Analytic Geometry
MANUSCRIPT

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Chapter 1

Analytic Geometry

tions. This is the generalization of the typical $y = f(x)$ construction, where instead of using one function, the same information is contained in two equations

$$x = g(t)$$
$$y = h(t) \ ,$$

where t is the parameter. Parametric equations are helpful because the form $y = f(x)$ may not be straightforwardly attained.

1.1 Parametric Systems

Kinematics

Let v_x, v_y be the components of the initial velocity of a moving body in the presence of uniform gravity without resistance. Starting from the position (x_0, y_0) and evolving with the time parameter t, the kinematic equations of motion read:

$$x(t) = x_0 + v_x t$$
$$y(t) = y_0 + v_y t - \frac{1}{2} g t^2$$

Ellipse

Consider the ellipse centered at the origin:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Defining a parameter ϕ in the domain $[0 : 2\pi]$, the same ellipse has the following parametric representation:

$$x = a\cos(\phi)$$
$$y = b\sin(\phi)$$

Hyperbola

Consider the hyperbola centered at the origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Defining a parameter ϕ in the domain $[0 : 2\pi]$, the same hyperbola has the following parametric representation:

$$x = a\cosh(\phi)$$
$$y = b\sinh(\phi)$$

Cycloid

The shape that solves the tautochrone problem, i.e. the cycloid, can *only* be represented by a pair of parametric equations. For a generating circle of radius R, we have

$$x = R\phi - R\sin(\phi)$$
$$y = R - R\cos(\phi) \ ,$$

where the parameter ϕ can be any real number. For one period of the cycloid, confine $0 \leq \phi \leq 2\pi$.

Involute

Unwind a string from a circle of radius R while maintaining tension. The endpoint of the string traces a shape called an involute. In terms of an 'unwinding parameter' ϕ, the involute is given by

$$x(\phi) = R\cos(\phi) + R\phi\sin(\phi)$$
$$y(\phi) = R\sin(\phi) - R\phi\cos(\phi) \ .$$

Lissajous Figures

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Scalar Multiplication

For a vector $\vec{m}$ that represents the slope of a straight line, along with a vector $\vec{b} = \langle 0, b \rangle$ as the y-intercept, the whole line can be represented by the vector

$$\vec{r}(\alpha) = \alpha\,\vec{m} + \vec{b}\,,$$

where the parameter α is any real number.

1.2 Polar Coordinate System

The polar coordinate system is a two-parameter apparatus. A point (x, y) in the plane requires two pieces information to specify:

$$x = x(r, \theta) = r\cos(\theta)$$
$$y = y(r, \theta) = r\sin(\theta)$$

It's also possible to frame r and θ in terms of each other, which is to say $r(\theta)$ and $\theta(r)$ are legal residents.

Of course, each can also be framed in terms of a general parameter t to make $r = r(t)$, $\theta = \theta(t)$. In this case, a point in the plane is represented by:

$$x = x(t) = r(t)\cos(\theta(t))$$
$$y = y(t) = r(t)\sin(\theta(t))$$

2 Parametric Derivatives

In a typical parametric system, we have separate entities $x(t)$ and $y(t)$ representing a single curve. Despite not having a $y = f(x)$ representation, we're still allowed to ask about the slope dy/dx of such a curve in the Cartesian plane. To this end, we simply employ the chain rule. For a parametric system, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} .$$

As per usual with the chain rule, you can see the dt-factor canceling on the right.

Of course, the above presumes that dx/dt is not zero at the point(s) of interest. The above can be easily inverted to get

$$\frac{dx}{dy} = \frac{dx/dt}{dy/dt} .$$

2.1 Slope in Polar Coordinates

Consider the polar coordinate system where r is known to be a function of θ:

$$\begin{aligned} x &= r(\theta)\cos(\theta) \\ y &= r(\theta)\sin(\theta) \end{aligned}$$

With respect to the parameter θ, we find:

$$\begin{aligned} dx/d\theta &= r'(\theta)\cos(\theta) - r(\theta)\sin(\theta) \\ dy/d\theta &= r'(\theta)\sin(\theta) + r(\theta)\cos(\theta) \end{aligned}$$

Then, by the chain rule, we have:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}$$

Divide by the cosine term to get a slightly neater formula:

$$\frac{dy}{dx} = \frac{r'(\theta)\tan(\theta) + r(\theta)}{r'(\theta) - r(\theta)\tan(\theta)}$$

Of course, the slope dy/dx is the instantaneous rise over run, which means the ratio of these lengths is the tangent of another angle ϕ measured from the positive x-axis:

$$\frac{dy}{dx} = \tan(\phi)$$

Psi Parameter

Staying in this picture, consider yet another angle ψ (Greek 'psi') measured from the position vector (rather than the x-axis) that terminates at the tangent line, which brings forth the identity

$$\theta + \psi - \phi = 0 .$$

Using the trig identity

$$\tan(\psi) = \frac{\tan(\phi) - \tan(\theta)}{1 + \tan(\phi)\tan(\theta)} ,$$

we find

$$\tan(\psi) = \frac{(dy/dx) - \tan(\theta)}{1 + (dy/dx)\tan(\theta)} ,$$

where dy/dx can be replaced by the slope formula. Simplifying like crazy, the above boils down to:

$$\tan(\psi) = \frac{r(\theta)}{r'(\theta)}$$

We can keep going with this. Take the reciprocal of the the equation and then notice the right side looks like a derivative

$$\cot(\psi) = \frac{r'(\theta)}{r(\theta)} = \frac{d}{d\theta}\left(\ln(r(\theta))\right) ,$$

and then this can be integrated to simplify the right side:

$$\int \cot(\phi - \theta)\, d\theta = \ln(r(\theta))$$

Logarithmic Spiral

While the above is a general statement, it's a bit unworkable in the general case. However, consider a regime where

$$\cot(\phi - \theta) = k ,$$

where k is constant. Then, the integral immediately becomes the equation of a *logarithmic spiral*:

$$r(\theta) = r_0\, e^{k\theta}$$

2.2 Parametric Second Derivative

The second derivative d^2y/dx^2 is straightforwardly stated:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Then, using the chain rule, we find:

$$\frac{d^2y}{dx^2} = \left(\frac{1}{dx/dt}\right)\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

3 Parametric Integrals

In standard integral calculus, we know the area A under the curve $y(x)$ between the limits x_0, x_1 is given by

$$A = \int_{x_0}^{x_1} y(x)\, dx\ .$$

In the parametric regime, begin instead with

$$\begin{aligned} x &= g(t) \\ y &= h(t)\ , \end{aligned}$$

wher $y(x)$ may be difficult or perhaps impossible to attain. Proceed by noting

$$dx = \frac{dg}{dt}\, dt = g'(t)\, dt\ ,$$

and also

$$\begin{aligned} x_0 &= g(t_0) \\ x_1 &= g(t_1)\ , \end{aligned}$$

i.e., the integration limits are indicated by t_0, t_1. In this set of variables, the area integral takes the form:

$$A = \int_{t_0}^{t_1} h(t)\, g'(t)\, dt$$

3.1 Parametric Arc Length

The typical starting point for the arc length calculation is given by the indefinite integral

$$S = \int dS = \int \sqrt{dx^2 + dy^2}\ ,$$

where the 'standard' move is to then factor out dx from the radical.

To handle the parametric regime, multiply dt/dt (a factor of one) into the integrand and simplify:

$$S = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In terms of the parameterization $x = g(t)$, $y = h(t)$, this is

$$S = \int \sqrt{\left(g'(t)\right)^2 + \left(h'(t)\right)^2}\, dt\ .$$

Integral of Speed

Of course, the above integrand can be condensed back down via

$$\frac{dS}{dt} = \sqrt{\left(g'(t)\right)^2 + \left(h'(t)\right)^2}\ ,$$

in which case dS/dt is interpreted as the speed:

$$S = \int \frac{dS}{dt}\, dt = \int v\, dt$$

3.2 Parametric Surface of Revolution

The surface area of a curve $y(x)$ rotated about the x axis is straightforwardly given by

$$A = \int_{x_0}^{x_1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\ .$$

This can be reverse-engineered by multiplying dt/dt into the integrand and simplifying:

$$A = \int_{t_0}^{t_1} 2\pi h(t) \sqrt{\left(g'(t)\right)^2 + \left(h'(t)\right)^2}\, dt$$

3.3 Polar Area Integral

The area integral in polar coordinates is a different beast than its Cartesian counterpart.

In the plane, consider a line that connects the origin $(0, 0)$ to some position (x, y), corresponding to polar coordinates (r, θ). Next, imagine a neighboring point located at $(x + \Delta x, y + \Delta y)$, or $(r + \Delta r, \theta + \Delta\theta)$, connected to the origin by a second line.

This setup constitutes two sides of a triangle with one vertex at $(0, 0)$ and two sides r, $r + \Delta r$. The third side of the triangle has length $\sqrt{\Delta x^2 + \Delta y^2}$. Further, it's straightforward to show that the area of such a triangle is

$$\Delta A = \frac{1}{2}(r + \Delta r)\, r \sin(\Delta\theta)\ .$$

In the differential limit, we may approximate

$$\begin{aligned} r + \Delta r &\approx r \\ \sin(\Delta\theta) &\approx d\theta\ , \end{aligned}$$

and the differential area of our very skinny triangle becomes

$$dA = \frac{1}{2} r^2 d\theta\ .$$

This is enough to write the formula for the area of a polar function $r(\theta)$ by integrating in the θ-variable:

$$A = \frac{1}{2} \int_{\theta_0}^{\theta_1} \left(r(\theta)\right)^2 d\theta$$

3.4 Polar Arc Length

Consider a point (x, y) in the Cartesian plane, having polar representation

$$x = r\cos(\theta)$$
$$y = r\sin(\theta) \ .$$

Also consider a neighboring point $(x+dx, y+dy)$, with

$$x + dx = (r + dr)\cos(\theta + d\theta)$$
$$y + dy = (r + dr)\sin(\theta + d\theta) \ .$$

Starting from the usual construction for arc length, we're interested in summing many consecutive lengths given by

$$dS = \sqrt{dx^2 + dy^2} \ .$$

With this in mind, use the angle-sum formulas on the pair of equations above to establish:

$$x + dx = (r + dr)\left(\cos(\theta)\cos(d\theta) - \sin(\theta)\sin(d\theta)\right)$$
$$y + dy = (r + dr)\left(\sin(\theta)\cos(d\theta) + \cos(\theta)\sin(d\theta)\right)$$

Of course, we can take

$$\cos(d\theta) \approx 1$$
$$\sin(d\theta) \approx d\theta$$
$$dr \cdot d\theta \approx 0$$

because differential quantities are vanishing, and the above simplifies to

$$dx = -y\, d\theta + \cos(\theta)\, dr$$
$$dy = x\, d\theta + \sin(\theta)\, dr \ .$$

Note that the ratio dy/dx returns the proper slope of a curve in polar coordinates.

Taking the sum $dx^2 + dy^2$ will eliminate most of the ugliness:

$$dx^2 + dy^2 = r^2 d\theta^2 + dr^2 \ ,$$

and now we have an equation for dS in polar coordinates:

$$dS = \sqrt{r^2\, d\theta^2 + dr^2}$$

Assuming that r occurs as a function $r(\theta)$, the term $d\theta$ can be pulled out of the root

$$dS = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \ ,$$

or more succinctly,

$$dS = \sqrt{r^2 + \left(r'(\theta)\right)^2}\, d\theta \ .$$

This is enough to write the formula for the arc length of a polar function $r(\theta)$ by integrating in the θ-variable:

$$S = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(r'(\theta)\right)^2}\, d\theta$$

4 Position and Basis Vectors

4.1 Cartesian Position Vector

In the Cartesian plane, a point (x, y) can be represented as a *position vector*

$$\vec{R} = \langle x, y \rangle \ .$$

The above is written in explicit 'bracket' notation, which reminds us that the vector $\vec{R}$ is made by going 'over by x', and then 'up by y'.

4.2 Cartesian Basis Vectors

There is an equivalent representation using basis vectors, which goes

$$\vec{R} = x\,\hat{x} + y\,\hat{y} \ ,$$

where $\hat{x}$, $\hat{y}$ are mutually-perpendicular vectors of magnitude 1. The Cartesian basis vectors have a few equivalent representations:

$$\hat{x} = \hat{i} = \hat{e}_x = \hat{e}_i = \langle 1, 0 \rangle$$
$$\hat{y} = \hat{j} = \hat{e}_y = \hat{e}_j = \langle 0, 1 \rangle$$

It's worth noting that the Cartesian basis vectors are fixed in their coordinate system. The vectors $\hat{x}$, $\hat{y}$ never change, and all derivatives of these are patently zero.

4.3 Polar Position Vector

Starting from the Cartesian position vector

$$\vec{R} = x\,\hat{x} + y\,\hat{y} \ ,$$

replace x, y with their equivalent representations in polar coordinates:

$$\vec{R} = r\cos(\theta)\,\hat{x} + r\sin(\theta)\,\hat{y}$$

From the above we can factor the radial term r, leaving a tangle of basis vectors and trig terms:

$$\vec{R} = r\left(\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\right)$$

Meanwhile, notice that the magnitude of

$$R = \sqrt{\vec{R} \cdot \vec{R}}$$

reduces to $R = r$. This simultaneously means that $\vec{R}$'s unit vector, i.e.

$$\hat{R} = \frac{\vec{R}}{R}$$

is equal to

$$\hat{R} = \cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\ .$$

As a matter of custom, the polar position vector is expressed in lowercase, i.e. $\vec{r}$. This is also true for the polar basis vector $\hat{r}$. The tightest way to write the position vector in polar coordinates is:

$$\vec{r} = r\,\hat{r}$$

4.4 Polar Basis Vectors

Starkly different from the fixed vectors $\hat{x}$, $\hat{y}$, the polar basis vector $\hat{r}$ is not fixed in the plane, and instead varies with θ via

$$\hat{r} = \cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\ .$$

In the same way that $\hat{r}$ points in the direction of increasing r, there ought to exist a second basis vector $\hat{\theta}$, always perpendicular to $\hat{r}$, that 'points' in the direction of increasing θ. This can be attained by rotating $\hat{r}$ by ninety degrees

$$\hat{\theta} = \cos\left(\theta + \frac{\pi}{2}\right)\hat{x} + \sin\left(\theta + \frac{\pi}{2}\right)\hat{y}\ ,$$

simplifying to

$$\hat{\theta} = -\sin(\theta)\,\hat{x} + \cos(\theta)\,\hat{y}\ .$$

Notice that the θ-basis vector is not needed to write the position $\vec{r}$.

Matrix Representation

A tight way to express the polar basis vectors $\hat{r}$, $\hat{\theta}$ uses matrix notation:

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}$$

Resolving Cartesian Basis

Given how $\hat{r}$, $\hat{\theta}$ depend on $\hat{x}$, $\hat{y}$, θ, it's useful to solve for $\hat{x}$, $\hat{y}$ in terms of $\hat{r}$, $\hat{\theta}$, θ instead. As a standard 2×2 matrix, the inverse is easy to write down:

$$\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\theta}\end{bmatrix}$$

Explicitly, this means:

$$\begin{aligned}\hat{x} &= \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}\\ \hat{y} &= \sin(\theta)\,\hat{r} + \cos(\theta)\,\hat{\theta}\end{aligned}$$

5 Intersections

5.1 Line Segments Intersecting

In the Cartesian plane, consider a line segment with endpoints located at $\vec{p}_A$, $\vec{p}_B$. Also consider a second line segment with endpoints located at $\vec{q}_A$, $\vec{q}_B$. The task is to determine whether line segments p, q are intersecting, or if parallel, whether the segments overlap.

In general, each line can be parameteized by a vector equation

$$\vec{y}_j = \vec{b}_j + \alpha_j \hat{t}_j\ ,$$

where $j = p, q$ toggles between segments. The vector $\vec{b}_j$ is any fixed point on the jth line segment (not necessarily the y-intercept). The unit vector $\hat{t}_j$ registers the slope of the line segment. The parameter α_j is a real number confined to the domain

$$\alpha_j^{\min} \le \alpha \le \alpha_j^{\max}$$

to assure the finite extent of each segment.

Intersection

The condition for intersection is given by $\vec{y}_p = \vec{y}_q$, or

$$\vec{b}_p + \alpha_p \hat{t}_p = \vec{b}_q + \alpha_q \hat{t}_q\ .$$

Letting

$$\begin{aligned}\Delta\vec{b} &= \vec{b}_p - \vec{b}_q\\ f &= \hat{t}_p \cdot \hat{t}_q\\ w_p &= \Delta\vec{b} \cdot \hat{t}_p\\ w_q &= \Delta\vec{b} \cdot \hat{t}_q\ ,\end{aligned}$$

the intersection condition yields a pair of equations

$$\begin{aligned}\alpha_q - \alpha_p f &= w_q\\ \alpha_q f - \alpha_p &= w_p\ ,\end{aligned}$$

allowing α_p, α_q to be isolated:

$$\begin{aligned}\alpha_p &= \frac{f w_q - w_p}{1 - f^2}\\ \alpha_q &= \frac{w_q - f w_p}{1 - f^2}\end{aligned}$$

For $f \neq 1$, the above gives the solution to the intersection of two infinite lines. For line segments, make sure

$$\alpha_{p,q}^{\min} \leq \alpha \leq \alpha_{p,q}^{\max}$$

is satisfied.

Overlap

If the two lines being compared are parallel, then $f = \hat{t} \cdot \hat{q} = \pm 1$, and the solutions for α become invalid. We must instead go back to:

$$\begin{aligned} \alpha_q &= \alpha_p f + w_q \\ \alpha_p &= \alpha_q f - w_p \end{aligned}$$

Taking only the first equation, we can set α_p to be either of $\alpha_p^{\min}$ or $\alpha_p^{\max}$, and α_q on the left side of the equation must be between each of these. A similar story applies to the bottom equation.

5.2 Line Intersecting Ellipse

Consider an ellipse located somewhere in the plane parameterized by ϕ such that

$$\vec{r}_e = \vec{r}_0 + \vec{a}\cos(\phi) + \vec{b}\sin(\phi) \ ,$$

where $\vec{r}_0$ is the center, and the mutually-perpendicular vectors $\vec{a}$, $\vec{b}$ orient the major and minor axes.

Also consider a straight line in the plane parameterized by α:

$$\vec{y} = \vec{y}_0 + \alpha\,\hat{v} \ ,$$

where $\vec{y}_0$ is a constant vector, and $\hat{v}$ is the unit tangent vector that tells us the slope of the line.

The case for intersections is given by:

$$\vec{r}_0 + \vec{a}\cos(\phi) + \vec{b}\sin(\phi) = \vec{y}_0 + \alpha\,\hat{v}$$

From prior knowledge of lines and ellipses, we can anticipate at most two solutions to the above when the line hits the ellipse in the belly, one solution for when the line is tangent to the ellipse, and zero solutions for when the line misses entirely.

To gain on this, multiply through the equation above by $\vec{a}$, and again by $\vec{b}$, and then isolate the trig terms:

$$\begin{aligned} \cos(\phi) &= \frac{\vec{a} \cdot (\vec{y}_0 - \vec{r}_0) + \alpha\,\vec{a} \cdot \hat{v}}{a^2} \\ \sin(\phi) &= \frac{\vec{b} \cdot (\vec{y}_0 - \vec{r}_0) + \alpha\,\vec{b} \cdot \hat{v}}{b^2} \end{aligned}$$

Let

$$\vec{q}_0 = \vec{y}_0 - \vec{r}_0 \ ,$$

and simplify further:

$$\begin{aligned} \cos(\phi) &= \frac{\hat{a} \cdot \vec{q}_0 + \alpha\,\hat{a} \cdot \hat{v}}{a} \\ \sin(\phi) &= \frac{\hat{b} \cdot \vec{q}_0 + \alpha\,\hat{b} \cdot \hat{v}}{b} \end{aligned}$$

To proceed, use the fundamental trig identity to write

$$1 = \left(\frac{\hat{a} \cdot \vec{q}_0 + \alpha\,\hat{a} \cdot \hat{v}}{a}\right)^2 + \left(\frac{\hat{b} \cdot \vec{q}_0 + \alpha\,\hat{b} \cdot \hat{v}}{b}\right)^2 \ .$$

Blooming out the algebra, the above takes the form

$$0 = A\alpha^2 + B\alpha + C \ ,$$

where:

$$\begin{aligned} A &= \left(\frac{\hat{a} \cdot \hat{v}}{a}\right)^2 + \left(\frac{\hat{b} \cdot \hat{v}}{b}\right)^2 \\ B &= \frac{2\,(\hat{a} \cdot \vec{q}_0)\,(\hat{a} \cdot \hat{v})}{a^2} + \frac{2\left(\hat{b} \cdot \vec{q}_0\right)\left(\hat{b} \cdot \hat{v}\right)}{b^2} \\ C &= \left(\frac{\hat{a} \cdot \vec{q}_0}{a}\right)^2 + \left(\frac{\hat{b} \cdot \vec{q}_0}{b}\right)^2 - 1 \end{aligned}$$

Finally, use the quadratic formula to establish

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \ .$$

In the special case $B^2 = 4AC$, the line is tangent to the ellipse. If the solutions become imaginary, the line misses the ellipse.

5.3 Circle Intersecting Circle

Let the position $\vec{r}_j$ on a circle of radius R_j centered at $\vec{C}_j$ be parameterized by ϕ_j:

$$\vec{r}_j = \vec{C}_j + R_j \left\langle \cos(\phi_j), \sin(\phi_j) \right\rangle$$

For the intersection of two such circles of different radii, we can write

$$\vec{D} = R_2 \left\langle \cos(\phi_2), \sin(\phi_2) \right\rangle - R_1 \left\langle \cos(\phi_1), \sin(\phi_1) \right\rangle \ ,$$

where

$$\vec{D} = \vec{C}_1 - \vec{C}_2 \ .$$

In component form, vector $\vec{D}$ reads

$$\begin{aligned} D_x &= R_2 \cos(\phi_2) - R_1 \cos(\phi_1) \\ D_y &= R_2 \sin(\phi_2) - R_1 \sin(\phi_1) \ . \end{aligned}$$

Next, solve for $\cos^2(\phi_1) + \sin^2(\phi_1)$ among the two equations to get

$$D_x \cos(\phi_2) + D_y \sin(\phi_2) = \frac{R_2^2 - R_1^2 + D^2}{2R_2} = E ,$$

where E is a constant. Similarly, isolate $\cos^2(\phi_2) + \sin^2(\phi_2)$ to find

$$D_x \cos(\phi_1) + D_y \sin(\phi_1) = \frac{R_2^2 - R_1^2 - D^2}{2R_1} = F ,$$

where F is also constant.

To summarize, the problem reduces to solving either of:

$$\begin{aligned} \frac{D_x}{E}\cos(\phi_2) + \frac{D_y}{E}\sin(\phi_2) = 1 \\ \frac{D_x}{F}\cos(\phi_1) + \frac{D_y}{F}\sin(\phi_1) = 1 \end{aligned}$$

Choosing the first equation, write this as

$$\frac{D_x}{E}\cos(\phi_2) \pm \frac{D_y}{E}\sqrt{1-\cos^2(\phi_2)} = 1 ,$$

equivalent to:

$$\left(\frac{D}{E}\right)^2 \cos^2(\phi_2) - \frac{2D_x}{E}\cos(\phi_2) + 1 - \left(\frac{D_y}{E}\right)^2 = 0$$

The term $\cos(\phi_2)$ can be isolated with the quadratic formula:

$$\cos(\phi_2) = \frac{D_x E \pm D_y E\sqrt{D^2/E^2 - 1}}{D^2}$$

An identical exercise in solving for $\sin(\phi_2)$ gives:

$$\sin(\phi_2) = \frac{D_y E \pm D_x E\sqrt{D^2/E^2 - 1}}{D^2}$$

Of course, the same two exercises can be done to isolate the terms involving ϕ_1. The result would have all terms E replaced with F.

To continue, we need to decide how to handle the multi-channel aspect of the solution, i.e. what to do with the $\pm$ symbols. Multiply the pair of equations by D_x/E, D_y/E respectively. The sum must come to one, so

$$\begin{aligned} 1 = &\frac{D_x^2 \pm D_y D_x\sqrt{D^2/E^2 - 1}}{D^2} \\ &+ \frac{D_y^2 \pm D_x D_y\sqrt{D^2/E^2 - 1}}{D^2} \end{aligned}$$

tells us

$$0 = (\pm 1 \pm 1)\,\frac{D_x D_y\sqrt{D^2/E^2 - 1}}{D^2} .$$

In other words, we can have the combinations $+-$, $-+$, but the pure cases $++$, $--$ are invalid.

Explicitly, one pair of solutions to the system reads

$$\begin{aligned} \cos\left(\phi_2^+\right) &= \frac{D_x E + D_y E\sqrt{D^2/E^2 - 1}}{D^2} \\ \sin\left(\phi_2^-\right) &= \frac{D_y E - D_x E\sqrt{D^2/E^2 - 1}}{D^2} , \end{aligned}$$

and the other pair of solutions has the signs outside the roots swapped. The intersection points of the two circles are finally:

$$\begin{aligned} \vec{X}^1_{\text{int}} &= \vec{C}_2 + R_2 \left\langle \cos\left(\phi_2^+\right), \sin\left(\phi_2^-\right)\right\rangle \\ \vec{X}^2_{\text{int}} &= \vec{C}_2 + R_2 \left\langle \cos\left(\phi_2^-\right), \sin\left(\phi_2^+\right)\right\rangle \end{aligned}$$

One Intersection

The condition for both intersection points overlapping at a single intersection point, i.e., when the two circles are tangent, occurs when $D = R_1 + R_2$, which is equivalent to $D = E$.

Zero Intersections

The circles clearly don't intersect in two cases: (i) the circles are sufficiently separated, or (ii) the smaller circle is inside the larger circle with no contact.

5.4 Circle Intersecting Points

Find the circle

$$(x-h)^2 + (y-k)^2 = R^2$$

that passes through three points in the plane $\vec{q}_j = \langle x_j, y_j\rangle$ with $j = 1, 2, 3$.

Proceed by blooming out the equation for the circle and substitute each data point:

$$\begin{aligned} q_1^2 - 2x_1 h - 2y_1 k &= R^2 - h^2 - k^2 \\ q_2^2 - 2x_2 h - 2y_2 k &= R^2 - h^2 - k^2 \\ q_3^2 - 2x_3 h - 2y_3 k &= R^2 - h^2 - k^2 \end{aligned}$$

Take the difference between the second and first equations

$$q_2^2 - q_1^2 + h(2x_1 - 2x_2) + k(2y_1 - 2y_2) = 0 ,$$

and also take the difference between the third and first equations:

$$q_3^2 - q_1^2 + h(2x_1 - 2x_3) + k(2y_1 - 2y_3) = 0$$

This is merely a linear system of two equations and two unknowns. Packing the above coefficients on h, k into new variables $a_{1,2}$, $b_{1,2}$, $c_{1,2}$, the above reads

$$a_1 + b_1 h + c_1 k = 0$$
$$a_2 + b_2 h + c_2 k = 0 \,.$$

Solving for h, k is a matter of elementary (linear) algebra. For results, we finally have:

$$h = \frac{a_1/b_1 - a_2/b_2}{c_2/b_2 - c_1/b_1}$$
$$k = \frac{a_1/c_1 - a_2/c_2}{b_2/c_2 - b_1/c_1}$$

To solve for the radius R, consider any vector $\vec{P}_j$ that extends from the center of the circle to any given point $\vec{q}_j$:

$$\vec{P}_j = \vec{q}_j - \langle h, k \rangle$$

The radius is the magnitude P.

5.5 Ellipse Intersecting Ellipse

Consider two ellipses in the same plane given by:

$$\vec{r}_j = \vec{k}_j + \vec{a}_j \cos(\phi_j) + \vec{b}_j \sin(\phi_j) \ ,$$

where $j = 1, 2$, and, as usual for an ellipse $\vec{a}_j \cdot \vec{b}_j = 0$. The condition for intersection of the two ellipses is

$$\vec{k}_1 + \vec{a}_1 \cos(\phi_1) + \vec{b}_1 \sin(\phi_1) = \vec{k}_2 + \vec{a}_2 \cos(\phi_2) + \vec{b}_2 \sin(\phi_1) \ .$$

Then, multiply $\vec{a}_1$, $\vec{b}_1$, separately into the above to generate two results:

$$\vec{a}_1 \cdot \vec{k}_1 + a_1^2 \cos(\phi_1) = \vec{a}_1 \cdot \vec{k}_2 + \vec{a}_1 \cdot \vec{a}_2 \cos(\phi_2) + \vec{a}_1 \cdot \vec{b}_2 \sin(\phi_1)$$
$$\vec{b}_1 \cdot \vec{k}_1 + b_1^2 \sin(\phi_2) = \vec{b}_1 \cdot \vec{k}_2 + \vec{b}_1 \cdot \vec{a}_2 \cos(\phi_2) + \vec{b}_1 \cdot \vec{b}_2 \sin(\phi_2)$$

Denoting

$$\Delta\vec{k} = \vec{k}_2 - \vec{k}_1$$
$$A_{jk} = \vec{a}_j \cdot \vec{a}_k$$
$$B_{jk} = \vec{b}_j \cdot \vec{b}_k$$
$$C_{jk} = \vec{a}_j \cdot \vec{b}_k$$

to keep the algebra tame, the above rearrange to

$$\cos(\phi_1) = \frac{\vec{a}_1 \cdot \Delta\vec{k} + A_{12} \cos(\phi_2) + C_{12} \sin(\phi_2)}{a_1^2}$$
$$\sin(\phi_1) = \frac{\vec{b}_1 \cdot \Delta\vec{k} + C_{21} \cos(\phi_2) + B_{12} \sin(\phi_2)}{b_1^2} \ .$$

Use the fundamental trig identity $\sin^2(\phi_1) + \cos^2(\phi_1) = 1$ to condense the two equations back into one, with the only unknown being ϕ_2. This is still a mess to solve, but can be done numerically in the general case, or analytically in certain special cases.

For a special case, consider the two ellipses

$$\vec{r}_1 = a \cos(\phi_1)\, \hat{x} + b \sin(\phi_1)\, \hat{y}$$
$$\vec{r}_2 = b \cos(\phi_2)\, \hat{x} + a \sin(\phi_2)\, \hat{y} \,.$$

For this we have $A_{12} = B_{12} = ab$, with everything else zero. The situation is then governed by

$$1 = \left(\frac{b}{a} \cos(\phi_2)\right)^2 + \left(\frac{a}{b} \sin(\phi_2)\right)^2 \ .$$

Solutions to the problem are then

$$\vec{r}_1 = \vec{r}_2 = \frac{ab}{\sqrt{a^2 + b^2}} \left(\pm\hat{x} \pm \hat{y}\right) \ .$$

5.6 Ellipse Intersecting Parabola

A parabolic curve

$$y(x) = Ax^2 + Bx + C$$

can be represented in vector notation via

$$\vec{y} = x\, \hat{x} + y(x)\, \hat{y} \,.$$

Let us find the intersection between such a parabola and the ellipse:

$$\vec{r} = \vec{r}_0 + \vec{a} \cos(\phi) + \vec{b} \sin(\phi)$$

The condition for intersection is $\vec{y} = \vec{r}$, or

$$x\, \hat{x} + y(x)\, \hat{y} = \vec{r}_0 + \vec{a} \cos(\phi) + \vec{b} \sin(\phi) \ .$$

Next, multiply $\vec{a}$, $\vec{b}$, separately into the above to generate two results,

$$x\, \vec{a} \cdot \hat{x} + y(x)\, \vec{a} \cdot \hat{y} = \vec{a} \cdot \vec{r}_0 + a^2 \cos(\phi)$$
$$x\, \vec{b} \cdot \hat{x} + y(x)\, \vec{b} \cdot \hat{y} = \vec{b} \cdot \vec{r}_0 + b^2 \sin(\phi) \ ,$$

or:

$$\cos(\phi) = \frac{x a_x + y(x)\, a_y - \vec{a} \cdot \vec{r}_0}{a^2}$$
$$\sin(\phi) = \frac{x b_x + y(x)\, b_y - \vec{b} \cdot \vec{r}_0}{b^2}$$

As expected, we're left with another mess, but the ϕ-parameter could be eliminated in a way similar to the cases above to proceed with the general case.

For a special case, suppose $\vec{r}_0 = 0$, and let $\vec{a} = a\,\hat{x}$, $\vec{b} = b\,\hat{y}$. Then, we have

$$1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y\,(x)}{b}\right)^2 \;,$$

or all in terms of one variable,

$$1 = \left(\frac{x}{a}\right)^2 + \left(\frac{Ax^2 + Bx + C}{b}\right)^2 \;.$$

5.7 Collision of Spheres

Consider two spheres of radius $R_{1,2}$ and mass $m_{1,2}$, each moving with uniform velocity $\vec{v}_{1,2}$. With this setup, the system has a total energy E (scalar) and linear momentum $\vec{P}$ (vector):

$$\begin{aligned} E &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ \vec{P} &= m_1\,\vec{v}_1 + m_2\,\vec{v}_2 \end{aligned}$$

If the spheres are to make contact via an elastic collision without exchanging mass, each sphere emerges with a new velocity vector $\vec{u}_{1,2}$ obeying

$$\begin{aligned} E &= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \\ \vec{P} &= m_1\,\vec{u}_1 + m_2\,\vec{u}_2 \;, \end{aligned}$$

which is to say energy and momentum are conserved throughout the collision. The task is to solve for $\vec{u}_1$, $\vec{u}_2$.

To make the problem easier, we can define a momentum exchange vector $\vec{q}$ such that

$$\begin{aligned} m_1\,\vec{u}_1 &= m_1\,\vec{v}_1 - \vec{q} \\ m_2\,\vec{u}_2 &= m_2\,\vec{v}_2 + \vec{q}\,. \end{aligned}$$

This pair of equations can recover the answers $\vec{u}_{1,2}$ from $\vec{q}$, so the whole problem becomes finding $\vec{q}$.

The two spheres exchange momentum at the point of contact, thus $\vec{q}$ is normal to each sphere's surface at that point. By the same token, $\vec{q}$ is parallel to the vector connecting the center of each sphere. Thus each vector on hand relates by:

$$\vec{q} = |q|\,\hat{n} = |q| \left(\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}\right)$$

Since the positions $\vec{r}_{1,2}$ are given, the direction of $\vec{q}$ is already clear, and the task reduces to finding $|q|$.

To proceed, square each momentum exchange equation to write

$$\begin{aligned} \frac{1}{2} m_1 u_1^2 &= \frac{1}{2} m_1 v_1^2 - \vec{v}_1 \cdot \vec{q} + \frac{q^2}{2m_1} \\ \frac{1}{2} m_2 u_2^2 &= \frac{1}{2} m_2 v_2^2 + \vec{v}_2 \cdot \vec{q} + \frac{q^2}{2m_2} \end{aligned}$$

Take the sum of these, and notice all kinetic energy terms cancel, leaving

$$0 = \vec{q} \cdot (\vec{v}_1 - \vec{v}_2) - q^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \;.$$

Using $\vec{q} = |q|\,\hat{n}$, finally solve for $|q|$:

$$|q| = \left(\frac{2m_1 m_2}{m_1 + m_2}\right) \hat{n} \cdot (\vec{v}_1 - \vec{v}_2)$$

6 Rotations

6.1 Rotated Vectors

Consider a vector

$$\vec{V} = V_x\,\hat{x} + V_y\,\hat{y}$$

whose magnitude is

$$V = \sqrt{V_x^2 + V_y^2} \;.$$

In terms of a parameter ϕ, the components of $\vec{V}$ can be written

$$\begin{aligned} V_x &= V \cos(\phi) \\ V_y &= V \sin(\phi) \;. \end{aligned}$$

Now, suppose that the angle parameter is increased by another angle θ such that

$$\phi \to \phi + \theta \;,$$

which has the effect of modifying the vector components to new values

$$\begin{aligned} V_x' &= V \cos(\phi + \theta) \\ V_y' &= V \sin(\phi + \theta) \;, \end{aligned}$$

which have been denoted V_x', V_y'. Right away, one sees that the magnitude

$$V = \sqrt{(V_x')^2 + (V_y')^2}$$

still holds, meaning we aren't changing the length of the vector.

Expanding out the trig terms in the components V_x', V_y' leads us to

$$\begin{aligned} V_x' &= V\left(\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)\right) \\ V_y' &= V\left(\sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta)\right) \;, \end{aligned}$$

which simplifies further:

$$\begin{aligned} V_x' &= V_x \cos(\theta) - V_y \sin(\theta) \\ V_y' &= V_x \sin(\theta) + V_y \cos(\theta) \end{aligned}$$

Rotation Matrix

In matrix notation, the above reads

$$\begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} ,$$

which is the same matrix needed to recover the Cartesian basis vectors from the polar ones. This is indeed the standard rotation matrix, denoted R:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix}$$

6.2 Rotated Coordinates

Consider a vector $\vec{A}$ living in the 'standard' Cartesian basis:

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y}$$

While leaving the vector unchanged, let us rotate the coordinate system by some angle θ so that the uv-plane replaces the xy-plane, with $\hat{u}$, $\hat{v}$ replacing the respective $\hat{x}$, $\hat{y}$ unit vectors.

In this construction, the unit vectors $\hat{u}$, $\hat{v}$ are totally analogous to $\hat{r}$, $\hat{\theta}$, and we can borrow the matrix that brings us from $\hat{x}$, $\hat{y}$ to the rotated system:

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

In terms of the rotated basis vectors, the original basis vectors are the inversion of the above:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}$$

Inverse Rotation Matrix

While we're here, it's worth noting that the inverse of R, namely

$$\tilde{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix}$$

is the inverse rotation matrix.

Consequence for Vectors

We're now ready to address what happens to the components of $\vec{A}$ in the rotated coordinate system. We frame the rotated vector as

$$\vec{A} = A_u\,\hat{u} + A_v\,\hat{v} ,$$

and the job is to solve for A_u, A_v. One way to proceed is to substitute $\hat{u}$, $\hat{v}$ and simplify:

$$\begin{aligned} \vec{A} &= A_u\left(\cos(\theta)\,\hat{x} + \sin(\theta)\,\hat{y}\right) \\ &\quad + A_v\left(-\sin(\theta)\,\hat{x} + \cos(\theta)\,\hat{y}\right) \\ \vec{A} &= \left(A_u\cos(\theta) - A_v\sin(\theta)\right)\hat{x} \\ &\quad + \left(A_u\sin(\theta) + A_v\cos(\theta)\right)\hat{y} \end{aligned}$$

So far, we've managed to show:

$$\begin{aligned} A_x &= A_u\cos(\theta) - A_v\sin(\theta) \\ A_y &= A_u\sin(\theta) + A_v\cos(\theta) \end{aligned}$$

Evidently, the coordinates can be related by the rotation matrix

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} A_u \\ A_v \end{bmatrix} ,$$

but we need the inverse of this to isolate the new components:

$$\begin{bmatrix} A_u \\ A_v \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

In other words, a positive rotation in the coordinate system looks like a negative rotation of the vector components.

7 Vector Derivatives

7.1 Rules of Differentiation

The rules for differentiating vector quantities are exactly analogous to those for scalars. For instance, if we have a position vector $\vec{r}(t)$, then the velocity vector is the time derivative by definition:

$$\vec{v}(t) = \frac{d}{dt}\left(\vec{r}(t)\right) = \lim_{h\to 0}\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

In the following, suppose c is a constant, $\lambda(t)$ is a function of time, and so too are the vectors $\vec{r}$, $\vec{s}$. Then, we always have:

$$\begin{aligned} \frac{d}{dt}\left(c\,\vec{r}\right) &= c\,\frac{d\vec{r}}{dt} \\ \frac{d}{dt}\left(\lambda\,\vec{r}\right) &= \frac{d\lambda}{dt}\vec{r} + c\,\frac{d\vec{r}}{dt} \\ \frac{d}{dt}\left(\vec{r} + \vec{s}\right) &= \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt} \\ \frac{d}{dt}\left(\vec{r}\cdot\vec{s}\right) &= \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt} \\ \frac{d}{dt}\left(\vec{r}\times\vec{s}\right) &= \frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt} \end{aligned}$$

7.2 Basis Vector Derivatives

Derivatives of the Cartesian basis vectors are all zero. As for polar coordinates, since $\hat{r}$, $\hat{\theta}$ are allowed to swivel about as θ changes, it makes sense to ask about the derivatives of these. Proceeding carefully, we find

$$\frac{d}{d\theta}\hat{r} = -\sin(\theta)\,\hat{x} + \cos(\theta)\,\hat{y} = \hat{\theta}$$

and

$$\frac{d}{d\theta}\hat{\theta} = -\cos(\theta)\,\hat{x} - \sin(\theta)\,\hat{y} = -\hat{r}\ .$$

7.3 Velocity

Supposing x, y are each functions of time, it follows that r, θ are also functions of time. For the Cartesian case, we easily differentiate the position with respect to time to get the velocity vector

$$\vec{V} = \frac{d}{dt}\vec{R} = \left(\frac{d}{dt}x(t)\right)\hat{x} + \left(\frac{d}{dt}y(t)\right)\hat{y}\ .$$

Velocity in Polar Coordinates

The story in polar coordinates is a little different. For shorthand, express the *angular velocity* as the time derivative of the θ coordinate as ω (Greek 'omega'):

$$\omega = \omega(t) = \frac{d}{dt}\theta(t)$$

Next, we'll need the time derivative of the *entire* position vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r(t)\ \hat{r}(t)\right)\ ,$$

which calls for the product rule:

$$\vec{v} = \left(\frac{d}{dt}r(t)\right)\hat{r}(t) + r(t)\,\frac{d}{dt}\hat{r}(t)$$

Expand the right-most term with the with the chain rule

$$\vec{v} = \frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\frac{d\hat{r}}{d\theta}\ ,$$

and simplify to get the velocity in polar coordinates:

$$\vec{v} = \frac{dr}{dt}\,\hat{r} + r\omega\,\hat{\theta}$$

Problem 1

If the magnitude $|\vec{r}| = r$ is constant in time, show that $\vec{r}$ and $\vec{v}$ are perpendicular. Hint: differentiate $\vec{r}\cdot\vec{r}$

7.4 Acceleration

The acceleration vector is the time derivative of the velocity vector. This is straightforward in Cartesian coordinates:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2}{dt^2}\vec{R} = \left(\frac{d^2}{dt^2}x(t)\right)\hat{x} + \left(\frac{d^2}{dt^2}y(t)\right)\hat{y}$$

Acceleration in Polar Coordinates

As expected, the acceleration in polar coordinates is messy. Starting the calculation, we have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\,\hat{r}\right) + \frac{d}{dt}\left(r\omega\,\hat{\theta}\right)\ .$$

Leaving the details for an exercise, the result comes out to

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega\right)\hat{\theta}\ .$$

Problem 2

Derive the acceleration vector in polar coordinates by taking two derivatives of $\vec{r} = r\,\hat{r}$.

7.5 Complex Number Analogy

Interestingly, the velocity and acceleration equations for polar coordinates can arise from taking derivatives of the complex number

$$z = r\,e^{i\theta}\ .$$

Assuming r, θ are functions of time, take a time-derivative of z to get something like the velocity:

$$\frac{d}{dt}z = \frac{dr}{dt}\,e^{i\theta} + r\omega\left(i\,e^{i\theta}\right)$$

Comparing this result to the velocity in polar coordinates, a direct analogy emerges:

$$e^{i\theta} \leftrightarrow \hat{r}$$
$$i\,e^{i\theta} \leftrightarrow \hat{\theta}$$

Evidently, the terms $e^{i\theta}$, $\hat{r}$ play similar roles in complex numbers and polar coordinates, which is no surprise since each is of the format $\langle\cos(\theta), \sin(\theta)\rangle$. Similar comments apply to the pair $i\,e^{i\theta}$, $\hat{\theta}$.

7.6 Differential Line Element

Starting from the velocity vector, whether it be the Cartesian representation or polar representation

$$\begin{aligned}\vec{V} &= \frac{dx}{dt}\,\hat{x} + \frac{dy}{dt}\,\hat{y} \\ \vec{v} &= \frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta}\ ,\end{aligned}$$

and multiply through by dt:

$$\begin{aligned}d\vec{R} &= \vec{V}\,dt = dx\,\hat{x} + dy\,\hat{y} \\ d\vec{r} &= \vec{v}\,dt = dr\,\hat{r} + r\,d\theta\,\hat{\theta}\end{aligned}$$

The terms $d\vec{R}$ (Cartesian), $d\vec{r}$ (polar) are each called the *differential line element*, also called $d\vec{S}$. For a point (x_0, y_0) in the plane, the differential line element provides a local coordinate system for moving to a nearby point $(x_0 + dx, y_0 + dy)$.

Notice that the differential line element always has units of length, hence the factor of r attached to the $d\theta$ term, which it itself dimensionless.

7.7 Differential Interval

The square of the differential line element is called the *differential interval*, always denoted dS^2, regardless of coordinate system. The reason for this is that the differential element is the same in all coordinate systems.

To check this, first write the differential interval in Cartesian coordinates, namely

$$dS^2 = (dx\,\hat{x} + dy\,\hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y}) = dx^2 + dy^2\ .$$

Then, for polar coordinates:

$$dS^2 = d\vec{r} \cdot d\vec{r} = dr^2 + r^2 d\theta^2$$

In other words, we have recovered

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2\ ,$$

which is a familiar identity relating Cartesian to polar coordinates.

The square root of the differential interval is the differential arc length. That is,

$$\sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}$$

are each equal to dS as it appears in arc length calculations

$$S = \int dS = \int \sqrt{d\vec{S} \cdot d\vec{S}}\ .$$

Misguided Missile

Source: TBD

A missile traveling at constant speed is homing in on a target at the origin. Do to an error in its circuitry, it is consistently misdirected by a constant angle α. Find its path. Show that if $|\alpha| < 90^\circ$, then it will eventually hit its target, taking $1/\cos(\alpha)$ as long as if it were correctly aimed.

Using polar coordinates, the velocity is

$$\vec{v} = -v\cos(\alpha)\,\hat{r} + v\sin(\alpha)\,\hat{\theta}\ ,$$

which means

$$\begin{aligned}\frac{dr}{dt} &= -v\cos(\alpha) \\ r\frac{d\theta}{dt} &= v\sin(\alpha)\ .\end{aligned}$$

Eliminating dt between the above and canceling v leads to

$$\frac{dr}{r} = -\cot(\alpha)\,d\theta\ ,$$

which can be integrated and simplified to

$$r(\theta) = r_0\,e^{-\cot(\alpha)\theta}\ ,$$

where the integration constant is recast as r_0. The path happens to be a logarithmic spiral.

The time taken for the correctly-aimed missile to reach the target is

$$T = \int dt = \frac{-1}{v}\int_{r_0}^{0} dr = \frac{r_0}{v}\ .$$

When the motion is not on a straight line, the time is instead

$$T' = \frac{-1}{v}\int_{r_0}^{0} dS\ .$$

Going from the above, it also follows that

$$dS = \frac{dr}{\cos(\alpha)}\ ,$$

and thus

$$T' = T/\cos(\alpha)\ ,$$

as we wanted to show.

7.8 Differential Area Element

The existence of the differential line element hints at the *differential area element*, which is a small 'patch' that covers part of the plane.

Cartesian Area Element

In Cartesian coordinates, it makes sense to propose

$$dA = dx\, dy$$

as the differential area element, which happens to be correct.

Let us derive this more carefully, though. Start with the differential line element

$$d\vec{R} = dx\, \hat{x} + dy\, \hat{y}\ ,$$

and then write two versions of this - one with $dy = 0$, the other with $dx = 0$:

$$\begin{aligned} d\vec{R}_1 &= dx\, \hat{x} + 0\, \hat{y} \\ d\vec{R}_2 &= 0\, \hat{x} + dy\, \hat{y} \end{aligned}$$

This defines two vectors that form the sides of a parallelogram (just a rectangle in this case). The magnitude of the cross product $d\vec{R}_1 \times d\vec{R}_2$ yields the area of said rectangle:

$$dA = \left| d\vec{R}_1 \times d\vec{R}_2 \right| = (dx\, \hat{x}) \times (dy\, \hat{y}) = dx\, dy$$

Polar Area Element

The reason for deriving the Cartesian area element in such a belabored way is to make easy the polar area element. For the polar case, we have

$$d\vec{r} = dr\, \hat{r} + r\, d\theta\, \hat{\theta}\ ,$$

and then

$$\begin{aligned} d\vec{r}_1 &= dr\, \hat{r} + 0\, \hat{\theta} \\ d\vec{r}_2 &= 0\, \hat{r} + r\, d\theta\, \hat{\theta}\ . \end{aligned}$$

In this case, $d\vec{r}_1$, $d\vec{r}_2$ form the sides of a parallelogram (not rectangular), whose area is the the magnitude of their cross product:

$$dA = |d\vec{r}_1 \times d\vec{r}_2| = r\, dr\, d\theta$$

8 Plane Curve Analysis

8.1 Tangent Vector

Consider a curve described by a position vector $\vec{r}\,(t)$ in Cartesian coordinates parameterized by a real number t:

$$\vec{r}\,(t) = x\,(t)\, \hat{x} + y\,(t)\, \hat{y}$$

The derivative of $\vec{r}$ with respect to the parameter t yields the *tangent vector* to the curve.

Velocity Vector

If t is taken as an accumulation of time, the tangent vector is the velocity:

$$\vec{v}\,(t) = \frac{d}{dt}\vec{r}\,(t) = \frac{dx}{dt}\,\hat{x} + \frac{dy}{dt}\,\hat{y} = v_x \hat{x} + v_y\, \hat{y}$$

The magnitude of the velocity is the speed,

$$v = \sqrt{v_x^2 + v_y^2} = \frac{dS}{dt}\ ,$$

and dividing the velocity by the speed returns the unit tangent vector:

$$\hat{T} = \frac{\vec{v}}{v} = \frac{v_x \hat{x} + v_y \hat{y}}{\sqrt{v_x^2 + v_y^2}}$$

Going further, we can divide out a factor of dt from the numerator and denominator to write

$$\hat{T} = \frac{dx\, \hat{x} + dy\, \hat{y}}{\sqrt{dx^2 + dy^2}}\ .$$

In this form, we see the right side is the ratio of the differential line element to the differential arc length. From this we have

$$\hat{T} = \frac{d\vec{S}}{dS}\ ,$$

which is the tightest definition for the tangent vector to a curve. The derivative of the position vector with respect to the arc length is the direction of 'motion'.

In terms of a standard parameter ϕ, the normalized tangent vector can be expressed as

$$\hat{T} = \cos(\phi)\, \hat{x} + \sin(\phi)\, \hat{y}\ ,$$

in which case the slope of the curve dy/dx is

$$\frac{dy}{dx} = \frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi)\ .$$

Note that the tangent vector also applies to non-parametric curves, i.e. the classic function $y = f\,(x)$. For this, the general equation for $\hat{T}$ can be configured as:

$$\hat{T} = \frac{\hat{x} + (dy/dx)\, \hat{y}}{\sqrt{1 + (dy/dx)^2}} = \frac{\hat{x} + y'\, \hat{y}}{\sqrt{1 + (y')^2}}$$

8.2 Normal Vector

Given the tangent vector $\hat{T}$ to a curve, one can imagine that differential changes in $\hat{T}$ are always perpendicular to the tangent, and are thus perpendicular to the curve:

$$\frac{d}{d\phi}\hat{T} = -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}$$

One can explicitly check that

$$\hat{T}\cdot\frac{d}{d\phi}\hat{T} = 0\,.$$

This implies the existence of the *normal vector*. In the general case, the normalized vector is crudely defined as

$$\vec{N} = \frac{d}{dt}\vec{T}\,,$$

where the parameter t is considered generic. The unit normal vector divides out its own magnitude

$$\hat{N} = \frac{1}{\left|d\vec{T}/dt\right|}\frac{d\vec{T}}{dt}\,,$$

and by the chain rule, we can essentially swap the t-parameter for the arc length:

$$\hat{N} = \frac{1}{\left|d\vec{T}/dS\right|}\frac{d\vec{T}}{dS}$$

Problem 1

Show that the tangent vector and normal vector always obey the orthogonality relation

$$\vec{T}\cdot\vec{N} = 0\,.$$

Hint: differentiate $\hat{T}\cdot\hat{T}$ with respect to arc length.

8.3 Acceleration Vector

The position vector

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y}$$

and the velocity vector

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \frac{dx}{dt}\,\hat{x} + \frac{dy}{dt}\,\hat{y}$$

imply the existence of the acceleration vector

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) = \frac{d^2x}{dt^2}\,\hat{x} + \frac{d^2y}{dt^2}\,\hat{y}\,.$$

While none of the above is news, we can frame these items in terms of tangent and normal vectors. Most easy is the velocity, which is

$$\vec{v}(t) = v(t)\,\hat{T}(t)\ .$$

Now recalculate the acceleration vector using the above definition, which gives

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{dv}{dt}\,\hat{T} + v\,\frac{d\hat{T}}{dt}\,,$$

or

$$\vec{a}(t) = \left(\frac{d^2S}{dt^2}\right)\hat{T} + \left|\frac{d\hat{T}}{dS}\right|\left(\frac{dS}{dt}\right)^2\hat{N}\,,$$

where $v = dS/dt$ and the definition of the normal vector have been used.

Problem 2

Derive the following:

$$\hat{N} = \frac{1}{v\left|d\hat{T}/dt\right|}\vec{a} - \frac{dv/dt}{v^2\left|d\hat{T}/dt\right|}\vec{v}$$

Traveling Basis Vectors

Since the tangent vector and normal vector are always perpendicular, these form a local set of of basis vectors to the curve:

$$\begin{aligned}\vec{v} &= v\,\hat{T}\\ \vec{a} &= a_T\,\hat{T} + a_N\,\hat{N}\end{aligned}$$

Problem 3

A particle has a known trajectory

$$r(t) = \frac{r_0}{\cos(\omega t)}\,,$$

where ω is constant. Find the velocity and acceleration vectors.

8.4 Curvature

The magnitude of the derivative of the normalized tangent vector with respect to arc length, i.e. $\left|d\hat{T}/dS\right|$, is called the curvature, denoted κ (Greek 'kappa'):

$$\kappa = \left|\frac{d\hat{T}}{dS}\right|$$

In terms of the curvature, the normal vector is

$$\hat{N} = \frac{1}{\kappa}\frac{d\hat{T}}{dS}\,,$$

and the acceleration vector is

$$\vec{a}(t) = \left(\frac{d^2S}{dt^2}\right)\hat{T} + \kappa\left(\frac{dS}{dt}\right)^2\hat{N}\,.$$

Interpreting Curvature

The curvature κ, having units of inverse length, can be regarded as one divided by the radius ρ of the circle that instantaneously approximates the curve. To see this, suppose, much like we do with a straight-line approximation, that the curve is approximated by a circle of radius $\rho(t)$ at some instant t.

To go around such a circle, the tangent vector

$$\hat{T} = \cos(\phi)\,\hat{x} + \sin(\phi)\,\hat{y}$$

runs the parameter ϕ from 0 to 2π. Meanwhile, the total arc length traveled throughout the trip is $2\pi\rho(t)$, and we establish

$$\frac{d\phi}{dS} = \frac{2\pi}{2\pi\rho(t)} = \frac{1}{\rho(t)} ,$$

which is harmlessly inverted:

$$\rho(t) = \frac{dS}{d\phi} = \frac{dS/dt}{d\phi/dt}$$

Proceed by calculating the time derivative of $\hat{T}$

$$\frac{d\hat{T}}{dt} = \frac{d\phi}{dt}\left(-\sin(\phi)\,\hat{x} + \cos(\phi)\right) ,$$

where the parenthesized portion is a unit normal vector. Taking the magnitude of the above allows $d\phi/dt$ to be isolated:

$$\left|\frac{d\hat{T}}{dt}\right| = \frac{d\phi}{dt}$$

Now rewrite the equation for $\rho(t)$:

$$\rho(t) = \frac{dS/dt}{\left|d\hat{T}/dt\right|} = \frac{1}{\left|d\hat{T}/dS\right|} = \frac{1}{\kappa(t)}$$

Calculating Curvature

Another way to isolate the curvature κ comes from the cross product between the velocity vector and the acceleration vector. For this, begin with

$$\vec{v}\times\vec{a} = \left(v\,\hat{T}\right)\times\left(\left(\frac{d^2S}{dt^2}\right)\hat{T} + \kappa\left(\frac{dS}{dt}\right)^2\hat{N}\right) .$$

The cross product distributes into both acceleration components, but $\hat{T}\times\hat{T}$ is automatically zero:

$$\hat{v}\times\hat{a} = v\kappa\left(\frac{dS}{dt}\right)^2\hat{T}\times\hat{N}$$

The terms v and dS/dt are the same, and the remaining cross product yields a unit vector oriented perpendicular to the plane of the curve:

$$\left|\hat{T}\times\hat{N}\right| = 1$$

This is enough to isolate the curvature κ in terms of the vectors of motion:

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{v^3}$$

Curvature in Polar Coordinates

For a polar curve

$$\vec{r}(t) = r(t)\,\hat{r} ,$$

the velocity and acceleration are

$$\begin{aligned}\vec{v} &= \frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta}\\ \vec{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} .\end{aligned}$$

The magnitude of the velocity is

$$v = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2} .$$

Anticipating the cross product $\vec{v}\times\vec{a}$, note that

$$\hat{r}\times\hat{r} = \hat{\theta}\times\hat{\theta} = 0 ,$$

and also

$$\hat{r}\times\hat{\theta} = -\hat{\theta}\times\hat{r} .$$

With this, we find

$$\begin{aligned}\vec{v}\times\vec{a} &= \left(v_r\hat{r} + v_\theta\hat{\theta}\right)\times\left(a_r\hat{r} + a_\theta\hat{\theta}\right)\\ &= \left(v_r a_\theta - v_\theta a_r\right)\left(\hat{r}\times\hat{\theta}\right) ,\end{aligned}$$

or, noting $\omega = d\theta/dt$,

$$|\vec{v}\times\vec{a}| = \left|r\frac{dr}{dt}\frac{d\omega}{dt} + 2\omega\left(\frac{dr}{dt}\right)^2 - r\omega\frac{d^2r}{dt^2} + r^2\omega^3\right| .$$

In the special case $t = \theta$, we have $dt = d\theta$, meaning $\omega = 1$ and $d\omega/dt = 0$. This is the setup for plotting polar curves $r = r(\theta)$ in the plane, and the corresponding curvature is:

$$\kappa = \frac{\left|r^2 + 2\left(dr/d\theta\right)^2 - r\left(d^2r/d\theta^2\right)\right|}{\left(r^2 + \left(dr/d\theta\right)^2\right)^{3/2}}$$

9 Bézier Curves

Here we develop a way of planning and drawing precise parametric curves in the plane. History has largely settled on the name *Bézier curves* to describe what follows.

9.1 Quadratic Case

In the Cartesian plane, consider three given points $\vec{p}_j = (x_j, y_j)$, where $j = 0, 1, 2$. Such given points are called *control points*.

Next, suppose there is a quadratic curve $\vec{r}\,(t)$ that passes through the first point, and passes through the last point, skipping the middle $j = 1$-point.

Letting t be a dimensionless parameter

$$0 \leq t \leq 1 \,,$$

we know

$$\begin{aligned} \vec{r}\,(0) &= \vec{p}_0 \\ \vec{r}\,(1) &= \vec{p}_2 \,, \end{aligned}$$

and that the slope $\vec{r}\,(t)$ is

$$\frac{d}{dt}\vec{r}\,(t) = \vec{v}\,(t) \ .$$

This describes an infinite family of curves so far, but now impose the restriction that $\vec{v}\,(0)$ is parallel to the difference $\vec{p}_1 - \vec{p}_0$. Also, let $\vec{v}\,(1)$ be parallel to the difference $\vec{p}_2 - \vec{p}_1$. For shorthand, let us write this as:

$$\begin{aligned} \vec{v}\,(0) &\propto \vec{p}_1 - \vec{p}_0 = \vec{p}_{01} \\ \vec{v}\,(1) &\propto \vec{p}_2 - \vec{p}_1 = \vec{p}_{12} \end{aligned}$$

If we wish for $\vec{r}\,(t)$ to be quadratic in form, then the velocity vector has at most linear dependence on the parameter t. While you can perhaps guess the form for $\vec{v}\,(t)$ already, lets us start a step back and define

$$\vec{a} = \frac{d}{dt}v\,(t) \ ,$$

which is constant for a quadratic curve.

In terms of two unknown parameters α, β, the constant vector $\vec{a}$ that characterizes the quadratic curve can be written

$$\vec{a} \propto \alpha\,\vec{p}_{01} + \beta\,\vec{p}_{12} \,.$$

Integrate $\vec{a}$ in the t-variable to get the velocity

$$\vec{v}\,(t) \propto \alpha t\,\vec{p}_{01} + \beta t\,\vec{p}_{12} + \vec{\gamma} \,,$$

where $\vec{\gamma}$ is an integration constant.

Imposing the $t = 0$ and $t = 1$ conditions on the velocity, we swiftly figure out the roles of α, β, γ, particularly

$$\begin{aligned} \alpha &= -A \\ \beta &= A \\ \vec{\gamma} &= A\,\vec{p}_{01} \,, \end{aligned}$$

where A is an overall proportionality constant. The velocity is given by:

$$\vec{v}\,(t) = A\left((1-t)\,\vec{p}_{01} + t\,\vec{p}_{12}\right)$$

Now we find the curve $\vec{r}\,(t)$ by integrating the velocity to write

$$\vec{r}\,(t) = A\left(\frac{-(1-t)^2}{2}\,\vec{p}_{01} + \frac{t^2}{2}\,\vec{p}_{12}\right) + \vec{\delta} \,,$$

where $\vec{\delta}$ is a constant.

The constants A, $\vec{\delta}$ are determined by the conditions at $t = 0$, $t = 1$. Writing each of these, we find

$$\begin{aligned} \vec{p}_0 &= \frac{A}{2}\left(-\vec{p}_1 + \vec{p}_0\right) + \vec{\delta} \\ \vec{p}_2 &= \frac{A}{2}\left(\vec{p}_2 - \vec{p}_1\right) + \vec{\delta} \,, \end{aligned}$$

implying

$$\begin{aligned} A &= 2 \\ \vec{\delta} &= \vec{p}_1 \,. \end{aligned}$$

Knowing the integration constants, we can go back and rewrite the velocity and its derivative:

$$\begin{aligned} \vec{v}\,(t) &= 2\,(1-t)\,\vec{p}_{01} + 2t\,\vec{p}_{12} \\ \vec{a} &= -2\,\vec{p}_{01} + 2\vec{p}_{12} \end{aligned}$$

In terms of the given $\vec{p}_j$, the position vector simplifies to

$$\vec{r}\,(t) = (1-t)^2\,\vec{p}_0 + 2t\,(1-t)\,\vec{p}_1 + t^2\,\vec{p}_2 \,,$$

and correspondingly:

$$\begin{aligned} \vec{v}\,(t) &= -2\,(1-t)\,\vec{p}_0 + 2\,(1-2t)\,\vec{p}_1 + 2t\,\vec{p}_2 \\ \vec{a} &= 2\left(\vec{p}_0 - 2\vec{p}_1 + \vec{p}_2\right) \end{aligned}$$

9.2 Cubic Case

Extending the quadratic case by one, consider four given points $\vec{p}_j = (x_j, y_j)$, where $j = 0, 1, 2, 3$. A curve $\vec{r}\,(t)$ passes through $\vec{p}_0$ at $t = 0$, and through $\vec{p}_3$ at $t = 1$. The velocity obeys

$$\begin{aligned} \vec{v}\,(0) &\propto \vec{p}_1 - \vec{p}_0 = \vec{p}_{01} \\ \vec{v}\,(1) &\propto \vec{p}_3 - \vec{p}_2 = \vec{p}_{23} \,. \end{aligned}$$

Unlike the quadratic case in where the derivative of $\vec{v}\,(t)$ is a constant $\vec{a}$, we now need an $\vec{a}\,(t)$ that is (at most) linearly dependent on t. Work toward this by defining two constants

$$\begin{aligned} \vec{p}_{012} &= 2\left(\vec{p}_0 - 2\vec{p}_1 + \vec{p}_2\right) \\ \vec{p}_{123} &= 2\left(\vec{p}_1 - 2\vec{p}_2 + \vec{p}_3\right) \ . \end{aligned}$$

Notice that $\vec{p}_{012}$ is same as $\vec{a}$ from the quadratic case.

Next, propose a vector $\vec{j}$ as the derivative of $\vec{a}\,(t)$

$$\frac{d}{dt}\vec{a}\,(t) = \vec{j}\,,$$

which is a constant. Similar to the quadratic case, write $\vec{j}$ as a linear combination of vectors

$$\vec{j} \propto \alpha\,\vec{p}_{012} + \beta\,\vec{p}_{123}\,,$$

for two (new) unknowns α, β.

It's important to have introduced two and only two unknowns at this stage, and of course all of the provided data points $\{(x_j, y_j)\}$ must occur in $\vec{j}$. While $\vec{j}$ could have been built in a variety of ways, the above is arguably the most natural.

Now the problem is analogous to the quadratic case with $\vec{j}$ playing $\vec{a}$'s role, etc. Transcribing the solution for α, β that works for the quadratic case, one writes

$$\begin{aligned} \vec{a}\,(t) &= B\left(2\,(1-t)\,\vec{p}_{012} + 2t\,\vec{p}_{123}\right) \\ \vec{j} &= B\left(-2\,\vec{p}_{012} + 2\vec{p}_{123}\right)\,, \end{aligned}$$

where B is a new proportionality constant.

Integrate $\vec{a}\,(t)$ to get the velocity

$$\vec{v}\,(t) = B\left(-\left(1-t\right)^2\vec{p}_{012} + t^2\,\vec{p}_{123}\right) + \vec{\gamma}\,,$$

where $\vec{\gamma}$ is a constant.

Integrate once more to get the position up to another constant $\vec{\delta}$:

$$\vec{r}\,(t) = B\left(\frac{-\left(1-t\right)^3}{3}\,\vec{p}_{012} + \frac{t^3}{3}\,\vec{p}_{123}\right) + t\,\vec{\gamma} + \vec{\delta}$$

To determine B and $\vec{\gamma}$, use $\vec{r}\,(0) = \vec{p}_0$ and $\vec{r}\,(1) = \vec{p}_3$, and then subtract one equation from the other to eliminate $\vec{\delta}$. This should result in

$$\vec{p}_3 - \vec{p}_0 = \vec{\gamma} + \frac{B}{3}\left(3\vec{p}_1 - 3\vec{p}_2 + \vec{p}_3 - \vec{p}_0\right)\,,$$

which begs the solution

$$\begin{aligned} B &= 3 \\ \vec{\gamma} &= -3\vec{p}_1 + 3\vec{p}_2\,. \end{aligned}$$

Staying with the velocity equation, substitute $\vec{\gamma}$ and simplify like mad to get

$$\vec{v}\,(t) = 3\left(1-t\right)^2\vec{p}_{01} + 6t\,(1-t)\,\vec{p}_{12} + 3t^2\,\vec{p}_{23}\,.$$

This form is much nicer than the $\vec{r}\,(t)$ derived a few lines above, so let's integrate the velocity again to get the position

$$\vec{r}\,(t) = \int \vec{v}\,(t)\,dt + \vec{\delta}\,,$$

where $\vec{\delta}$ is the same integration constant. Eliminate $\vec{\delta}$ delta using what's known about $t = 0$, $t = 1$, and get the final position:

$$\begin{aligned} \vec{r}\,(t) &= \left(1-t\right)^3\vec{p}_0 \\ &\quad + 3t\left(1-t\right)^2\vec{p}_1 + 3t^2\left(1-t\right)\vec{p}_2 + t^3\,\vec{p}_3 \end{aligned}$$

9.3 General Case

We've done enough work to hopefully spot a pattern in the quadratic and cubic cases to extend the equations to fourth-order cases and beyond. To set another shorthand notation, define

$$1 - t = w\,.$$

Quadratic

For the quadratic case:

$$\begin{aligned} \vec{v}_2\,(t) &= 2w\,\vec{p}_{01} + 2t\,\vec{p}_{12} \\ \vec{r}_2\,(t) &= w^2\,\vec{p}_0 + 2tw\,\vec{p}_1 + t^2\,\vec{p}_2 \end{aligned}$$

Cubic

For the cubic case:

$$\begin{aligned} \vec{v}_3\,(t) &= 3w^2\,\vec{p}_{01} + 6tw\,\vec{p}_{12} + 3t^2\,\vec{p}_{23} \\ \vec{r}_3\,(t) &= w^3\,\vec{p}_0 + 3tw^2\,\vec{p}_1 + 3t^2w\,\vec{p}_2 + t^3\,\vec{p}_3 \end{aligned}$$

Binomial Coefficients

Looking at the pattern in the numeric coefficients in each position vector $\vec{r}\,(t)$, namely

$$\begin{aligned} \text{quadratic:}\, &\{1, 2, 1\} \\ \text{cubic:}\, &\{1, 3, 3, 1\}\,, \end{aligned}$$

these are none other than the binomial coefficients.

The n, kth binomial coefficient is given by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

Quickly demonstrating for the cubic case, the above produces:

$$\begin{aligned} \binom{3}{0} &= \frac{3!}{0!\,(3-0)!} = 1 \\ \binom{3}{1} &= \frac{3!}{1!\,(3-1)!} = 3 \\ \binom{3}{2} &= \frac{3!}{2!\,(3-2)!} = 3 \\ \binom{3}{3} &= \frac{3!}{3!\,(3-3)!} = 1 \end{aligned}$$

Using the so-called 'choose' notation, the respective position vectors are:

$$\vec{r}_2(t) = \binom{2}{0} w^2\,\vec{p}_0 + \binom{2}{1} tw\,\vec{p}_1 + \binom{2}{2} t^2\,\vec{p}_2$$

$$\begin{aligned}\vec{r}_3(t) &= \binom{3}{0} w^3\,\vec{p}_0 + \binom{3}{1} tw^2\,\vec{p}_1 \\ &\quad + \binom{3}{2} t^2 w\,\vec{p}_2 + \binom{3}{3} t^3\,\vec{p}_3\end{aligned}$$

General Equations

Each of the above lends to summation notation. Converting to this, we find, after restoring $w = 1 - t$:

$$\vec{r}_n(t) = \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j}\, t^j\, \vec{p}_j$$

Since both the quadratic and the cubic cases are represented by the above, it takes little to imagine that higher n are also accommodated.

In the quadratic and cubic cases above, recall that the center of each derivation it a constant that involves all of the provided control points. For the quadratic case, we discovered

$$\vec{a} = \frac{d^2}{dt^2}\vec{r}_2(t) = 2\left(-\vec{p}_{01} + \vec{p}_{12}\right) ,$$

and for the cubic case,

$$\vec{j} = \frac{d^3}{dt^3}\vec{r}_3(t) = 6\left(-\vec{p}_{012} + \vec{p}_{123}\right) .$$

Going from the pattern, one should suspect that the fourth-order case obeys

$$\vec{k} = \frac{d^4}{dt^4}\vec{r}_4(t) = 24\left(-\vec{p}_{0123} + \vec{p}_{1234}\right) ,$$

and so on for higher orders.

Problem 1

Write down the fourth-order curve $\vec{r}_4(t)$ and determine exactly what is meant by $\vec{p}_{0123}$, $\vec{p}_{1234}$.

10 Planetary Motion

Early Progress

The 'modern' understanding of planetary motion arguably began with Johannes Kepler (1571 - 1630), whose career predates the invention of calculus and Newton's laws of motion by decades. Already familiar with the Heliocentric model of the solar system, Kepler studied meticulously-recorded charts of night sky measurements recorded by Tycho Brahe (1546 - 1601).

Paying attention to the positions of observable planets in the night sky, Kepler astonishingly figured out that planetary orbits were elliptical in shape with the sun at a focus. This became known as Kepler's first law, which survives to this day among two other laws written by Kepler.

Aware of Kepler's first law, Newton proposed the existence of a law of mutual Earth-sun attraction that gives rise to elliptical planetary orbits. In the modern vector notation, he began with something like

$$\vec{F} = F(r)\,\hat{r} ,$$

and the quest was to find whatever $F(r)$ is.

Using the calculus of his own invention, Newton found the answer to be a unified force depending on the masses involved and the inverse square of the distance separating them. We know this as Newton's law of universal gravitation.

The plan here is to develop the equations of planetary motion using a similar approach, at least in spirit, to Newton.

Shell Theorem

One assumption we'll make early on, which happens to be *true*, and will be proven with triple integration, is *any object can be considered as a point mass located at the object's center of mass.* For instance, if we need to calculate the gravitational attraction between two asteroids, the shape of each does not matter. Only the center-to-center distance and the mass of each body is important.

Newton's Second Law

The one-dimensional version of Newton's second law

$$m\frac{d^2}{dt^2}x(t) = -\frac{d}{dx}U(x)$$

generalizes to more dimensions where the force and acceleration become vectors:

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

I avoided saying exactly how $-dU/dx$ becomes $\vec{F}$. Note that in one dimension,

$$F = -\frac{dU}{dx}$$

is true by definition, but the three dimensional version of this requires a vector derivative operator. The exact details aren't needed in order to proceed.

Newton's Third Law

The classic phrase, *for every action, there is an equal an opposite reaction*, is Newton's third law. It means that the force from object 1 onto object 2 is exactly opposite of the force from object 2 onto object 1. This is concisely stated via vectors:

$$\vec{F}_{12} = -\vec{F}_{21}$$

10.1 Two-Body Problem

Consider two bodies in space, one of mass m_1 at position $\vec{r}_1(t)$, and the other of mass m_2 at position $\vec{r}_2(t)$. The force imposed onto body 1 by body 2 is given by

$$m_1 \frac{d^2}{dt^2}\vec{r}_1(t) = m_1 \frac{d}{dt}\vec{v}_1(t) = \vec{F}_{12} \,,$$

and the force imposed onto particle 2 by particle 1 is given by

$$m_2 \frac{d^2}{dt^2}\vec{r}_2(t) = m_2 \frac{d}{dt}\vec{v}_2(t) = \vec{F}_{21} \,.$$

This setup is called the *two-body problem.*

Center of Mass

In the two-body system, the *center of mass* is defined as a point in space $\vec{R}(t)$ such that

$$\vec{R}(t) = \frac{m_1\vec{r}_1(t) + m_2\vec{r}_2(t)}{m_1+m_2} \,.$$

The time derivative of the center of mass gives a quantity called the *center of velocity*:

$$\vec{V}(t) = \frac{d}{dt}\vec{R}(t) = \frac{m_1\vec{v}_1(t) + m_2\vec{v}_2(t)}{m_1+m_2} \,.$$

Taking the time derivative of the center of velocity gives something interesting:

$$\begin{aligned}\frac{d^2}{dt^2}\vec{R}(t) &= \frac{m_1\left(d\vec{v}_1(t)/dt\right) + m_2\left(d\vec{v}_2(t)/dt\right)}{m_1+m_2} \\ &= \frac{\vec{F}_{12}+\vec{F}_{21}}{m_1+m_2} = \frac{\vec{F}_{12}-\vec{F}_{12}}{m_1+m_2} = 0\end{aligned}$$

Evidently, the second derivative of the center of mass is precisely zero because $\vec{F}_{12} = -\vec{F}_{21}$, regardless of how the forces act. This means that two bodies, while free to move individually, are not accelerating anywhere as a group. Moreover, this result proves that the center of velocity $\vec{V}$ is a constant $\vec{V}_0$.

Relative Displacement

If the distance separating the two bodies is r, define a vector

$$\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t)$$

with $|\vec{r}| = r$, capturing the relative displacement between the two.

Listing this with the center of mass $\vec{R}(t)$, we have a system of two equations that can be solved for $\vec{r}_1(t)$, $\vec{r}_2(t)$ separately: (We know everything is a function of t by now, so drop the extra notation.)

$$\begin{aligned}\vec{r}_1 &= \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{m_1+m_2}\vec{r}\end{aligned}$$

Reduced Mass

From the equations above, multiply through by m_1, m_2, respectively, and take two time derivatives:

$$\begin{aligned}m_1\frac{d^2\vec{r}_1}{dt^2} &= m_1\cancel{\frac{d^2\vec{R}}{dt^2}} + \frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{r}}{dt^2} \\ m_2\frac{d^2\vec{r}_2}{dt^2} &= m_2\cancel{\frac{d^2\vec{R}}{dt^2}} - \frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{r}}{dt^2}\end{aligned}$$

These results say the same thing, as the left sides are $\vec{F}_{12}$, $\vec{F}_{21}$, respectively, and the right sides differ by the proper negative sign.

Evidently, we have

$$\vec{F}_{12} = \frac{m_1m_2}{m_1+m_2}\frac{d^2\vec{r}}{dt^2} \,.$$

That is, there is only one force equation to worry about, and thus one position to worry about if we work with the relative displacement vector $\vec{r}$ rather than two explicit position vectors $\vec{r}_{1,2}$.

The price we pay is the mass term became a mess. This group of symbols is called the *reduced mass*:

$$m_* = \frac{m_1m_2}{m_1+m_2}$$

Representing the effective mass of the total system as m_*, the two-body problem is summarized in one equation:

$$\vec{F}_{12} = m_*\frac{d^2\vec{r}}{dt^2} = m_*\vec{a}$$

A handy identity involving the reduced mass, somewhat reminiscent of resistors in parallel, goes as:

$$\frac{1}{m_*} = \frac{1}{m_1} + \frac{1}{m_2}$$

10.2 Angular Momentum

Alongside the notion of forces, we'll need to put the ideas of angular momentum to use. In particular, we can show that the angular momentum of the two-body system is constant, and find what it is.

By definition, the angular momentum $\vec{L}$ of the two-body system reads

$$\vec{L} = m_* \vec{r} \times \vec{v} \, ,$$

where $\vec{r}$ is the relative displacement vector, and $\vec{v}$ is its time derivative. Now calculate the time derivative of $\vec{L}$:

$$\begin{aligned} \frac{d}{dt}\vec{L} &= m_* \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) \\ &= m_* \left(\vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \right) \\ &= \vec{r} \times \vec{F} \end{aligned}$$

For the remaining cross product to vanish, we go back to Newton's original assumption that

$$\vec{F} = F\left(r\right) \hat{r} \, ,$$

which means the force vector and the displacement vector are parallel. Using this, we see that the derivative of $\vec{L}$ resolves to zero.

Without knowing the exact motion of the two-body system, we can still write a formula for the angular momentum. For some $r\left(t\right)$, $\theta\left(t\right)$, we have, in polar coordinates:

$$\begin{aligned} \vec{r} &= r\,\hat{r} \\ \vec{v} &= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} \end{aligned}$$

Remembering $\hat{r} \times \hat{r}$ is zero, we then have

$$\vec{L} = m_* r^2 \frac{d\theta}{dt}\left(\hat{r} \times \hat{\theta}\right) \, .$$

The angular momentum is a constant vector that points perpendicular to the plane of motion. We take its magnitude

$$L = m_* r^2 \frac{d\theta}{dt}$$

as a constant of motion in the two-body system.

It's easy to show that the position vector and the angular momentum vector are always perpendicular. Starting with the definition of $\vec{L}$, project $\vec{r}$ into both sides:

$$\vec{r} \cdot \vec{L} = m_* \vec{r} \cdot \left(\vec{r} \times \vec{v}\right) \, ,$$

and then make use of the triple product:

$$\vec{r} \cdot \vec{L} = m_* \vec{v} \cdot \left(\vec{r} \times \vec{r}\right) = 0$$

10.3 Inverse-Square Acceleration

We've made it this far without knowing the magnitude gravitational force $F\left(r\right)$, although we have harmlessly assumed that gravity acts in a straight line. Here we will derive the proper gravitational force by using Kepler's first law as a starting point.

In detail, Kepler noticed that the orbit of any planet around the sun takes an elliptical form described by

$$r\left(\theta\right) = \frac{r_0}{1 + e\cos\left(\theta\right)} \, ,$$

where e is the *eccentricity* of the orbit, and r_0 is a positive characteristic length. Notice that $r\left(\theta\right)$ as written places the origin (the sun) at the *right* focus of the ellipse. Reverse the sign on the cosine term for the sun at the left focus.

To really get started, take the time derivative of the (constant) angular momentum of the two-body system:

$$0 = \frac{dL}{dt} = m_* r\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)$$

Perhaps you recognize the parenthesized term as being identically the $\hat{\theta}$-component of the acceleration vector in polar coordinates. In terms of L, the acceleration vector is

$$\vec{a} = \left(\frac{d^2 r}{dt^2} - \frac{L^2}{m_*^2 r^3}\right)\hat{r} + \frac{1}{m_* r}\cancel{\left(\frac{dL}{dt}\right)}\hat{\theta} \, .$$

We need the polar form of the ellipse to calculate d^2r/dt^2. For this, we find, after simplifying,

$$\frac{dr}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{r_0}{1 + e\cos\left(\theta\right)}\right) = \frac{L}{m_* r_0} e \sin\left(\theta\right) \, ,$$

and keep going to the second derivative:

$$\frac{d^2 r}{dt^2} = \frac{L^2}{m_*^2 r^2 r_0} e\cos\left(\theta\right) = \frac{L^2}{m_*^2 r^2}\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

The full acceleration vector then reads

$$\vec{a} = \frac{L^2}{m_*^2 r^2}\left(\cancel{\frac{1}{r}} - \frac{1}{r_0} - \cancel{\frac{1}{r}}\right)\hat{r} \, ,$$

which simplifies nicely:

$$\vec{a} = \frac{-L^2}{m_*^2 r_0}\frac{\hat{r}}{r^2}$$

This finally reveals the nature of $F\left(r\right)$. The r-dependence is present as $-1/r^2$, hence the name inverse-square acceleration.

Going back to the equations that led to the reduced mass, i.e.

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = m_* \frac{d^2\vec{r}}{dt^2}$$
$$m_2 \frac{d^2\vec{r}_2}{dt^2} = -m_* \frac{d^2\vec{r}}{dt^2} ,$$

we can solve for the absolute acceleration of each body:

$$\vec{a}_1 = \frac{m_*}{m_1} \vec{a}$$
$$\vec{a}_2 = \frac{-m_*}{m_2} \vec{a}$$

Eliminate $\vec{a}$ between the two equations to recover Newton's third law:

$$m_1 \vec{a}_1 + m_2 \vec{a}_2 = 0$$

10.4 Universal Gravitation

Enough ground work has been done to finally write Newton's universal law of gravitation.

Recall the absolute acceleration of each body $\vec{a}_1$, $\vec{a}_2$, and replace the reduced mass m_* and acceleration $\vec{a}$ with expanded forms:

$$\vec{a}_1 = \left(\frac{m_2}{m_1+m_2}\right) \frac{-L^2}{m_*^2 r_0} \frac{\hat{r}}{r^2}$$
$$\vec{a}_2 = \left(\frac{-m_1}{m_1+m_2}\right) \frac{-L^2}{m_*^2 r_0} \frac{\hat{r}}{r^2}$$

To get rid of some clutter, let us group the coefficients that occur identically in both equations into an auxiliary constant γ such that

$$\gamma = \frac{1}{m_1+m_2}\left(\frac{L^2}{m_*^2 r_0}\right) = \frac{L^2}{m_* m_1 m_2 r_0} ,$$

and then we can forget about γ (momentarily) by trading the equality signs for proportionality symbols:

$$\vec{a}_1 \propto \; m_2 \frac{(-\hat{r})}{r^2}$$
$$\vec{a}_2 \propto -m_1 \frac{(-\hat{r})}{r^2}$$

What we see is the acceleration of body 1 being proportional to the mass of body 2, and vice versa.

Multiply each equation through by m_1, m_2, respectively to turn accelerations into forces:

$$\vec{F}_{12} = m_1 \vec{a}_1 \propto \; m_1 m_2 \frac{(-\hat{r})}{r^2}$$
$$\vec{F}_{21} = m_2 \vec{a}_2 \propto -m_1 m_2 \frac{(-\hat{r})}{r^2}$$

Of course, these are saying the same thing due to Newton's third law, so in summary:

$$\vec{F}_{12} \propto -m_1 m_2 \frac{\hat{r}}{r^2}$$

This comprises all of the ingredients for building the gravitational force. We have a force vector acting along the line connecting two bodies whose strength is proportional the product of the masses and inversely proportional to the square of the separation.

Newton decided to introduce a new proportionality constant G, named after 'gravity', to turn the above back into an equation. We take the gravitational force, finally, to be:

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r^2} \hat{r}$$

Note that the force vector bears the 12-subscript and not the other way around. The subscript is often omitted because the unit vector $\hat{r}$ has an implied 12-subscript that goes back to the definition of $\vec{r}$.

To reconcile the constants γ and G, it's easy to work out that

$$G = \gamma m_1 m_2 .$$

Eliminating γ, we can also write

$$G = \frac{L^2}{m_* r_0} .$$

While this calculation was set up in the context of planetary motion, note that the gravitational force is in fact *universal*, which is to say that every pair of particles in the universe obeys the same law.

10.5 Equations of Motion

With the law of universal gravitation on hand, we should be able to run the analysis in reverse by starting with $\vec{F}_{12}$ and finishing with the shape of the ellipse, along with all other allowed possibilities.

Acceleration

Use

$$L = m_* r^2 \frac{d\theta}{dt}$$

to eliminate $1/r^2$ in the force vector:

$$\vec{F}_{12} = -G m_1 m_2 \frac{m_*}{L} \frac{d\theta}{dt} \hat{r}$$

Also replace $\vec{F}_{12}$ to keep simplifying

$$m_1 \vec{a}_1 = m_* \vec{a} = -G m_1 m_2 \frac{m_*}{L} \frac{d\theta}{dt} \hat{r} ,$$

and solve for the relative acceleration:

$$\vec{a} = -G \frac{m_1 m_2}{L} \frac{d\theta}{dt} \hat{r}$$

Velocity

To proceed, replace the acceleration vector as the derivative of the relative velocity by $\vec{a} = d\vec{v}/dt$. Also replace $\hat{r}$ via $-\hat{r} = d\hat{\theta}/d\theta$ to get

$$\frac{d\vec{v}}{dt} = G\frac{m_1 m_2}{L}\frac{d\theta}{dt}\frac{d\hat{\theta}}{d\theta} ,$$

simplifying with the chain rule to:

$$d\vec{v} = G\frac{m_1 m_2}{L}\, d\hat{\theta}$$

Integrate both sides of the above to get a vector equation for the velocity

$$\vec{v}\,(t) = G\frac{m_1 m_2}{L}\,\hat{\theta}\,(t) + \vec{v}_0 ,$$

where $\vec{v}_0$ is the integration constant. Letting $\theta = 0$ correspond with the positive x-axis, it must be that $\vec{v}_0 = v_0\,\hat{y}$.

Position

To goal is get hold of a position equation $r\,(\theta)$. To get closer, calculate the full angular momentum vector:

$$\begin{aligned} \vec{L} &= m_* \vec{r} \times \vec{v} \\ &= m_* \vec{r} \times \left(G\frac{m_1 m_2}{L}\,\hat{\theta} + v_0\,\hat{y}\right) \\ &= m_* G\frac{m_1 m_2}{L} r\left(\hat{r} \times \hat{\theta}\right) + m_* v_0 r\,(\hat{r} \times \hat{y}) \end{aligned}$$

To handle the cross products, note that

$$\begin{aligned} \left|\hat{r} \times \hat{\theta}\right| &= 1 \\ \left|\hat{r} \times \hat{y}\right| &= \left|\cos(\theta)\right| , \end{aligned}$$

and we can work with just magnitudes:

$$L = m_* G\frac{m_1 m_2}{L} r + m_* v_0 r \cos(\theta)$$

To help simplify this, recall the proportionality factor γ that preceded G

$$\gamma = \frac{L^2}{m_* m_1 m_2 r_0} ,$$

and work to isolate r:

$$\frac{\gamma r_0}{G} = r\left(1 + \frac{\gamma r_0}{G}\frac{m_* v_0}{L}\cos(\theta)\right)$$

The combination $\gamma r_0/G$ is another characteristic length which we'll call R_0:

$$R_0 = \frac{r_0 \gamma}{G}$$

Solving for r finally gives the result

$$r\,(\theta) = \frac{R_0}{1 + (m_* R_0 v_0/L)\cos(\theta)} .$$

With $r\,(\theta)$ known, the position vector is straightforwardly written:

$$\vec{r} = r\,(\theta)\,\hat{r}$$

Eccentricity

Comparing the above to the general form of a conic section in polar coordinates, we pick out the eccentricity to be

$$e = \frac{m_* R_0 v_0}{L} .$$

Circular orbits arise from the special case $v_0 = 0$. Another special case is $e = 1$ for a parabolic trajectory. For all $e < 1$, the orbit is strictly an ellipse. For $e > 1$, the path (also technically an orbit) is hyperbolic.

This surely nails the case shut for Kepler's first law. All results reinforce the fact that planetary orbits occur on ellipses with the sun at a focus.

The eccentricity can be expressed by a variety of combinations of terms. For a version without L, one can find

$$e = \frac{\sqrt{R_0}\,v_0}{\sqrt{G\,(m_1 + m_2)}} ,$$

or, if you need to get rid of R_0:

$$e = \frac{v_0 L}{G m_1 m_2}$$

In terms of the eccentricity, the equations of motion can be simplified. For the position, we simply have

$$r\,(\theta) = \frac{R_0}{1 + e\cos(\theta)} .$$

For the velocity and acceleration, shuffle the constants around to establish

$$\frac{G m_1 m_2}{L} = \frac{v_0}{e} ,$$

which is only defined for non-circular orbits. With this, we have:

$$\begin{aligned} \vec{v} &= v_0\left(\frac{\hat{\theta}}{e} + \hat{y}\right) \\ \vec{a} &= \frac{-v_0}{e}\frac{d\theta}{dt}\,\hat{r} \end{aligned}$$

10.6 Runge-Lorenz Vector

The two-body problem exhibits conservation of angular momentum via the constant vector $\vec{L}$. There is, in fact, another constant vector of motion lurking about called the *Runge-Lorenz* vector

$$\vec{Z} = \vec{v} \times \vec{L} - Gm_1 m_2 \, \hat{r} \; .$$

Constant of Motion

Take a time derivative to prove $\vec{Z}$ is constant:

$$\begin{aligned} \frac{d}{dt}\vec{Z} &= \frac{d}{dt}\left(\vec{v} \times \vec{L}\right) - Gm_1 m_2 \frac{d\hat{r}}{dt} \\ &= \frac{d\vec{v}}{dt} \times \vec{L} + \vec{v} \times \cancel{\frac{d\vec{L}}{dt}} - Gm_1 m_2 \frac{d\hat{r}}{dt} \end{aligned}$$

Keep simplifying with

$$\frac{d\vec{v}}{dt} = \frac{1}{m_*}\vec{F} = -G\frac{m_1 m_2}{m_* r^2}\,\hat{r} \; ,$$

and also with $\vec{L} = m_* \vec{r} \times \vec{v}$, so we have

$$\frac{d}{dt}\vec{Z} = Gm_1 m_2 \left(-\frac{\hat{r} \times (\vec{r} \times \vec{v})}{r^2} - \frac{d\hat{r}}{dt} \right) \; .$$

Replace $\vec{v}$ with its polar expression and note that

$$\vec{r} \times \vec{v} = \vec{r} \times \left(\frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta} \right) = r^2 \frac{d\theta}{dt}\left(\hat{r} \times \hat{\theta}\right) \; ,$$

and furthermore, using the BAC-CAB formula:

$$\hat{r} \times (\vec{r} \times \vec{v}) = r^2 \frac{d\theta}{dt}\,\hat{r} \times \left(\hat{r} \times \hat{\theta}\right) = -r^2 \frac{d\theta}{dt}\,\hat{\theta}$$

Summarizing, we find

$$\frac{d}{dt}\vec{Z} = Gm_1 m_2 \left(\frac{d\theta}{dt}\,\hat{\theta} - \frac{d\hat{r}}{dt} \right) = 0$$

as proposed.

Perigee

With $\vec{Z}$ known to be constant, we're free to evaluate it at any point along the trajectory. Choose a point $\vec{r}_p = r_p\,\hat{x}$ that has $\vec{v}_p \cdot \vec{r}_p = 0$, called a *perigee*:

$$\begin{aligned} \vec{Z} &= \vec{v}_p \times \vec{L} - Gm_1 m_2\,\hat{x} \\ &= \vec{v}_p \times (m_* r_p\,\hat{x} \times \vec{v}_p) - Gm_1 m_2\,\hat{x} \\ &= \left(m_* r_p v_p^2 - Gm_1 m_2\right)\hat{x} \end{aligned}$$

At the perigee, the velocity v_p is momentarily equal to $r_p\,d\theta/dt$, which we'll call

$$v_p = r_p\,\omega_p \; .$$

In the same notation, the angular momentum is

$$L = m_* r_p^2 \omega_p = m_* r_p v_p \; ,$$

and the vector $\vec{Z}$ becomes

$$\vec{Z} = \left(\frac{L^2}{m_* r_p} - Gm_1 m_2 \right) \hat{x} \; .$$

We can keep simplifying. Replace L^2 with the expression involving γ:

$$\vec{Z} = Gm_1 m_2 \left(\frac{\gamma r_0}{G r_p} - 1 \right) \hat{x} \; .$$

Note that $\gamma r_0/G$ is the same combination we identified as the characteristic length of a conic trajectory, namely R_0, thus

$$\vec{Z} = Gm_1 m_2 \left(\frac{R_0}{r_p} - 1 \right) \hat{x} \; .$$

The ratio R_0/r_p can be calculated by setting $\theta = 0$ in the polar equation $r(\theta)$ for a conic section:

$$r_p = \frac{R_0}{1 + (R_0 m_* v_0/L)} = \frac{R_0}{1+e}$$

Finally, the simplest form for $\vec{Z}$ is:

$$\vec{Z} = Gm_1 m_2\, e\,\hat{x}$$

What $\vec{Z}$ tells us, apart from containing all information about the trajectory, is that all gravitational trajectories contain at least one perigee, defining the x-axis of the coordinate system about which the motion is symmetric.

Apogee

The perigee is also known as the nearest distance attained between the two bodies. For an elliptical orbit or hyperbolic orbit, the perigee is given by $\theta = 0$:

$$r_{\text{perigee}} = \frac{R_0}{1+e}$$

For elliptical orbits, there is also the notion of *apogee*, which is the furthest distance attained between the two bodies. Set $\theta = \pi$ to find

$$r_{\text{apogee}} = \frac{R_0}{1-e}$$

<u>Problem 1</u>

Take derivatives of

$$r(\theta) = \frac{R_0}{1 + e\cos(\theta)}$$

to verify the locations of the perigee and apogee.

<u>Problem 2</u>

Show that:

$$e = \left| \frac{r_p - r_a}{r_p + r_a} \right|$$

Conic Trajectory

The Runge-Lorenz vector

$$\vec{Z} = \vec{v} \times \vec{L} - G m_1 m_2 \, \hat{r} \, ,$$

together with its particular expression

$$\vec{Z} = G m_1 m_2 \, e \, \hat{x}$$

can be used together to quickly recover the polar equation for conic sections by projecting the position vector across the equation and simplifying:

$$\begin{aligned} \vec{r} \cdot \vec{Z} &= \vec{r} \cdot \left(\vec{v} \times \vec{L} \right) - G m_1 m_2 \, \vec{r} \cdot \hat{r} \\ r Z \cos(\theta) &= \vec{L} \cdot (\vec{r} \times \vec{v}) - G m_1 m_2 r \\ G m_1 m_2 r e \cos(\theta) &= \frac{L^2}{m_*} - G m_1 m_2 r \end{aligned}$$

Now solve for $r(\theta)$ and simplify more:

$$\begin{aligned} r(\theta) &= \left(\frac{L^2}{G m_1 m_2 m_*} \right) \frac{1}{1 + e \cos(\theta)} \\ &= \left(\frac{\gamma r_0}{G} \right) \frac{1}{1 + e \cos(\theta)} \\ &= \frac{R_0}{1 + e \cos(\theta)} \end{aligned}$$

Relation to Ellipse

An ellipse is classified by two perpendicular lengths we know as the semi-major and semi-minor axes, denoted a, b, respectively. By studying the ellipse, it's straightforward to show that

$$a = \frac{R_0}{1 - e^2} \, ,$$

and also

$$b = \frac{R_0}{\sqrt{1 - e^2}} \, .$$

The a-equation can be derived by taking the difference between $r(0)$ and $r(\pi)$, i.e. the distance between the perigee and apogee. This pair of points defines the distance $2a$.

The b-equation can be derived by finding r_*, θ_* that correspond to $y = b$, the highest point on the ellipse:

$$\begin{aligned} 0 &= \frac{d}{d\theta} (y(\theta)) = \frac{d}{d\theta} (r(\theta) \sin(\theta)) \Big|_{r_*, \theta_*} \\ &= \left(\frac{R_0 \, e \sin^2(\theta)}{(1 + e \cos(\theta))^2} + \frac{R_0 \cos(\theta)}{1 + e \cos(\theta)} \right) \Bigg|_{r_*, \theta_*} \\ &= \frac{r_*^2}{R_0} (e + \cos(\theta_*)) \end{aligned}$$

Evidently, we have

$$\cos(\theta_*) = -e \, .$$

Taking this with

$$\begin{aligned} b &= r_* \sin(\theta_*) \\ r_* &= \sqrt{e^2 a^2 + b^2} \end{aligned}$$

is enough to finish the job. Note that similar relationships can be drawn for hyperbolic orbits.

Problem 3

Show that $\vec{r} \cdot \vec{v} = 0$ is true only at the apogee and perigee.

Dimensionless Runge-Lorenz

The Runge-Lorenz vector can be made into a dimensionless vector $\vec{\epsilon}$ by dividing $G m_1 m_2$ across the whole equation

$$\vec{\epsilon} = \frac{\vec{v} \times \vec{L}}{G m_1 m_2} - \hat{r} \, ,$$

where by the properties of $\vec{Z}$, we also know

$$\vec{\epsilon} = e \, \hat{x} \, .$$

With this setup, write

$$\hat{r} + e \, \hat{x} = \frac{\vec{v} \times \vec{L}}{G m_1 m_2} \, ,$$

and then project $\vec{r}$ into each side to recover the equation of a conic section:

$$r(1 + e \cos(\theta)) = \frac{\vec{r} \cdot \left(\vec{v} \times \vec{L} \right)}{G m_1 m_2} = R_0$$

10.7 Kepler's Laws

We spent a good effort developing the nature of gravitational orbits, and it would be difficult to imagine doing this without all of the modern advantages, particularly calculus and vectors. Somehow, Kepler was able to find enough pattern in sixteenth-century astronomical data to work out three correct laws of planetary motion. The data itself was recorded by astronomer Tycho Brahe over a span of at least thirty years.

Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the sun at a focus.

This law we know very well by now, as did Newton. For the sun at the right focus (reverse the sign for the left focus), a planetary orbit looks like

$$r(t) = \frac{R_0}{1 + e\cos(\theta(t))} \,,$$

where e is the eccentricity.

Law of Equal Areas (1609)

A line drawn between the sun and the planet sweeps out equal areas in equal times.

This is an amazing thing to notice from looking at charts of numbers. It turns out that this law is actually stating the conservation of angular momentum, although Kepler wouldn't have known so.

To derive the law in familiar language, recall the setup for the area integral in polar coordinates, particularly

$$A = \frac{1}{2}\int_{\theta_0}^{\theta_1} r^2 \, d\theta \,.$$

In differential form, this same notion reads

$$dA = \frac{1}{2}r^2 d\theta \,.$$

Or, by the chain rule, we can also write

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \,.$$

Notice, though, that $r^2 d\theta/dt$ is also present in the angular momentum

$$L = m_* r^2 \frac{d\theta}{dt} \,,$$

which can only mean

$$\frac{dA}{dt} = \frac{L}{2m_*} \,,$$

thus dA/dt is constant. This is the literal mathematical statement of 'equal areas swept in equal times'.

Problem 4

For a body moving on a path $r = f(\theta)$ obeying Kepler's second law, show that the acceleration is:

$$\vec{a} = \frac{L^2}{m_* r^3}\left(\frac{f''(\theta)}{f(\theta)} - 2\left(\frac{f'(\theta)}{f(\theta)}\right)^2 - 1\right)\hat{r}$$

Problem 5

Show that Kepler's second law works for straight-line motion.

Harmonic Law (1618)

The square of the period of a planet is directly proportional to the cube of the semi-major axis of the orbit.

Years after his first two discoveries, Kepler discerned yet another relationship for linking the time scale of the orbit to its length scale. While Kepler only knew of the proportionality between the period T and the semi-major axis a, we can do better by finding the associated constant.

Integrate the area equation for a full period of the orbit:

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{L}{2m_*}\int_0^T dt = \frac{L}{2m_*}T$$

The area is simply πab, so we find

$$T = \pi ab\frac{2m_*}{L} \,.$$

Replace b using $b = a\sqrt{1-e^2}$, and eliminate L using $L^2 = Gm_* r_0$:

$$T = 2\pi a^2\sqrt{1-e^2}\,\frac{\sqrt{m_*}}{\sqrt{Gr_0}}$$

To deal with the r_0 term, recall two identities previously used

$$\begin{aligned} R_0 &= \gamma r_0/G \\ a &= R_0/\left(1-e^2\right) \,, \end{aligned}$$

and reason that

$$\sqrt{r_0} = \sqrt{a}\sqrt{1-e^2}\sqrt{m_1 m_2} \,.$$

The period is, after simplifying,

$$T = \frac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}} \,.$$

10.8 Energy Considerations

With the fine details of planetary motion finished, it's worth pointing out that the notion of 'energy' was not used at all. To develop some of this now, recall that in one dimension, the force relates to the potential energy by

$$F = -\frac{d}{dx}U(x) \,.$$

Planetary motion, on the other hand, requires three dimensions to express the force, or two dimensions if we already know the plane of the motion. This is why the force is a vector:

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\,\hat{r}$$

Notice, though, that the force is dependent on one spacial quantity, the length, which to say the force is effectively one-dimensional.

Gravitational Potential Energy

Since the gravitational force acts in strictly the radial direction, it stands to reason that the gravitational potential energy $U(r)$ relates to the force by:

$$\vec{F}(\vec{r}) = -\frac{d}{dr}(U(r))\,\hat{r}$$

This is just like the one-dimensional Newton's law $F = -dU/dx$, except the force is a vector, balanced by $\hat{r}$ on the right.

To solve for $U(r)$, project $\hat{r}$ into both sides of the above to get

$$\frac{Gm_1 m_2}{r^2} = \frac{d}{dr}(U(r)) \ ,$$

solved by:

$$U(r) = -\frac{Gm_1 m_2}{r}$$

This is the total gravitational potential energy stored between the two masses m_1, m_2.

For a more formal definition, turn Newton's second law into a definite integral in the variable $d\vec{r}$ to get

$$\int_{r_0}^{r_1} \vec{F}(r)\cdot d\vec{r} = -\int_{r_0}^{r_1} \frac{d}{dr}U(r)\,\hat{r}\cdot d\vec{r} \,,$$

where the integral on the right is redundant to the derivative, leaving $U(r)$ evaluated at the endpoints:

$$\int_{r_0}^{r_1} \vec{F}(r)\cdot d\vec{r} = -\left(U(r_1) - U(r_0)\right)$$

Set r_0 to infinity to recover the previous form.

Kinetic Energy

Containing two objects in total, the kinetic energy energy T of the two-body system is

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \ .$$

What we need, however, is to express the kinetic energy in terms of the relative velocity

$$\vec{v} = \vec{v}_1 - \vec{v}_2 \ .$$

Working out the algebra for this is left as an exercise, but the effort results in

$$T = \frac{1}{2}m_* v^2 + \frac{1}{2}(m_1 + m_2)\,V_0^2 \ ,$$

where V_0 is the (constant) center of velocity of the whole system. It's harmless to set this term to zero.

Conservation of Energy

The total energy of the two-body system is the sum of the kinetic and the potential contributions:

$$E = T + U = \frac{1}{2}m_* v^2 - \frac{Gm_1 m_2}{r}$$

As it turns out, the energy of the system is constant.

To prove this, begin with Newton's second law

$$\vec{F} = -\frac{Gm_1 m_2}{r^2}\,\hat{r} \ ,$$

and project the velocity vector into each side:

$$\vec{v}\cdot\vec{F} = -\frac{Gm_1 m_2}{r^2}\left(\vec{v}\cdot\hat{r}\right)$$

Replace $\vec{F}$ on the left and $\vec{v}$ on the right

$$m_*\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right) = -\frac{Gm_1 m_2}{r^2}\left(\frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta}\right)\cdot\hat{r} \ ,$$

which simplifies to

$$\frac{1}{2}m_*\frac{d}{dt}\left(\vec{v}\cdot\vec{v}\right) = -\frac{Gm_1 m_2}{r^2}\frac{dr}{dt} \ .$$

(Note we didn't really need the polar expression for the velocity. The r-component of the velocity is always dr/dt.) The right side can be undone with the chain rule:

$$\frac{d}{dt}\left(\frac{1}{2}m_* v^2\right) = \frac{d}{dt}\left(\frac{Gm_1 m_2}{r}\right)$$

Finally, we have found

$$\frac{d}{dt}(T + U) = 0 \ ,$$

as expected.

The Apocalypse Problem

If a planet were suddenly stopped in its orbit, supposed circular, it would fall into the sun in a time which is $\sqrt{2}/8$ times the period of the planet's revolution.

To prove this, begin with the total energy of the system

$$-\frac{Gm_1 m_2}{a} = \frac{1}{2}m_*\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1 m_2}{r(t)} \ ,$$

where a is the radius of the orbit. Solve for dr/dt to get

$$\frac{dr}{dt} = \sqrt{\frac{2G(m_1 + m_2)}{a}}\sqrt{\frac{a}{r} - 1} \ ,$$

which can be separated into two equal integrals:

$$\int_a^0 \frac{dr}{\sqrt{a/r-1}} = \sqrt{\frac{2G\left(m_1+m_2\right)}{a}} \int_0^{t_*} dt \,,$$

where t_* is the answer we're after.

To solve the r-integral, choose the peculiar substitution

$$\begin{aligned} r &= a\cos^2\left(\theta\right) \\ dr &= -2a\cos\left(\theta\right)\sin\left(\theta\right)d\theta \,, \end{aligned}$$

and the above reduces to

$$2a\int_{\pi/2}^{\pi} \cos^2\left(\theta\right)d\theta = t_*\sqrt{\frac{2G\left(m_1+m_2\right)}{a}} \,.$$

The remaining θ-integral resolves to $\pi/4$. Solving for t_* gives

$$t_* = \frac{\sqrt{2}}{8}\left(\frac{2\pi a^{3/2}}{\sqrt{G\left(m_1+m_2\right)}}\right) = \frac{\sqrt{2}}{8}\,T \,,$$

as stated. This is about 0.1768 years, or just over two months, supposing there are twelve months per year on that planet.

10.9 Solid Sphere

We've taken on assumption (correctly) the shell theorem, which says a gravitational body with finite size can be treated as a point located at its center of mass.

With the shell theorem, we can calculate the gravitational force inside a uniform sphere of mass M and radius R at any distance $r < R$ from the center. A uniform sphere has the same density throughout, which we'll call λ:

$$\lambda = \frac{M}{4\pi R^3/3}$$

Force Inside Solid Sphere

At a distance $r < R$ from the center, according to the theorem, all of the sphere's mass that is located further from the center than r can be ignored. Only the sphere's mass obeying $r < R$ contributes to the force at distance r. This portion is called the *enclosed mass*. The enclosed mass is written $m\left(r\right)$, given by

$$m\left(r\right) = \lambda\frac{4}{3}\pi r^3 \,.$$

If the test particle has mass m_0, the magnitude of the force on the test particle is

$$F\left(r\right) = -\frac{Gm_0 m\left(r\right)}{r^2} = -\frac{Gm_0 M r}{R^3} \,.$$

Due to the r^3 factor that enters the numerator, the usual r^{-2} factor is replaced by r. The gravitational force inside a sphere grows linearly with distance until $r = R$.

As a vector, the force inside the solid sphere reads

$$\vec{F}\left(r\right) = -\frac{Gm_0 M}{R^3}\,\vec{r} \,.$$

Energy Inside Solid Sphere

The gravitational potential energy inside a solid sphere is not $U \propto -1/r$. To find the proper answer, first define

$$\lim_{r\to\infty} U\left(r\right) = 0$$

which assumes there is no energy when infinitely far from the solid sphere, assumed centered at the origin.

Starting from infinity, let a test particle of mass m_0 approach the solid sphere, eventually penetrating the its surface, stopping at r_1. The energy spent during approach is broken into two integrals:

$$U\left(r_1\right) = -\int_\infty^R \vec{F}_{\text{out}}\cdot d\vec{r} - \int_R^{r_1} \vec{F}_{\text{in}}\cdot d\vec{r} \,,$$

where $\vec{F}_{\text{out}}$, $\vec{F}_{\text{int}}$ are the forces felt by m_0 outside and inside the sphere, respectively.

Carrying out the integrals and simplifying, one finds

$$U\left(r_1\right) = \frac{Gm_0 M}{2}\left(\frac{r_1^2 - 3R^2}{R^3}\right) \,.$$

Note that the special point $r_1 = R$ corresponds to being on the sphere's surface, and the potential energy takes a familiar form

$$U\left(R\right) = -\frac{Gm_0 M}{R} \,.$$

10.10 Energy and Orbit

Escape Velocity

In a two-body system with gravity being the only force present, suppose we imparted an initial carefully-chosen *escape velocity* v_e along the line between the bodies such that the kinetic energy goes to zero as the separation becomes infinite.

As a two-body problem, we can apply conservation of energy to write

$$\frac{1}{2}m_* v_0^2 - \frac{Gm_1 m_2}{d} = \frac{1}{2}m_* v^2 - \frac{Gm_1 m_2}{r} = 0 \,,$$

where d is the initial separation between the bodies. The total energy is zero by definition.

From the energy statement, we can easily solve for the escape velocity from a starting separation d:

$$v_e = \sqrt{\frac{2Gm_1 m_2}{m_* d}} = \sqrt{\frac{2G\left(m_1+m_2\right)}{d}}$$

Parabolic Orbit

Suppose now that a two-body system has zero total energy

$$E = 0 \, .$$

but the motion is not strictly along the line connecting the two bodies. In this special case, the system is *always* at escape velocity. This does not mean the escape velocity is constant. The distance d is playing the role of r in the v_e equation.

To develop this, recall that the velocity for a parabolic orbit can be written

$$\vec{v} = v_0 \left(\hat{\theta} + \hat{j} \right) \, ,$$

which means

$$v^2 = \vec{v} \cdot \vec{v} = 2 v_0^2 \frac{R_0}{r} \, .$$

Using the escape velocity in place of v allows us to write

$$\frac{2G\left(m_1 + m_2\right)}{r} = 2 v_0^2 \frac{R_0}{r} \, ,$$

or

$$v_0^2 = \frac{G\left(m_1 + m_2\right)}{R_0} \, .$$

Elliptical Orbit

Elliptical orbits are called *bound* orbits, and have negative total energy:

$$E < 0$$

Interestingly, if we take a parabolic orbit with $E = 0$ and subtract a little energy from the total (by some external means), then the parabola becomes an ellipse by having the second focus come in from infinity.

We ought to be able to prove the total energy is negative for an elliptical orbit. Start with the total energy

$$E = \frac{1}{2} m_* v^2 - \frac{G m_1 m_2}{r} \, ,$$

and substitute v^2 using

$$\vec{v} = \frac{v_0}{e} \left(\hat{\theta} + e \, \hat{y} \right) \, ,$$

which excludes the case of circles. Proceeding carefully, find

$$\begin{aligned} v^2 &= \frac{v_0^2}{e^2} \left(\frac{2R_0}{r} - 1 + e^2 \right) \\ &= \frac{2 G m_1 m_2}{m_* r} - \frac{G m_1 m_2}{m_* R_0} \left(1 - e^2\right) \, , \end{aligned}$$

so the kinetic term is

$$E_{kin} = \frac{G m_1 m_2}{r} - \frac{G m_1 m_2}{2 R_0} \left(1 - e^2\right) \, .$$

The total energy sums the potential plus the kinetic, which happens to contain equal and opposite $1/r$-like terms, leaving just the constant:

$$E = \frac{-G m_1 m_2}{2 R_0} \left(1 - e^2\right) = \frac{-G m_1 m_2}{2a} \, ,$$

in terms of v_0,

$$E = -\frac{1}{2} m_* v_0^2 \left(\frac{1 - e^2}{e^2} \right) \, .$$

Hyperbolic Orbit

Hyperbolic orbits are called *unbound* orbits, and have positive total energy:

$$E > 0$$

The analysis of this situation follows exactly like the elliptical case. For the total energy, you can see $e > 1$ simply flips the sign to make

$$E = \frac{G m_1 m_2}{2a} = \frac{1}{2} m_* v_0^2 \left(\frac{e^2 - 1}{e^2} \right) \, .$$

Circular Orbit

For circular orbits, we need to go back to the velocity equation

$$\vec{v} = \frac{G m_1 m_2}{L} \, \hat{\theta} \, ,$$

which has no v_0-term.

The angular momentum is

$$L = m_* R^2 \frac{d\theta}{dt} = m_* a^2 \frac{2\pi}{T} \, ,$$

where T is the period of the orbit and R is the radius. Simplifying gives

$$L = m_* \sqrt{G\left(m_1 + m_2\right) R} \, ,$$

and then the square of the velocity is:

$$v^2 = \frac{G\left(m_1 + m_2\right)}{R}$$

The time derivative of $\vec{v}$ gives the a familiar equation for the acceleration

$$\vec{a} = -\frac{G m_1 m_2}{L} \frac{d\theta}{dt} \, \hat{r} \, ,$$

which for circular orbits simplifies to

$$\vec{a} = \frac{-v^2}{R}\,\hat{r}\;,$$

as expected for circular motion in general.

The energy of a circular orbit is

$$E = \frac{1}{2}\frac{Gm_1m_2}{R} - \frac{Gm_1m_2}{R} = \frac{-Gm_1m_2}{2R}\;,$$

thus the kinetic energy is half the potential energy, and the total is negative.

Eccentricity and Orbit

Begin with the Runge-Lorenz vector and replace $\vec{L}$ using its definition:

$$\vec{Z} = \vec{v} \times (m_* \vec{r} \times \vec{v}) - Gm_1m_2\,\hat{r}\;,$$

and square the whole equation:

$$\begin{aligned}\vec{Z}\cdot\vec{Z} &= |\vec{v} \times (m_*\vec{r} \times \hat{v})|^2 \\ &\quad - 2Gm_1m_2\,\vec{v} \times (m_*\vec{r} \times \hat{v})\cdot\hat{r} + G^2m_1^2m_2^2\end{aligned}$$

For the first term on the right, notice $\vec{v}$ is perpendicular to $\vec{r}\times\vec{v}$, so

$$|\vec{v} \times (m_*\vec{r} \times \hat{v})| = m_* r v^2\,|\sin(\phi)|\;,$$

where ϕ is the angle between $\vec{r}$ and $\vec{v}$.

For the second term, the scalar triple product can be rewritten

$$\vec{v} \times (m_*\vec{r} \times \hat{v})\cdot\hat{r} = m_*\,(\vec{r}\times\vec{v})\cdot(\hat{r}\times\vec{v})\;.$$

The remaining vectors are parallel and the whole quantity simplifies to

$$\vec{v} \times (m_*\vec{r} \times \hat{v})\cdot\hat{r} = m_* r v^2 \sin^2(\phi)\;.$$

Rewriting $\vec{Z}\cdot\vec{Z}$ with this in mind, we have

$$\begin{aligned}Z^2 &= \left(m_* r v^2\right)^2 \sin^2(\phi) \\ &\quad - 2Gm_1m_2m_* r v^2 \sin^2(\phi) + G^2m_1^2m_2^2\;,\end{aligned}$$

or

$$\frac{Z^2}{G^2m_1^2m_2^2} = 1 + \sin^2(\phi)\left(q^2 - 2q\right)\;,$$

where

$$q = \frac{m_* r v^2}{Gm_1m_2}\;.$$

simplifying this further gives

$$\frac{Z^2}{G^2m_1^2m_2^2} = \cos^2(\phi) + \sin^2(\phi)\,(1-q)^2$$

Finally, note that the left size is actually the square of the eccentricity, giving, after restoring q:

$$e^2 = \cos^2(\phi) + \sin^2(\phi)\left(1 - \frac{rv^2}{G\,(m_1+m_2)}\right)^2$$

This is an enlightening result. For $\phi = 0$ the motion is purely radial and uninteresting. For all other cases, we see the combination of variables being suspiciously like to the escape velocity. Swapping this in gives

$$e^2 = \cos^2(\phi) + \sin^2(\phi)\left(1 - \frac{2v^2}{v_e^2}\right)^2$$

We see if $v = v_e$, then the eccentricity is precisely one, which is consistent with what we know of parabolic orbits. Similarly we see the cases $v < v_e$ and $v > v_e$ give $e < 1$ and $e > 1$ respectively, which is the signature of elliptic and hyperbolic orbits. A circular orbit has $\phi = \pi/2$.

10.11 Gravity Near Earth

Students of classical physics find out early that the force due to gravity near Earth's surface is a vector pointing straight down

$$\vec{F}_g = -mg\,\hat{y}\;,$$

with corresponding potential energy

$$U(y) = mgy\;,$$

where y is the height above the surface (or a location near it), and g is the local gravitation constant:

$$g = \frac{9.8\,m}{s^2}$$

On the other hand, we just went through all the pains of showing that the gravitational force is

$$\vec{F}(\vec{r}) = -\frac{Gm_1m_2}{r^2}\,\hat{r}$$

with potential energy

$$U(r) = -\frac{Gm_1m_2}{r}\;.$$

Clearly, these two pictures must be reconciled. To do so, let r be replaced by the quantity $R + y$, where R is a constant distance we'll take to be the radius of the Earth, and y is the effective height, approximately from sea level. What we assume throughout is that $y \ll R$.

Without loss of generality, we can assume all displacements are one dimensional and thus $\hat{r} = \hat{y}$. This identifies m_1 for the mass of the Earth, and m_2 for the mass of a test projectile.

With these restrictions, the force and energy become:

$$\vec{F}(y) = -\frac{Gm_1m_2}{(R+y)^2}\,\hat{y}$$
$$U(y) = -\frac{Gm_1m_2}{(R+y)}$$

Next apply binomial expansion to each denominator, particularly:

$$(R+y)^{-2} \approx \frac{1}{R^2} - \frac{2y}{R^3} + \frac{3y^2}{R^4} - \cdots$$
$$(R+y)^{-1} \approx \frac{1}{R} - \frac{y}{R^2} + \frac{y^2}{R^3} - \cdots$$

To first order, the above equations become

$$\vec{F}(y) \approx -\frac{Gm_1m_2}{R^2}\left(1 - \frac{2y}{R}\right)\hat{y}$$
$$U(y) \approx -\frac{Gm_1m_2}{R}\left(1 - \frac{y}{R}\right) .$$

We want the force equation to be constant, and it just happens that the quantity $2y/R$ is negligible, so the effective force at the surface is

$$\vec{F}_g = -\frac{Gm_1m_2}{R^2}\,\hat{y}$$

This tells where g comes from:

$$g = \frac{Gm_{\text{Earth}}}{R^2_{\text{Earth}}} .$$

For the potential energy, we have

$$U(y) = U_0 + mgy ,$$

where U_0 is the potential energy at $y = 0$, often defined to be zero, and the unscripted mass m is that of a test particle (not the Earth).

Note that the first-order potential term is maintained despite y/R being a very small number. The reason for this not just to recover the form mgy, but also the first derivative must equal a constant, which is what we asked of the force.

11 Three Dimensions

11.1 Cartesian Coordinates

The extension of the Cartesian coordinate system to three dimensions is straightforward. The xy plane is upgraded to the xyz volume, which requires three coordinates to specify any point in the system as shown in Figure 1.1.

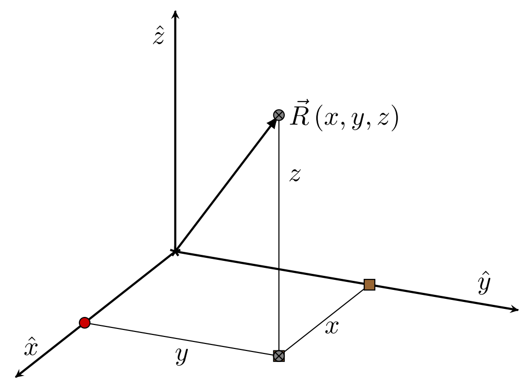

Figure 1.1: Cartesian coordinate system.

The position vector becomes

$$\vec{R} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} ,$$

which requires three mutually-perpendicular basis vectors:

$$\hat{x} = \hat{i} = \hat{e}_x = \hat{e}_i = \langle 1, 0, 0 \rangle$$
$$\hat{y} = \hat{j} = \hat{e}_y = \hat{e}_j = \langle 0, 1, 0 \rangle$$
$$\hat{z} = \hat{k} = \hat{e}_z = \hat{e}_k = \langle 0, 0, 1 \rangle$$

Quantities like velocity and acceleration simply take on a new z-component:

$$\vec{v} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$
$$\vec{a} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z}$$

The same goes for the differential line element

$$d\vec{S} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

and the differential interval

$$dS^2 = dx^2 + dy^2 + dz^2 .$$

Cartesian Area Element

In three dimensions, the notion of 'area element' takes on three meanings. An area can be 'facing' along the x, y, or z direction. A differential patch of area parallel to the xy-plane is written

$$dA_z = dx\,dy .$$

Similarly, a differential patch of area parallel to the yz-plane is

$$dA_x = dy\,dz ,$$

and finally, differential patch of area parallel to the zx-plane is

$$dA_y = dz\,dx .$$

Cartesian Volume Element

Each of the differential area elements can be turned into a differential volume element by multiplying by dz, dx, dy, respectively:

$$\begin{aligned} dz\, dA_z &= dx\, dy\, dz \\ dx\, dA_x &= dy\, dz\, dx \\ dy\, dA_y &= dz\, dx\, dy \end{aligned}$$

Each of these describes the same *Cartesian volume element*:

$$dV = dx\, dy\, dz$$

11.2 Rotations

...

11.3 Planes

A line is given by $y = mx + b$ in Cartesian coordinates. This construction is convenient and ergonomic, but is more versatile if written

$$Ax + By + C = 0\,.$$

This is still the equation of a line, but the y-variable is treated on equal footing to the x-variable. The line passes through the origin if $C = 0$.

Equation of a Plane

The notion of the straight line can be extended by one dimension to *planes* in three-dimensional space. The equation of a plane in Cartesian coordinates is

$$ax + by + cz + d = 0\,.$$

The variables a, b, c together govern the overall slant of the plane. The plane passes through the origin if $d = 0$.

Vector Representation

Interestingly, observe that the quantity $ax + by + cz$ looks like the dot product of two vectors. Define a vector $\vec{P}$ to hold the coefficients

$$\vec{P} = a\,\hat{x} + b\,\hat{y} + c\,\hat{z}\,,$$

and project this into the position vector

$$\vec{R} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

to find

$$\vec{R} \cdot \vec{P} = -d\,.$$

11.4 Normal Vector to a Plane

Consider any given point $\vec{R}_0$ that is known to be in the plane. For any point $\vec{R}$ that is also in the plane, the difference

$$\Delta\vec{R} = \vec{R} - \vec{R}_0$$

is a tangent vector that stays embedded in the plane.

This is enough to define the notion of the normal vector to the plane $\vec{N}$ such that

$$\Delta\vec{R} \cdot \vec{N} = 0\,.$$

Peel apart the $\Delta\vec{R}$ term to write:

$$\vec{R} \cdot \vec{N} = \vec{R}_0 \cdot \vec{N}$$

The above looks very much like the equation of the plane when written

$$\vec{R} \cdot \vec{P} = -d\,.$$

Subtract the two equations to get

$$\vec{R} \cdot \left(\vec{N} - \vec{P}\right) = \vec{R}_0 \cdot \vec{N} + d\,,$$

and then notice d is also equivalent to $-\vec{R}_0 \cdot \vec{P}$. The above becomes:

$$\vec{R} \cdot \left(\vec{N} - \vec{P}\right) = \vec{R}_0 \cdot \left(\vec{N} - \vec{P}\right)$$

This result needs to be true for all $\vec{R}$, which can only mean the parenthesized quantities are zero, which means $\vec{N} = \vec{P}$. The normal vector is the list of coefficients on x, y, and z:

$$\vec{N} = a\,\hat{x} + b\,\hat{y} + c\,\hat{z}$$

Two Planes Intersecting

Consider two non-parallel planes in three-dimensional space

$$\begin{aligned} ax + by + cz + d &= 0 \\ Ax + By + Cz + D &= 0\,. \end{aligned}$$

Somewhere out in the Cartesian volume is a line of intersection between the planes. To find such a line, take the cross product of the normal vector of each:

$$\vec{L} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a & b & c \\ A & B & C \end{vmatrix} = \langle L_x, L_y, L_z \rangle$$

The vector $\vec{L}$ indicates the intersection of each plane. To establish this, note that the following pair of equations must be true:

$$\begin{aligned} \vec{N}_1 \cdot \vec{L} &= 0 \\ \vec{N}_2 \cdot \vec{L} &= 0 \end{aligned}$$

Substitute the proposed form for $\vec{L}$,

$$\vec{N}_1 \cdot \left(\vec{N}_1 \times \vec{N}_2\right) = 0$$
$$\vec{N}_2 \cdot \left(\vec{N}_1 \times \vec{N}_2\right) = 0 \, ,$$

and notice each as a triple product to finish the job:

$$\vec{N}_2 \cdot \left(\vec{N}_1 \times \vec{N}_1\right) = 0$$
$$\vec{N}_1 \cdot \left(\vec{N}_2 \times \vec{N}_2\right) = 0$$

11.5 Point Intersecting Triangle

Consider three points in Cartesian space that mark the vertices of a triangle

$$\vec{P}_j = x_j \, \hat{x} + y_j \, \hat{y} + z_j \, \hat{z} \, ,$$

where $j = 1, 2, 3$. Also consider a fourth point in space

$$\vec{Q} = Q_x \, \hat{x} + Q_y \, \hat{y} + Q_j \, \hat{z}$$

that may or may not be embedded in the triangle. The job is to find out whether this is so.

Improvising the Normal Vector

From the coordinates given, define a pair of tangent vectors $\vec{U}$, $\vec{V}$ such that

$$\vec{U} = \vec{P}_2 - \vec{P}_1$$
$$\vec{V} = \vec{P}_3 - \vec{P}_1 \, .$$

Since both vectors are in the same plane, their cross product will by definition point perpendicular to the plane, which is a normal vector:

$$\vec{N} = \vec{U} \times \vec{V}$$

The details of the cross product are straightforward and aren't necessary to spell out here.

Improvising a Coordinate System

The pair of vectors $\vec{U}$, $\vec{V}$, despite not being mutually perpendicular, can *still* be used as the basis for a coordinate system embedded in the plane of the triangle. Indeed, any point $\vec{p}$ in the plane cab ne expressed as a linear combination of the basis vectors

$$\vec{p} = \alpha \, \vec{U} + \beta \, \vec{V} \, ,$$

where α (Greek 'alpha') and β (Greek 'beta') are two real-valued parameters.

Since the normal vector adds a third dimension to the picture, it follows that any point in the three-dimensional Cartesian space can be located by the vector

$$\vec{R} = \alpha \, \vec{U} + \beta \, \vec{V} + \gamma \, \vec{N}$$

using a third parameter γ.

In this improvised coordinate system, the point $\vec{P}_1$ is considered to be the origin. That is, if all parameters are zero, points $\vec{p}$ and $\vec{R}$ land back at $\vec{P}_1$.

The basis vectors need not be normalized, although it is good practice to do so. Either way, the parameters take care of all scaling, and the magnitudes U, V, N are simple to calculate when needed.

Intersection Condition

The point $\vec{Q}$, despite being handed to us in Cartesian coordinates, can also be represented by the vector $\vec{R}$ for some choice of α, β, γ. If $\vec{Q}$ is in the same plane as the three points $\vec{P}_j$ provided, then the γ-parameter should be zero.

Let us propose that point $\vec{Q}$ is located in the *normalized* $\hat{U}$, $\hat{V}$, $\hat{N}$ system with new coefficients α, β, γ (that are different that those that occur in the non-normalized version). Tracing from the origin, we have

$$\vec{Q} = \vec{P}_1 + \alpha \, \hat{U} + \beta \, \hat{V} + \gamma \, \hat{N} \, ,$$

and project $\hat{N}$ into both sides to get

$$\vec{Q} \cdot \hat{N} = \vec{P}_! \cdot \hat{N} + \alpha \cancel{\hat{U} \cdot \hat{N}} + \beta \cancel{\hat{V} \cdot \hat{N}} + \gamma \, \hat{N} \cdot \hat{N} \, ,$$

and this lets us solve for gamma, which ought to be zero if $\vec{Q}$ is within the triangle:

$$\gamma = \hat{N} \cdot \left(\vec{Q} - \vec{P}_1\right)$$

Containment Condition

Supposing point $\vec{Q}$ satisfies $\gamma = 0$, the job now is to figure out whether $\vec{Q}$ is inside or outside the boundaries of the triangle defined by $\vec{P}_j$. Define a third tangent vector $\vec{W}$ that satisfies

$$\vec{U} + \vec{V} + \vec{W} = 0 \, ,$$

which is like a trip around the triangle. In accordance with the right hand rule, this trip around the triangle should go in the counterclockwise direction.

Now comes the important observation. In order for $\vec{Q}$ to be inside the triangle, $\vec{Q}$ must occur to the 'left' of the line formed by each vector $\vec{U}$, $\vec{V}$, $\vec{W}$ simultaneously. (We say 'left' and not 'right' due to the counterclockwise flow of the tangent vectors.)

To see this, write out

$$\vec{U} \times \left(\vec{Q} - \vec{P}_1\right) = \delta_U \hat{N}$$

for some parameter δ_U. Since $\vec{Q} - \vec{P}_1$ also lies in the plane, the above simplifies

$$U \left| \vec{Q} - \vec{P}_1 \right| \sin(\phi) = \delta_U \ ,$$

where ϕ is the angle formed between $\vec{U}$ and $\vec{Q} - \vec{P}_1$.

If $\vec{Q}$ is to the 'left' of the line made by $\vec{U}$ as described, then $\sin(\phi)$ is a positive quantity, and δ_U is also positive. Repeat for the V- and W-cases to decide if the point is inside the triangle. All δ-parameters must be positive.

11.6 Plane from Three Points

Three points not on the same line define an infinite plane. From the points

$$\begin{aligned} \vec{P}_1 &= 1\,\hat{x} + 0\,\hat{y} + 1\,\hat{z} \\ \vec{P}_2 &= 0\,\hat{x} + 1\,\hat{y} + 1\,\hat{z} \\ \vec{P}_3 &= 1\,\hat{x} + 1\,\hat{y} + 0\,\hat{z} \ , \end{aligned}$$

find the equation of the implied plane and report the result as

$$ax + by + cz + d = 0 \ .$$

Determining the plane implied by three points is identical to finding all points $\vec{Q}$ coplanar to the triangle. Proceed by defining a pair of vectors $\vec{U}$, $\vec{V}$:

$$\begin{aligned} \vec{U} &= \vec{P}_2 - \vec{P}_1 = -1\,\hat{x} + 1\,\hat{y} + 0\,\hat{z} \\ \vec{V} &= \vec{P}_3 - \vec{P}_1 = 0\,\hat{x} + 1\,\hat{y} - 1\,\hat{z} \end{aligned}$$

For a change of taste, calculate the normal vector using determinant/bracket notation:

$$\vec{N} = \vec{U} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle -1, -1, -1 \rangle$$

In other words, the normal vector is

$$\vec{N} = -1\,\hat{x} - 1\,\hat{y} - 1\,\hat{z} \ .$$

Next take any point known to be on the plane, such as $\vec{P}_1$, and propose the vector $\vec{Q}$ is always in the plane if

$$\left(\vec{Q} - \vec{P}_1 \right) \cdot \vec{N} = 0$$

is satisfied. In particular

$$((x-1)\,\hat{x} + (y-0)\,\hat{y} + (z-1)) \cdot \hat{N} = 0 \ ,$$

or, after simplifying:

$$x + y + z - 2 = 0$$

12 Non-Cartesian Coordinates

12.1 Cylindrical Coordinates

Another way of mapping the three-dimensional Cartesian space is with *cylindrical coordinates*. In this system, illustrated in Figure 1.2, the xy plane is replaced by the $\rho\phi$ plane, with radius ρ and angle ϕ playing their 'usual' roles in polar coordinates. The z-direction is handled much as in the ordinary Cartesian system.

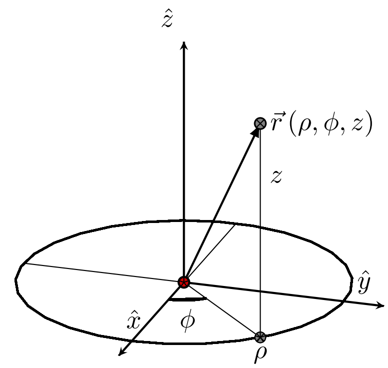

Figure 1.2: Cylindrical coordinate system.

Cylindrical Position Vector

In cylindrical coordinates, the position vector is:

$$\vec{r}\,(\rho, \phi, z) = \rho\,\hat{\rho} + z\,\hat{z} \ ,$$

or in terms of Cartesian directions,

$$\vec{r} = \rho \cos(\phi)\,\hat{x} + \rho \sin(\phi)\,\hat{y} + z\,\hat{z} \ .$$

As an extension of two already well-studied systems, it's worth mentioning but not writing that time derivatives of the position vector yield the velocity and acceleration. The $\hat{\rho}$ and $\hat{\phi}$ unit vectors are exactly analogous to $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ has no derivative.

Cylindrical Basis Vectors

The basis vectors $\hat{\rho}$, $\hat{\phi}$ are analogous to those in plane polar coordinates. In particular:

$$\begin{aligned} \hat{\rho} &= \cos(\phi)\,\hat{x} + \sin(\phi)\,\hat{y} \\ \hat{\phi} &= -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y} \end{aligned}$$

The derivatives follow the familiar pattern:

$$\begin{aligned} \frac{d\hat{\rho}}{d\phi} &= \hat{\phi} \\ \frac{d\hat{\phi}}{d\phi} &= -\hat{\rho} \end{aligned}$$

Cylindrical Line Element

The differential version of the cylindrical position vector $d\vec{S} = d\vec{r}$ gives the *cylindrical line element*:

$$d\vec{S} = d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

Differential Interval

The differential interval in cylindrical coordinates is straightforwardly calculated from the line interval:

$$dS^2 = d\vec{S}\cdot d\vec{S} = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

Differential Arc Length

The differential arc length is the positive square root of the differential interval:

$$dS = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2}$$

Cylindrical Area Element

Like the Cartesian case, the notion of area element takes three meanings. For a patch of area swept by constant ρ with small changes in z and ϕ, the area is

$$dA_\rho = \rho\,d\phi\,dz\ .$$

A patch of area that keeps ϕ constant is

$$dA_\phi = d\rho\,dz\ .$$

A patch of area that has z constant has area

$$dA_z = \rho\,d\rho\,d\phi\ .$$

Note that each of these can be constructed from various cross products of special cases of the cylindrical line element:

$$\begin{aligned} dA_\rho &= \left|\rho\,d\phi\,\hat{\phi}\times dz\,\hat{z}\right| \\ dA_\phi &= \left|d\rho\,\hat{\phi}\times dz\,\hat{z}\right| \\ dA_z &= \left|d\rho\,\hat{\rho}\times \rho\,d\phi\,\hat{\phi}\right| \end{aligned}$$

Cylindrical Volume Element

The volume element in cylindrical coordinates is the product of components of the line element:

$$dV = \rho\,d\rho\,d\phi\,dz$$

More rigorously, dV can be written as the triple product

$$dV = d\rho\,\hat{\rho}\cdot\left(\rho\,d\phi\,\hat{\phi}\times dz\,\hat{z}\right)$$

or any of its permutations.

12.2 Spherical Coordinates

The *spherical coordinate system* is another way to locate any point in three-dimensional space. There are three parameters: (i) a distance r from the origin, (ii) an angle θ measured from the positive z-axis, (iii) a polar parameter ϕ.

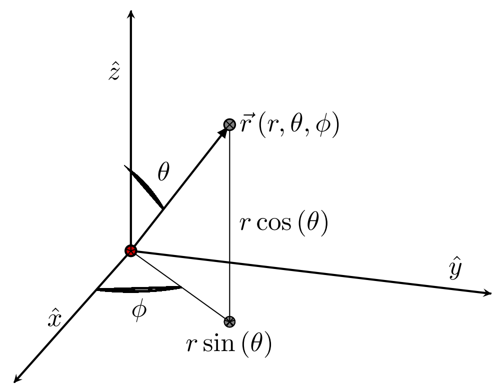

Figure 1.3: Spherical coordinate system.

Spherical Position Vector

In spherical coordinates, the position vector is

$$\begin{aligned} \vec{r}\,(r,\theta,\phi) = r\sin(\theta)\cos(\phi)\,\hat{x} \\ + r\sin(\theta)\sin(\phi)\,\hat{y} \\ + r\cos(\theta)\,\hat{z}\ , \end{aligned}$$

which can be discerned by analyzing Figure 1.3. Explicitly, this is stating

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta)\ . \end{aligned}$$

It's a worthwhile exercise to check that

$$\sqrt{\vec{r}\cdot\vec{r}} = \sqrt{r^2} = r$$

still holds, which means to make sure all of the trigonometric terms cancel inside the square root.

Spherical Basis Vectors

The position vector in spherical coordinates can be abbreviated

$$\vec{r} = r\,\hat{r}\,(r,\theta,\phi)\ ,$$

which tells us the radial basis vector:

$$\hat{r} = \frac{\vec{r}}{r} = \langle\sin(\theta)\cos(\phi)\,,\sin(\theta)\sin(\phi)\,,\cos(\theta)\rangle$$

We need to find two more basis vectors, namely $\hat{\phi}$, $\hat{\theta}$ for spherical coordinates. To figure out $\hat{\phi}$, notice that $r\sin(\theta)$ is the projected radius in the xy-plane, and ϕ is free to vary without affecting the projected radius. This means $\hat{\phi}$ has no radial component and no z-component, and is therefore analogous to the polar basis vector in the two-dimensional case. We conclude

$$\hat{\phi} = -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y} \; ,$$

or in tighter notation:

$$\hat{\phi} = \langle -\sin(\phi)\,, \cos(\phi)\,, 0 \rangle$$

The $\hat{\theta}$ unit vector chops downward in the direction of increasing θ, and is confined to the plane that hangs under the position vector. As a ninety-degree rotation from $\hat{r}$, we can quickly write

$$\begin{aligned} \hat{\theta}(\theta, \phi) = \sin\left(\theta + \frac{\pi}{2}\right)\cos(\phi)\,\hat{x} \\ + \sin\left(\theta + \frac{\pi}{2}\right)\sin(\phi)\,\hat{y} \\ + \cos\left(\theta + \frac{\pi}{2}\right)\hat{z} \; , \end{aligned}$$

simplifying to

$$\hat{\theta} = \langle \cos(\theta)\cos(\phi)\,, \cos(\theta)\sin(\phi)\,, -\sin(\theta) \rangle \; .$$

Basis Vector Orthogonality

Despite their messiness, the unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ are easily shown to have unit length

$$\hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{\theta} \cdot \hat{\theta} = 1 \; ,$$

and to be mutually perpendicular

$$\begin{aligned} \hat{r} \cdot \hat{\theta} = \hat{r} \cdot \hat{\phi} = 0 \\ \hat{\theta} \cdot \hat{\phi} = \hat{\theta} \cdot \hat{r} = 0 \\ \hat{\phi} \cdot \hat{r} = \hat{\phi} \cdot \hat{\theta} = 0 \; , \end{aligned}$$

and subordinate to the cross product:

$$\begin{aligned} \hat{r} \times \hat{\theta} = \hat{\phi} \\ \hat{\theta} \times \hat{\phi} = \hat{r} \\ \hat{\phi} \times \hat{r} = \hat{\theta} \end{aligned}$$

Basis Vector Derivatives

Various derivatives of the basis vectors establish relationships between them. In particular:

$$\begin{aligned} d\hat{r}/d\theta &= \hat{\theta} \\ d\hat{r}/d\phi &= \sin(\theta)\,\hat{\phi} \\ d\hat{\theta}/d\theta &= -\hat{r} \\ d\hat{\theta}/d\phi &= \cos(\theta)\,\hat{\phi} \\ d\hat{\phi}/d\theta &= 0 \\ d\hat{\phi}/d\phi &= -\sin(\theta)\,\hat{r} - \cos(\theta)\,\hat{\theta} \end{aligned}$$

Matrix Representation

The basis vectors in spherical coordinates are nicely organized in matrix notation

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = R_{r\theta\phi} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \; ,$$

where:

$$R_{r\theta\phi} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix}$$

Resolving Cartesian Basis

The process for isolating $\hat{x}$, $\hat{y}$, $\hat{z}$ in terms of $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ is equivalent to inverting the matrix $R_{r\theta\phi}$:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = R^{-1}_{r\theta\phi} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$$

Leaving the details to the reader, the required matrix is:

$$R^{-1}_{r\theta\phi} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \cos(\theta)\cos(\phi) & -\sin(\phi) \\ \sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{bmatrix}$$

Spherical Velocity Vector

The time derivative of $\vec{r} = r\,\hat{r}(r, \theta, \phi)$ gives the velocity vector in spherical coordinates. Doing this carefully, one should find:

$$\vec{v}(t) = \frac{dr}{dt}\,\hat{r} + r\frac{d\theta}{dt}\,\hat{\theta} + r\sin(\theta)\frac{d\phi}{dt}\,\hat{\phi}$$

Eliminating the dt-term by the chain rule gives the differential line element

$$d\vec{S} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin(\theta)\,d\phi\,\hat{\phi} \; ,$$

and the square of the differential line element is the differential interval:

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\,d\phi^2$$

The positive root of the differential interval is the differential arc length:

$$dS = \sqrt{dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\,d\phi^2}$$

Spherical Area Element

There are three ways to write an area element in spherical coordinates - one for each coordinate being fixed. Fixing r, the patch of spherical shell swept out is

$$dA_r = \left| r\, d\theta\, \hat{\theta} \times r \sin(\theta)\, d\phi\, \hat{\phi} \right| = r^2 \sin(\theta)\, d\theta\, d\phi\, .$$

The other two area elements come from fixing θ, ϕ respectively:

$$dA_\theta = \left| dr\, \hat{r} \times r \sin(\theta)\, d\phi\, \hat{\phi} \right| = r \sin(\theta)\, dr\, d\phi$$
$$dA_\phi = \left| dr\, \hat{r} \times r d\theta\, \hat{\theta} \right| = r\, dr\, d\theta$$

Spherical Volume Element

The volume element in spherical coordinates is the product of components of the line element:

$$dV = r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

More rigorously, dV can be written as the triple product

$$dV = dr\, \hat{r} \cdot \left(r\, d\theta\, \hat{\theta} \times r \sin(\theta)\, d\phi\, \hat{\phi} \right)$$

or any of its permutations.

13 Curves in Three Dimensions

13.1 Generalizing Plane Curves

In three dimensions, the position vector representing a parametric curve can be written

$$\vec{r}(t) = x(t)\, \hat{x} + y(t)\, \hat{y} + z(t)\, \hat{z}\, .$$

This is just a generalization of the two-dimensional case, where of course, we've used the Cartesian representation for simplicity, but any three-dimensional system works just as well.

Derivatives

The position vector admits time derivatives to yield the velocity vector and acceleration vector, so long as the parametric equations $x(t)$, $y(t)$, $z(t)$ are differentiable:

$$\vec{v}(t) = \left(\frac{d}{dt} x(t) \right) \hat{x} + \left(\frac{d}{dt} y(t) \right) \hat{y} + \left(\frac{d}{dt} z(t) \right) \hat{z}$$
$$\vec{a}(t) = \left(\frac{d^2}{dt^2} x(t) \right) \hat{x} + \left(\frac{d^2}{dt^2} y(t) \right) \hat{y} + \left(\frac{d^2}{dt^2} z(t) \right) \hat{z}$$

Tangent Vector

The (unit) tangent vector $\hat{T}$ is given by the velocity vector divided by the speed

$$\hat{T} = \frac{\vec{v}}{v}\, ,$$

and the velocity vector can also be written in terms of the speed and the tangent vector:

$$\vec{v}(t) = v\, \hat{T}$$

Another equation for the tangent vector reads

$$\hat{T} = \frac{d\vec{S}}{dS}\, ,$$

where $d\vec{S}$ is the differential line element, and dS is the differential arc length.

Curvature

The notion of curvature straightforwardly applies in three dimensions from the definition, particularly

$$\kappa = \left| \frac{d\hat{T}}{dS} \right|\, .$$

The tight formula for curvature also survives migration to three dimensions:

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{v^3}$$

Normal Vector

The (unit) normal vector $\hat{N}$ also needs no modification from the two-dimensional case:

$$\hat{N} = \frac{1}{\kappa} \frac{d\hat{T}}{dS}$$

Acceleration Vector

In terms of the tangent vector, the formula for the acceleration vector remains in tact as well:

$$\vec{a}(t) = \left(\frac{d^2 S}{dt^2} \right) \hat{T} + \kappa \left(\frac{dS}{dt} \right)^2 \hat{N}$$

13.2 The Binormal Vector

For a curve in three dimensions, the tangent vector $\hat{T}$ and normal vector $\hat{N}$ form the basis vectors for a plane, which itself has a normal vector called the *binormal.* The binormal vector is a unit vector $\hat{B}$, and is defined by

$$\hat{B} = \hat{T} \times \hat{N}\, .$$

There are two cyclic permutations of the definition that contain the same information:

$$\hat{T} = \hat{N} \times \hat{B}$$
$$\hat{N} = \hat{B} \times \hat{T}$$

Torsion

We can learn more about $\hat{B}$ by writing two straightforward consequences of its definition, namely

$$\hat{B} \cdot \hat{T} = 0$$
$$\hat{B} \cdot \hat{B} = 1 \, ,$$

and apply an arc length derivative to each:

$$\frac{d\hat{B}}{dS} \cdot \hat{T} + \hat{B} \cdot \frac{d\hat{T}}{dS} = 0$$
$$2\hat{B} \cdot \frac{d\hat{B}}{dS} = 0$$

Evidently, the derivative $d\hat{B}/dS$ is perpendicular to both $\hat{T}$ and $\hat{B}$. This can only mean $d\hat{B}/dS$ is parallel to the normal vector:

$$\frac{d\hat{B}}{dS} = -\tau \, \hat{N}$$

The proportionality constant τ (Greek 'tau'), defined with a minus sign, is called the *torsion* along the curve.

13.3 Serret-Frenet Formulas

The equations for $d\hat{T}/dS$ and $d\hat{B}/dS$ constitute two of the three *Serret-Frenet* formulas. To complete the set we need to find $d\hat{N}/dS$. By brute force, we have:

$$\begin{aligned} \frac{d\hat{N}}{dS} &= \frac{d}{dS}\left(\hat{B} \times \hat{T}\right) \\ &= -\tau \, \hat{N} \times \hat{T} + \kappa \, \hat{B} \times \hat{N} \\ &= \tau \hat{B} - \kappa \hat{T} \end{aligned}$$

The TNB Frame

The tangent, normal, and binormal vectors constitute three coordinate axes that trace along the curve, sometimes called the TNB frame.

Since the derivative of each vector is a linear combination in the TNB frame, the Serret-Frenet formulas lend nicely to matrix notation:

$$\frac{d}{dS}\begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix}$$

The matrix containing the κ, τ coefficients is skew-symmetric.