A\cos\left(\frac{qB}{m}\tau + \phi_0\right)$$

where A and ϕ_0 are integration constants. Integrating once more gives equations for $y\left(\tau\right)$, $z\left(\tau\right)$, namely:

$$y\left(\tau\right) = -A\frac{m}{qB}\cos\left(\frac{qB}{m}\tau + \phi_0\right) + y_0 \qquad\qquad z\left(\tau\right) = A\frac{m}{qB}\sin\left(\frac{qB}{m}\tau + \phi_0\right) + z_0$$

Meanwhile, the t- and x- components of the four-velocity are

$$U^t\left(\tau\right) = c\frac{dt}{d\tau} \qquad\qquad U^x\left(\tau\right) = \dot{x}_0\,.$$

The norm of the four-velocity always resolves to $-c^2$, allowing us to solve for $t\left(\tau\right)$ by integration (of the gamma factor):

$$t = \tau\sqrt{1 + \frac{\dot{x}_0^2 + A^2}{c^2}}$$

Motionless Particle

Consider a point charge of magnitude q fixed at the origin in a reference frame q^μ. The electric and magnetic fields $\vec{E}$, $\vec{B}$ for all points in space are given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}}{|\vec{x}|^3} \qquad\qquad \vec{B} = 0 \,.$$

A boosted reference frame moving along the x-direction at speed v observes the same point charge. Using the Lorentz transformation to find the electric and magnetic fields $(\vec{E})'$, $(\vec{B})'$ at all points in the boosted frame, we must have

$$(\vec{E})' = \frac{q}{4\pi\epsilon_0}\frac{\left(\gamma\left(x'-vt'\right),\gamma y',\gamma z'\right)}{\left(\gamma^2\left(x'-vt'\right)^2+y'^2+z'^2\right)^{3/2}}$$

$$(\vec{B})' = \frac{v}{c^2}\frac{q}{4\pi\epsilon_0}\frac{\left(0,\gamma z',-\gamma y'\right)}{\left(\gamma^2\left(x'-vt'\right)^2+y'^2+z'^2\right)^{3/2}} \,.$$

For the case of ultra-relativistic motion $v \to c$, the primed fields are

$$(\vec{E})' = \delta\left(x'-ct'\right)\frac{2q}{4\pi\epsilon_0}\frac{\left(0,y',z'\right)}{y'^2+z'^2}$$

$$(\vec{B})' = \delta\left(x'-ct'\right)\frac{v}{c^2}\frac{2q}{4\pi\epsilon_0}\frac{\left(0,z',-y'\right)}{y'^2+z'^2} \,.$$

Note that the above fields are confined to a two-dimensional plane that extends orthogonally to the direction of motion, called an electromagnetic shock wave, whose derivation is facilitated by

$$\lim_{\beta\to 1}\frac{1}{\sqrt{1-\beta^2}}f\left(\frac{x-ct}{1-\beta^2}\right) = \delta\left(x-ct\right)\int_{-\infty}^{\infty} f\left(w\right)dw \,,$$

along with the theory of distributions.

8.5 Currents

Similar to the rest mass, the quantity of charge is invariant in all reference frames. On the other hand, the notion of current involves charge density and velocity, and must be treated to accommodate special relativity.

Four-Current

Multiply the rest-frame charge density ρ_0 into the velocity four-vector to write the *four-current*:

$$J^\mu = \left(\rho_0\gamma c, \rho_0\gamma\vec{v}\right) = \left(\rho c, \rho\vec{v}\right) = \left(\rho c, \vec{J}\right)$$

The non-relativistic continuity equation is the four-gradient of the four-current:

$$\partial_\mu J^\mu = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Boosted Current

In the same way that space and time interweave in special relativity, the current and the density become mixed under general coordinate transformations. In a boosted reverence frame moving at speed v in the x-direction, the components of $J^{\mu'}$ are given by the Lorentz transformation

$$J^{\mu'} = \Lambda^{\mu'}_{\nu} J^{\nu}$$

such that

$$\rho' = \gamma \left(\rho - vJ^x/c^2\right) \qquad\qquad J^{x'} = \gamma \left(J^x - v\rho\right) \, .$$

Gauss-Ampere Law

The key objects of electromagnetism are the four-potential A^μ, the electromagnetic field tensor $F^{\mu\nu}$, and the four-current J^μ. Assembling these into a (scalar) Lagrangian density (and adjusting ahead for units), let us take

$$\mathcal{L}_{EM} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu$$

as a starting point. Applying the Euler-Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu A_\mu\right)} \right) = 0 \, ,$$

we find

$$\begin{aligned}
\mu_0 J^\mu &= \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu A_\mu\right)} \right) \\
&= -\frac{1}{4} \eta^{\alpha\rho} \eta^{\beta\sigma} \partial_\nu \left(\frac{\partial}{\partial \left(\partial_\nu A_\mu\right)} \left(F_{\alpha\beta} F_{\rho\sigma}\right) \right) \\
&= -\frac{1}{4} \eta^{\alpha\rho} \eta^{\beta\sigma} \partial_\nu \left(F_{\rho\sigma} \left(\delta^\nu_\alpha \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\alpha \right) + F_{\alpha\beta} \left(\delta^\nu_\rho \delta^\mu_\sigma - \delta^\nu_\sigma \delta^\mu_\rho \right) \right) \\
&= -\frac{1}{4} \partial_\nu \left(F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right) \\
-\mu_0 J^\mu &= \partial_\nu F^{\nu\mu} \, ,
\end{aligned}$$

a result known as the *Gauss-Ampere law* in index notation.

8.6 Maxwell's Two Equations

Using index notation, the set of four Maxwell's equations can be reduced to two equations.

Four-Curl

The electromagnetic field tensor obeys a cyclic derivative equation (an analog to the Bianchi identity from Riemann geometry)

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \, ,$$

also known as the *four-curl* of the electromagnetic field strength tensor. Note that when two indices are equal, the above yields $0 = 0$, and permutations in the indices yield no new information.

Choosing

$$\rho = 1 \qquad \mu = 2 \qquad \nu = 3\,,$$

we get the familiar statement about the divergence of any magnetic field, namely

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = \vec{\nabla} \cdot \vec{B} = 0\,.$$

Meanwhile, writing three instances with $(0,1,2)$, $(0,2,3)$, $(0,3,1)$ gives

$$\partial_t B_z + \partial_x E_y - \partial_y E_x = 0$$

$$\partial_t B_x + \partial_y E_z - \partial_z E_y = 0$$

$$\partial_t B_y + \partial_z E_x - \partial_x E_z = 0\,,$$

which add to deliver another of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,.$$

The Gauss-Ampere law isn't a new law, but in fact contains the two Maxwell's equations *not* contained in the four-curl of F. Writing out the law explicitly,

$$\partial_\nu F^{\nu\mu} = -\mu_0 J^\mu$$

breaks into:

$$\partial_\nu F^{\nu\mu} = \begin{bmatrix} -\vec{\nabla} \cdot \vec{E}/c \\ c^{-2}\partial\vec{E}/\partial t - \vec{\nabla} \times \vec{B} \end{bmatrix} = \begin{bmatrix} -\mu_0 \rho c \\ -\mu_0 \vec{J} \end{bmatrix} = -\mu_0 J^\mu$$

8.7 Gauge Fixing

The equations of electromagnetism stem from derivatives the four-potential A^μ. The four-potential is not uniquely determined, but is set by the *gauge* of the theory being applied.

Lorenz Gauge

The continuity equation $\partial_\mu J^\mu = 0$ automatically implies that the four-potential may be translated by a harmonic scalar function f^μ that obeys $\partial^\mu \partial_\mu f = 0$, namely

$$A^\mu \rightarrow A^\mu + f^\mu\,,$$

with no physical consequences, i.e. $F_{\mu\nu}$ remains unchanged:

$$(F')_{\mu\nu} = \partial_\mu (A_\nu + f_\nu) - \partial_\nu (A_\mu + f_\mu) = F_{\mu\nu}$$

The translation $A^\mu \rightarrow A^\mu + f^\mu$ may also be written

$$\partial_\mu A^\mu = 0\,,$$

known as the *Lorenz gauge-fixing condition.* (Note 'Lorenz' is not a misspelling of 'Lorentz' - these are two different names if this subject wasn't confusing already.) In ordinary vector notation, the Lorenz gauge-fixing condition is written

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \, ,$$

which greatly simplifies the corresponding equations of motion (previously written) to inhomogeneous wave equations

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$
$$-\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \nabla^2 \vec{A} = -\mu_0 \vec{J} + \vec{\nabla}\cancel{\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)} \, ,$$

more succinctly written in terms of the d'Alembert operator as

$$\Box A^\mu = \mu_0 J^\mu \, .$$

Coulomb Gauge

In contrast to the Lorenz gauge, which makes no reference to a preferred reference frame, we may also work in the so-called Coulomb gauge, also known as the 'radiation' or 'transverse' gauge-fixing condition by setting

$$\vec{\nabla} \cdot \vec{A} = 0 \, .$$

The corresponding equations of motion become:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$
$$-\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \nabla^2 \vec{A} = -\mu_0 \vec{J} + \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)$$

It can be shown that the final term in the above can be restated as μ_0 times the longitudinal component of the current density J_l such that $\vec{\nabla} \times \vec{J_l} = 0$. The transverse component of the current density J_t has the property $\vec{\nabla} \cdot \vec{J_t} = 0$ and $\vec{J} = \vec{J_l} + \vec{J_t}$ according to Helmholtz's theorem. The latter equation is thus:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J_t}$$

8.8 Electromagnetic Stress-Energy Tensor

The most general form of the stress-energy tensor for a covariant vector field A_σ was found to be

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\sigma\right)}\partial^\nu A_\sigma + \eta^{\mu\nu}\mathcal{L} + \partial_\rho \mathcal{K}^{[\rho\mu]\nu} \, ,$$

where the $\mathcal{K}$-term assures gauge invariance and the zero-trace property of $T^{\mu\nu}$. Using the Lagrangian density

$$\mathcal{L}_{EM} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu} \, ,$$

along with setting

$$\partial_{\rho} \mathcal{K}^{[\rho\mu]\nu} = \frac{1}{\mu_0} F^{\mu\rho} \partial_{\rho} A^{\nu} \, ,$$

the electromagnetic stress-energy tensor reads

$$T^{\mu\nu}_{EM} = \frac{1}{\mu_0} \left(F^{\mu\rho} F^{\nu}_{\rho} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) - \eta^{\mu\nu} J^{\rho} A_{\rho} \, .$$

Properties

For spaces enclosing no charges or currents ($J^{\mu} = 0$), the homogeneous electromagnetic stress-energy tensor is easily shown to be symmetric, have zero divergence, and have zero trace:

$$T^{\mu\nu}_{EM} = T^{\nu\mu}_{EM} \qquad \partial_{\mu} T^{\mu\nu}_{EM} = 0 \qquad (T_{EM})^{\mu}_{\mu} = 0$$

Components

Determining the components of $T^{\mu\nu}_{EM}$ is a matter of brute-force calculation based on our previous achievements. In space not enclosing charges or currents, the homogeneous version ($J^{\lambda} = 0$) reads

$$T^{\mu\nu} = \begin{bmatrix} \epsilon_0 E^2/2 + B^2/2\mu_0 & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix} \, ,$$

where S_j are components of the *Poynting vector*

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \, ,$$

and any purely spatial components are contained in the *Maxwell stress tensor* σ_{ij}:

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \delta_{ij}$$

9 Matter and Fluids

From the point of view of fields, manifolds, and spacetime geometry, matter is modeled as a continuous fluid that ignores quantum mechanics. Accounting for special relativity, the notion of material volume is subject to length contraction, however the number of particles per unit volume is a Lorentz invariant.

9.1 Number-Flux Four-vector

At any point in space mapped by coordinates q^μ, there exists a dimensionless matter density scalar $n(q^\mu)$. Coupling this with the velocity four-vector gives the *number-flux four-vector*,

$$N^\mu = n(q^\mu)\, U^\mu(q^\mu) \ .$$

In the non-relativistic limit, N^μ reduces to

$$N^\mu = \Big(c\, n(\vec{x}, t)\, , \vec{J}(\vec{x}, t)\Big) \ ,$$

where $\vec{J}$ is the non-relativistic number-flux three-vector. Taking the divergence of N, we find

$$\partial_\mu N^\mu = \frac{\partial}{\partial t} n(\vec{x}, t) + \vec{\nabla} \cdot \vec{J}(\vec{x}, t) \ ,$$

identical in form to the electric and gravitational continuity equations. To ensure conservation of matter, we set the equation equal to zero:

$$\partial_\mu N^\mu = 0$$

9.2 Stress-Energy Tensor for Matter

By definition, the stress-energy tensor $T^{\mu\nu}$ is the flux of the four-momentum $P^\mu = (E/c, \vec{p})$ across a surface of constant $q^\mu = (ct, \vec{q})$. Using suitable notation for neutral matter, the components of $T^{\mu\nu}$ are:

$$\begin{aligned}
T^{00} &= \frac{\Delta E}{d^3 q} = \text{Energy Density} \\
T^{i0} &= \frac{\Delta p^i}{d^3 q} = \text{Momentum Density} \\
T^{0i} &= \frac{\Delta E}{\Delta t} \frac{1}{\Delta q^j \Delta q^k} = \text{Energy Flux} \\
T^{ij} &= \frac{\Delta p^i}{\Delta t} \frac{1}{\Delta q^i \Delta q^k} = \text{Shear Stress or Momentum Flux} \\
T^{ii} &= \frac{\Delta p^i}{\Delta t} \frac{1}{\Delta q^j \Delta q^k} = \text{Pressure}
\end{aligned}$$

9.3 Isolated Particle

A single particle of mass m that traces out the trajectory $\vec{q}_p$ with velocity components V^μ has

$$T^{\mu\nu} = m\frac{V^\mu V^\nu}{\sqrt{1 - V^2/c^2}}\delta\left(\vec{q} - \vec{q}_p\right) \ .$$

Since we know $E = \gamma mc^2$, we may instead write

$$T^{\mu\nu} = E\frac{V^\mu V^\nu}{c^2}\delta\left(\vec{q} - \vec{q}_p\right) \ ,$$

where E may also be replaced via $E^2 = p^2c^2 + m^2c^4$. At rest, $T^{\mu\nu}$ simplifies to

$$T^{\mu\nu} = mc^2\delta\left(\vec{q}_p\right)\delta^{00} \ .$$

9.4 Perfect Fluid

A continuum of matter in thermodynamic equilibrium is known as a *perfect fluid*. Treating the four-velocity as a vector field, each point in space caries a stress-energy tensor given by

$$T^{\mu\nu} = \left(\rho + p/c^2\right) U^\mu U^\nu + \eta^{\mu\nu} p \ ,$$

where ρ is the mass-density, and $\vec{p}$ is the isotropic pressure. In its rest frame, the stress-energy tensor for a perfect fluid has only diagonal components:

$$T^{\mu\nu} = \text{diag}\left(\rho c^2, p, p, p\right)$$

Boosted Fluid

Consider a perfect fluid at rest in the coordinate system q^μ. A boosted system moving at speed $v = \beta c$ along the x-direction mapped by $q^{\mu'}$ that observes the fluid will carry a different stress-energy tensor given by

$$T^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}T^{\mu\nu} \ .$$

The non-zero components of $T^{\mu'\nu'}$ work out to:

$$(T')^{00} = \Lambda^0_0\Lambda^0_0 T^{00} + \Lambda^0_1\Lambda^0_1 T^{11} \qquad (T')^{11} = \Lambda^1_1\Lambda^1_1 T^{11} + \Lambda^1_0\Lambda^1_0 T^{00}$$

$$(T')^{00} = \frac{\rho c^2 + p\beta^2}{1 - \beta^2} \qquad (T')^{11} = \frac{p + \rho v^2}{1 - \beta^2}$$

10 Einstein's Field Equation

10.1 Riemannian Geometry

Index notation and tensors are a natural language for Riemannian geometry, which applies calculus on multi-dimensional curved manifolds admitting a general metric tensor $g_{\mu\nu}$.

Covariant Derivative

To begin, the notion of the partial derivative must be replaced by the covariant derivative in order to ensure that tangent vectors remain on the manifold via:

$$\partial_\nu V^\mu \qquad \rightarrow \qquad D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\rho} V^\rho \ ,$$

where the connection coefficients $\Gamma^\rho_{\mu\nu}$ (not a tensor!) are given in terms of the metric tensor as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \ .$$

Geodesics

The un-accelerated equation of motion is the geodesic equation

$$0 = \frac{dU\rho}{d\tau} + \Gamma^\rho_{\mu\nu} U^\mu U^\nu \ .$$

Curvature

For curved space, the Riemann curvature tensor

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \ ,$$

is nonzero, where a contraction over the top and bottom-middle indices yields the more accessible Ricci tensor

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \ .$$

The Ricci tensor can be projected into a scalar by contracting indices via

$$g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu = R \ .$$

Einstein Tensor

A particular combination of Ricci-objects combine to a new tensor that has zero covariant derivative, called the Einstein tensor:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \qquad\qquad D_\nu G^{\mu\nu} = 0$$

10.2 Weak Curvature

A weakly-curved space admits a perturbed Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} ,$$

where $h_{\mu\nu}$ is the non-flat component. In the non-relativistic limit, the geodesic equation tells us

$$\frac{d^2\vec{q}}{dt^2} = \frac{c^2}{2}\vec{\nabla} h_{00} ,$$

which implies a curved surface is analogous to acceleration. Finally, note it's readily shown that the time-component of the Ricci tensor reads

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00} .$$

10.3 Curvature and Gravity

With all the elements laying around, Einstein eventually postulated that perhaps the acceleration that arises due to curvature is *gravity*. In the weak-field case, it would immediately follow that

$$h_{00} = -\frac{2}{c^2}V(\vec{r}) \qquad\qquad R_{00} = \frac{1}{c^2}\nabla^2 V(\vec{r}) ,$$

where $V(\vec{r})$ is the gravitational potential scalar. Of course, the Laplacian operator ∇^2 acting on the gravitational potential resolves to $4\pi G\rho_m$, where G is the Newton's gravitation constant, and ρ_m is the mass-density. That is,

$$R_{00} = \frac{4\pi G}{c^2}\rho_m$$

10.4 Curvature and Matter

Knowing that $G^{\mu\nu}$ has zero covariant derivative on curved manifolds, Einstein further speculated that the general stress-energy tensor $T^{\mu\nu}$, which must also have zero covariant derivative, is in fact proportional to $G^{\mu\nu}$ via

$$D_\nu G^{\mu\nu} = D_\nu T^{\mu\nu} = 0 \qquad \rightarrow \qquad G^{\mu\nu} \propto T^{\mu\nu} ,$$

where we simply have to find the proportionality constant κ in

$$G^{\mu\nu} = \kappa T^{\mu\nu} .$$

To proceed, take the low-index version of the Einstein tensor and contract with $g^{\mu\nu}$, giving

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \kappa g^{\mu\nu}T_{\mu\nu} .$$

Note that $g^{\mu\nu}R_{\mu\nu}$ is simply the Ricci scalar R, and the product $g^{\mu\nu}g_{\mu\nu}$ counts the dimension of the manifold, which we take to be four. Also, note that the product $g_{\mu\nu}T^{\mu\nu}$ is the trace of the stress energy-tensor, denoted T. With this in mind, the above reduces to

$$-R = \kappa T ,$$

such that the field $G_{\mu\nu}$ now reads

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \kappa T = \kappa T_{\mu\nu} \ .$$

Isolating the 00-component, we have

$$R_{00} + \frac{1}{2} \kappa T = \kappa T_{00} \ .$$

Since all we're doing is solving for the constant κ, we're free to work in the non-relativistic limit. Taking the perfect fluid at rest, it follows that $T_{00} = \rho_m c^2$, and the trace of T is $-\rho_m c^2$. Evidently then,

$$R_{00} = \frac{1}{2} \kappa \rho_m c^2 \ .$$

Comparing R_{00} to the result that arises from weak curvature, κ may finally be isolated:

$$\frac{1}{2} \kappa \rho_m c^2 = \frac{4\pi G}{c^2} \rho_m \qquad \rightarrow \qquad \kappa = \frac{8\pi G}{c^4}$$

Finally, we write the mighty Einstein Field equation

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \ ,$$

telling us that the presence of matter and energy cause curvature in spacetime, and that curved spacetime is the literal interpretation of gravity.